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$\lim +$, δ^+, and Non-Permutability of β-Steps

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ABSTRACT

Using a human-oriented formal example proof of the $\lim +$-theorem (that the sum of limits is the limit of the sum), we exhibit a non-permutability of β-steps and δ^+-steps (according to Smullyan's classification), which is not visible with non-liberalized δ-rules and dissolves into a problem of mere inefficiency with further liberalized δ-rules, such as the δ^{++}-rules. Beside a careful presentation of the human-oriented search for a formal proof of $(\lim +)$, our main intention is to show where sequent and tableau calculi are in conflict with human-oriented proof construction.

1. Motivation

In the theoretical part of an advanced senior-level lecture course on mathematics assistance systems (Autexier et al., 2004/05), I proved in a human-oriented sequent calculus that the sum of limits is the limit of the sum; for short: $(\lim +)$.

In this paper I revisit this proof with the aim to clarify and emphasize some issues regarding the (non-) permutability of inference steps in computer-assisted proof search.

Mathematics assistance systems are human-oriented interactive theorem provers with strong automation support. They aim at a synergetic interplay between mathematician and machine. Isabelle/HOL (Nipkow et al., 2002; Paulson, 1990), Coq (Bertot and Castéran, 2004), PVS (Owre, 2009), Ωmega (Siekmann et al., 2002), and QuodLibet (Avenhaus et al., 2003; Wirth, 2009) are some of the systems approaching this long term goal.

Computer-assisted proof construction does not necessarily aim at finding exactly those proofs (up to isomorphism) that would be found by working mathematicians in the absence of computer

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assistance. To the contrary, computer-assisted proof search with powerful automated theorem provers often results in proofs that would hardly be chosen by working mathematicians.

In the domain of the machines, the actual path taken by a proof hardly matters if a (trustworthy) computer program proves an obvious lemma completely automatically; and in the rare case that an open conjecture is proved automatically by a proving system, the mathematicians simply have to find a way on how to deal with this result: They may ignore the proof, take it for granted, or invent an *ad hoc* procedure on how to learn from the proving system.

In the human domain, computer assistance in theorem proving may not be needed, especially if a traditional mathematician is interested only in those problems that he can solve without machine assistance and if he omits the proofs of the many tedious technical lemmas of everyday mathematical practice as trivial (or delegates them to his students).

If, however, cooperation between mathematician and machine is intended, we find these two domains as well, but now with a higher degree of mutual dependence: If a powerful automated theorem prover requires a human assistant, then it is today unfortunately still so that this assistant cannot be just a working mathematician, but must be some well-trained expert, seasoned with the machine-oriented peculiarities of the proving system. To improve this situation in the future, on top of our agenda are more intelligent ways to fold lower level proof states into more concise human-readable forms of representation as well as more human-oriented calculi for machine search (because the automation of proof search will always fail on the lowest logic level from time to time). Currently, however, the only generally possible form of cooperation between a working mathematician and a theorem proving system seems to be that the system follows the proof ideas of the mathematician and that it is the task of the machine to assist the mathematician, and not vice versa.

Regarding *reductive calculi* (such as sequent, tableau, or matrix calculi), one of the functions of my lectures within the course was to show

- where sequent and tableau calculi are in conflict with human-oriented proof construction, and
- why *matrix* calculi (or *indexed formula trees* (Autexier, 2003; Wallen, 1990)) cannot only be seen as a clever implementation of *sequent* calculi,[1] but – more important in our context – why matrix calculi are also needed to be able to follow the proof search of a working mathematician more closely.[2]

I approached this goal by giving the students an idea of the premature commitments enforced by some sequent and tableau calculi, which may require a mathematician to deviate from his intended proof plans and proof-search heuristics.

In his fascinating book (Wallen, 1990), Lincoln A. Wallen criticized the non-permutability of γ- and δ-steps[3] in sequent calculi (cf. Section 2 of this paper). I explained how this non-permutability can be overcome by replacing the (non-liberalized) δ-rule (which we will call δ^--*rule*) with the liberalized δ^+-rule (Hähnle and Schmitt, 1992). Along the proof of $(\lim +)$, I then showed that with the δ^+-rule, however, another non-permutability becomes visible, now of the β- and δ^+-steps.

Before the liberalization, this non-permutability was *hidden behind the non-permutability of the* γ*-and* δ^-*-steps*.[4] Moreover, after further liberalization of the δ-rule, this non-permutability dissolves

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1 Because of the locality of sequent calculi, however, sequent calculi are easier to present and understand than matrix calculi.

2 The problem with sequent and tableau calculi will have become clear in Section 4.9. How matrix calculi can help us to follow the proof search of a working mathematician more closely can be found in Autexier (2003).

3 This is to be understood according to Raymond M. Smullyan's classification and uniform notation of reductive inference rules as α, β, γ, and δ (Smullyan, 1968).

4 Anonymous reviewers of previous versions of this paper wrote:

> "…a very interesting insight, namely that different non-permutabilities can hide each other."
> "To speak of a 'hidden non-permutability' is, at best, a questionable interpretation with hindsight."

The verbalization captures the state of affairs (described in more detail in Section 5.2) from the point of view of a working mathematician: Comparing a calculus and its improved version, if the improved version still has a weakness of design – namely an enforcing of a premature commitment that was not visible before because of a further weakness of the simple version – then the weakness of the simple version may well be said to have "hidden" the remaining weakness of the improved version.

into a problem of mere inefficiency. This may be the reason why the non-permutability problems of β-steps are not only unpublished, but had even been denied by experts of the field.

Beside this hard result on non-permutability, following the lecture, in this paper we will address several soft aspects of formal calculi for human–machine interaction. Moreover – to the best of our knowledge for the first time – we publish a more or less readable, complete, and human-oriented proof of a mathematical standard theorem in a standard general-purpose formal calculus (Sections 3 and 4). We discuss the non-permutabilities of this example proof (Section 5), prove the non-permutability of its crucial β- and δ^+-steps (Section 6), and discuss related problems (Section 7).

2. Weaknesses in design of reductive calculi

As already explained in Wallen (1990), the search space of sequent or tableau calculi may suffer from the following weaknesses in design:

- *irrelevance*,
- enforcement of *notational redundancy*, and
- *non-permutability*.

Unless explicitly stated otherwise, the points described in the following apply to sequent and tableau calculi alike.

Irrelevance means, e.g., that when proving the sequent

$$A, \quad \neg(B \wedge \mathsf{Loves}(\mathsf{Romeo}, y_0^\gamma)), \quad \mathsf{Loves}(\mathsf{Romeo}, \mathsf{Juliet})$$

with A and B some big formulas, we may try to prove A or $\neg B$ for a long time, although this is not relevant if they are false.

Note that, in this paper, *sequents* are just lists of formulas understood disjunctively. (This is the simplest form of sequents that will do for two-valued logics.)

We call *free γ-variables* (after the γ-steps, which may introduce new ones) (written as y_0^γ) what has the standard names of "meta" (Nipkow et al., 2002) or "free" variables (Fitting, 1996). Indeed, free γ-variables must be distinguished from the true meta-variables and the other kinds of free variables we will need.

The means to avoid irrelevance is focusing on *connections*, just as the one between $\neg\mathsf{Loves}(\mathsf{Romeo}, y_0^\gamma)$ and $\mathsf{Loves}(\mathsf{Romeo}, \mathsf{Juliet})$. In practice of mathematics assistance systems, it is often necessary, however, to expand connectionless parts to support the speculation of lemmas, which then provide a "connection" that is not syntactically obvious, but closes the branch nevertheless. This is especially the case for inductive theorem proving for theoretical (Kreisel, 1965) and practical (Schmidt-Samoa, 2006a,b,c) reasons.

Notational Redundancy means (in a sequent-calculus proof) that the offspring sequents repeat the formulas of their ancestor sequents again and again. This is partly overcome in the corresponding tableau calculi. But even tableau proofs repeat the subformulas of their *principal formulas*[5] as *side formulas* again and again. *Structure sharing* can overcome this redundancy and does not differ much for sequent, tableau, or matrix calculi, because information on branch, γ-multiplicity, and fairness has to be stored anyway.

Non-permutability is the subject of this paper. Very roughly speaking, it means that the *order* of inference *steps* (i.e. applications of reductive inference rules) may be crucial for a proof to succeed.

[5] The notions of a *principal formula* and a *side formula* were introduced in Gentzen (1935) and refined in Schmidt-Samoa (2006c). Very roughly speaking, the principal formula of a reductive inference rule is the formula that is taken to pieces by that rule, and the side formulas are the resulting pieces. In Fig. 1, the principal formulas are the formulas above the lines except the ones in Γ, Π, and the side formulas are the formulas below the lines except the ones in Γ, Π.

Roughly speaking, permutability of two steps S_1 and S_0 simply means the following:

> *In a closed proof tree where S_0 precedes S_1 and where S_1 was already applicable before S_0, we can do the step S_1 before S_0 and expand the resulting new subtree into a closed proof tree nevertheless.*

This informal definition of non-permutability will suffice until Section 6.2, where the reader can find the details of a technically involved formal definition.

When several formulas in a sequent classify as principal formulas of α-, β-, γ-, or δ-steps, the search space is typically non-confluent. Therefore, a bad order of application of these inference steps may require the search procedure to backtrack or to construct a proof on a level of γ-multiplicity that is higher than necessary or higher than what a mathematician would expect. Note that in the latter case, a human user has little chance to cooperate successfully in proof construction: Which mathematician would give a system the hint to apply a lemma *twice* when he knows that one application suffices?[6]

For example, when we do a γ-step first and a δ-step second, a proof may fail on the given level of γ-multiplicity, whereas it succeeds when we apply the δ-step first and the γ-step second. For sequent calculi without free variables (Gentzen, 1935), this is exemplified in Wallen (1990, Chapter 1, Section 4.3.2). The reason for this non-permutability is simply that, for the first alternative, the γ-step cannot instantiate its side formula with the parameter introduced by the δ-step because of the *eigenvariable condition*.

This non-permutability is *not* overcome with the introduction of free γ-variables, resulting in the so-called "free-variable" calculi (Hähnle and Schmitt, 1992; Fitting, 1996; Wirth, 2004): The reason for the non-permutability is now that (for the first alternative) the *variable-condition* blocks the free γ-variable y^γ introduced by the γ-step against the instantiation of any term containing the *free δ^--variable* x^{δ^-} introduced by the δ^--step. In inference systems that use Skolemization instead of variable-conditions, however, this non-permutability is isomorphically expressed as follows: y^γ becomes an argument of the Skolem term $x^{\delta^-}(\ldots y^\gamma \ldots)$ introduced by the δ^--step, which then causes unification of y^γ and $x^{\delta^-}(\ldots y^\gamma \ldots)$ to fail by the occur check.

This non-permutability is overcome in Wallen (1990, Chapter 2) with a matrix calculus which generates variable-conditions that are equivalent to the effect of *Outer Skolemization*. *A fortiori*, this non-permutability is overcome by the replacement of the δ^--steps with δ^+-steps, because δ^+-steps (Hähnle and Schmitt, 1992) extend a variable-condition only equivalently to *Inner Skolemization* (which is an improvement over Outer Skolemization, i.e. less blockings (or less occurrences of free γ-variables in Skolem-terms); cf. (Nonnengart, 1996) or (Wirth et al., 2009, Note 59)).

Optimization Problems where a badly chosen order of inference steps does not cause a failure of the proof attempt (at the given level of γ-multiplicity) but only an increase in proof size, are not subsumed under the notion of non-permutability. A typical optimization problem is the following: The size of a proof crucially depends on the β-steps being applied not too early and in the right order. This is obvious from a working mathematician's point of view: "Do not start a case analysis before it is needed and make the nested case assumptions in an order that unifies identical argumentations!"

Thus, assuming an any-time behavior of a semi-decision procedure for closedness running in parallel (because *simultaneous rigid E-unification* is not co-semi-decidable (Degtyarev and Voronkov, 1998)), the *folklore heuristics* is roughly as follows:

- Step 1: Apply all α- and δ-steps, guaranteeing termination by deleting their principal formulas from the child sequents (either directly syntactically in sequent calculi, or indirectly by some bookkeeping for search control in tableau calculi).
- Step 2: If a γ-rule is applicable to a principal formula that has not reached the current threshold for γ-multiplicity in some branch, do such a γ-step, namely the one with the most promising connections, and then go to Step 1.
- Step 3: If a β-rule is applicable, then apply the most promising one, deleting its principal formula from the child sequents, and then go to Step 1.
- Step 4: If a γ-rule is applicable, then increase the threshold for γ-multiplicity and go to Step 2.

[6] Cf. the discussion in Section 1.

3. Background required for the example proof

Before we go on with this abstract expert-style discussion in Section 5, we do the proof of $(\lim +)$ in Section 4. To this end, we now present a sub-calculus of the calculus of Wirth (2004), whose development was driven by the integration of FERMAT's *descente infinie* into state-of-the-art deduction, with human-orientedness as the second design goal. The calculus uses variable-conditions instead of SKOLEMization. Variable-conditions are isomorphic to SKOLEMization in the relevant aspects of this paper, but admit the usage of simple variables instead of huge SKOLEM terms. This improves the readability of our formal proof significantly. We assume the following sets of *variables* to be disjoint:

V_γ	*free γ-variables*,	i.e. the free variables of Fitting (1996)
V_δ	*free δ-variables*,	i.e. nullary parameters, instead of SKOLEM functions
$\mathsf{V}_{\mathrm{bound}}$	*bound variables*,	i.e. variables to be bound, cf. below

We use '$\uplus$' for the union of disjoint classes. We partition the free δ-variables into *free δ^--variables* and *free δ^+-variables*: $\mathsf{V}_\delta = \mathsf{V}_{\delta^-} \uplus \mathsf{V}_{\delta^+}$. We define the *free variables* by $\mathsf{V}_{\mathrm{free}} := \mathsf{V}_\gamma \uplus \mathsf{V}_\delta$ and the *variables* by $\mathsf{V} := \mathsf{V}_{\mathrm{bound}} \uplus \mathsf{V}_{\mathrm{free}}$. Finally, the *rigid*[7] *variables* by $\mathsf{V}_{\gamma\delta^+} := \mathsf{V}_\gamma \uplus \mathsf{V}_{\delta^+}$. We use $\mathcal{V}_k(\Gamma)$ to denote the set of variables from V_k occurring in Γ.

We do not permit binding of variables that already occur bound in a term or formula; that is: $\forall x.\ A$ is only a formula if no binder on x already occurs in A. The simple effect is that our formulas are easier to read and our γ- and δ-rules can replace *all* occurrences of the bound variable x. Moreover, we assume that all binders have minimal scope (i.e. $\forall x.\ A \vee B$ reads $(\forall x.\ A) \vee B$, and not $\forall x.\ (A \vee B)$).

Let σ be a *substitution*. We say that σ is a *substitution on X* if $\mathrm{dom}(\sigma) \subseteq X$. We denote with "$\Gamma\sigma$" the result of replacing each occurrence of every variable $x \in \mathrm{dom}(\sigma)$ in Γ with $\sigma(x)$. Unless stated otherwise, we tacitly assume that all occurrences of variables from $\mathsf{V}_{\mathrm{bound}}$ in a term or formula or in the range of a substitution are *bound occurrences* (i.e. that a variable $x \in \mathsf{V}_{\mathrm{bound}}$ occurs only in the scope of a binder on x) and that each substitution σ satisfies $\mathrm{dom}(\sigma) \subseteq \mathsf{V}_{\mathrm{free}}$, so that no bound occurrences of variables can be replaced and no additional variable occurrences can become bound (i.e. captured) when applying σ.

Definition 1 (*Variable-Condition, σ-Update, R-Substitution*)**.**
A *variable-condition* is a subset of $\mathsf{V}_{\mathrm{free}} \times \mathsf{V}_{\mathrm{free}}$.
Let R be a variable-condition and σ be a substitution. The *σ-update of R* is

$$R \quad \cup \quad \{\, (z^{\mathrm{free}}, x^{\mathrm{free}}) \mid x^{\mathrm{free}} \in \mathrm{dom}(\sigma) \wedge z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}})) \,\}.$$

σ is an *R-substitution* if σ is a substitution and the σ-update R' of R is *well-founded*; i.e. for every nonempty set B, there is a $b \in B$ such that there is no $a \in B$ with $a\, R'\, b$. □

Note that $(x^{\mathrm{free}}, y^{\mathrm{free}}) \in R$ is intended to mean that an R-substitution σ must not replace x^{free} with a term in which y^{free} could ever occur. This is guaranteed when the σ-updates R' of R are always required to be well-founded. Indeed, for $z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}}))$, we get $z^{\mathrm{free}}\, R'\, x^{\mathrm{free}}\, R'\, y^{\mathrm{free}}$, blocking z^{free} against terms containing y^{free}. In practice, a σ-update of R can always be chosen to be finite. In this case, it is well-founded iff it is acyclic.

3.1. Inference rules for reduction within a proof tree

In Fig. 1, the inference rules for reductive reasoning within a tree are presented in sequent style. Note that GENTZEN would have inverted the inference rules such that passing the line means consequence. In our case, passing the line means reduction, and trees grow downward.

[7] Contrary to free δ^--variables (which are true parameters in the sense that they cannot be instantiated in purely reductive calculi), free δ^+-variables are indeed *rigid* in the sense that we may globally instantiate some of them simultaneously in the whole proof forest, provided that we can prove that their associated choice-conditions are met by this instantiation; cf. Section 3.1. Thus, though introduced by δ-rules, the δ^+-variables are very close to γ-variables in the sense that we could define the γ-variables as δ^+-variables with empty choice-conditions, i.e. with identically true choice-conditions. This closeness is actually implemented as identity in Wirth (2011).

Let A and B be formulas. Let Γ and Π be sequents, i.e. disjunctive lists of formulas. Let $x \in \mathrm{V}_{\mathrm{bound}}$ be a bound variable, and let $\mathcal{F}$ be the current proof forest, such that $\mathcal{V}(\mathcal{F})$ contains all variables already in use, especially those from Γ, Π, and A. Note that $\overline{A}$ is the *conjugate* of the formula A, i.e. B if A is of the form $\neg B$, and $\neg A$ otherwise.

α**-rules** $\frac{\alpha}{\alpha_0}$:

$$\frac{\Gamma\ \neg\neg A\ \Pi}{A\ \Gamma\ \Pi} \qquad \frac{\Gamma\ (A{\vee}B)\ \Pi}{A\ B\ \Gamma\ \Pi} \qquad \frac{\Gamma\ \neg(A{\wedge}B)\ \Pi}{\overline{A}\ \overline{B}\ \Gamma\ \Pi} \qquad \frac{\Gamma\ (A{\Rightarrow}B)\ \Pi}{\overline{A}\ B\ \Gamma\ \Pi} \qquad \frac{\Gamma\ (A{\Leftarrow}B)\ \Pi}{A\ \overline{B}\ \Gamma\ \Pi}$$

β**-rules** $\frac{\beta}{\beta_1 \quad \beta_2}$:

$$\frac{\Gamma\ (A{\wedge}B)\ \Pi}{\begin{array}{c}A\ \Gamma\ \Pi\\ B\ \Gamma\ \Pi\end{array}} \qquad \frac{\Gamma\ \neg(A{\vee}B)\ \Pi}{\begin{array}{c}\overline{A}\ \Gamma\ \Pi\\ \overline{B}\ \Gamma\ \Pi\end{array}} \qquad \frac{\Gamma\ \neg(A{\Rightarrow}B)\ \Pi}{\begin{array}{c}A\ \Gamma\ \Pi\\ \overline{B}\ \Gamma\ \Pi\end{array}} \qquad \frac{\Gamma\ \neg(A{\Leftarrow}B)\ \Pi}{\begin{array}{c}\overline{A}\ \Gamma\ \Pi\\ B\ \Gamma\ \Pi\end{array}}$$

γ**-rules** $\frac{\gamma}{\gamma_0(t)}$: Let t be any term (by default a new free γ-variable):

$$\frac{\Gamma\quad \exists x.A\quad \Pi}{A\{x{\mapsto}t\}\quad \Gamma\quad \exists x.A\quad \Pi} \qquad \frac{\Gamma\quad \neg\forall x.A\quad \Pi}{\overline{A\{x{\mapsto}t\}}\quad \Gamma\quad \neg\forall x.A\quad \Pi}$$

δ^-**-rules** $\frac{\delta}{\delta^-_0(x^{\delta^-})}$: Let $x^{\delta^-} \in \mathrm{V}_{\delta^-} \setminus \mathcal{V}(\mathcal{F})$ be a new free δ^--variable:

$$\frac{\Gamma\quad \forall x.A\quad \Pi}{A\{x{\mapsto}x^{\delta^-}\}\quad \Gamma\quad \Pi} \qquad \mathcal{V}_{\gamma\delta^+}(\Gamma\ \ \forall x.A\ \ \Pi) \times \{x^{\delta^-}\}$$

$$\frac{\Gamma\quad \neg\exists x.A\quad \Pi}{\overline{A\{x{\mapsto}x^{\delta^-}\}}\quad \Gamma\quad \Pi} \qquad \mathcal{V}_{\gamma\delta^+}(\Gamma\ \ \neg\exists x.A\ \ \Pi) \times \{x^{\delta^-}\}$$

δ^+**-rules** $\frac{\delta}{\delta^+_0(x^{\delta^+})}$: Let $x^{\delta^+} \in \mathrm{V}_{\delta^+} \setminus \mathcal{V}(\mathcal{F})$ be a new free δ^+-variable:

$$\frac{\Gamma\quad \forall x.A\quad \Pi}{A\{x{\mapsto}x^{\delta^+}\}\quad \Gamma\quad \Pi} \qquad \begin{array}{l}\{(x^{\delta^+},\ \overline{A\{x{\mapsto}x^{\delta^+}\}})\}\\ \mathcal{V}_{\mathrm{free}}(\forall x.A) \times \{x^{\delta^+}\}\end{array}$$

$$\frac{\Gamma\quad \neg\exists x.A\quad \Pi}{\overline{A\{x{\mapsto}x^{\delta^+}\}}\quad \Gamma\quad \Pi} \qquad \begin{array}{l}\{(x^{\delta^+},\ A\{x{\mapsto}x^{\delta^+}\})\}\\ \mathcal{V}_{\mathrm{free}}(\neg\exists x.A) \times \{x^{\delta^+}\}\end{array}$$

Fig. 1. The reductive rules of our calculus.

Of course, all rules in Fig. 1 are *sound*. Moreover, they even satisfy the stronger property of being *solution preserving for the rigid variables* in the sense of Wirth (2004, Section 2.4): A solution to the rigid variables of the child sequents (i.e. the lower sequents, the premises) is a solution to the parent sequent as well (i.e. the upper sequent, the conclusion). For our proof of (lim +) this means that (updating our global variable-condition R) we can globally apply any R-substitution on any subset of V_γ to the whole proof forest without destroying the soundness of the instantiated proof steps in any proof tree.

Instead of an eigenvariable condition, the δ^-- and δ^+-rules come with a binary relation on variables to the lower right, which must be added to the current variable-condition R. The δ^+-rules come with an additional relation to the upper right, which has to be added to the *R-choice-condition* C. This choice-condition is an optional part of the calculus. It may store a structure-sharing representation of an ε-term (Hilbert and Bernays, 1968/70; Giese and Ahrendt, 1999; Wirth, 2008, 2011) for a free δ^+-variable, which may restrict the possible values of this variable. As they play only a marginal rôle in the example proof of Section 4, we do not have to discuss choice-conditions here. Note, however, that without a choice-condition, the δ^+-rules would only be sound but not solution preserving; cf. Example 3 in Section 5.3.

Indeed, the calculus contains *different kinds of δ-rules in parallel*. Therefore, the δ^--rules have to refer to the free δ^+-variables (introduced by the δ^+-rules) in their variable-conditions, and vice versa.[8]

[8] Examples 2.6, 2.9, 2.19, and 2.50 of Wirth (2004) show that these references are necessary for soundness (and that we even have to consider the transitive closure of the introduced variable-conditions (without actually having to compute it)). If we have only one kind of free δ-variables, then the variable-conditions introduced by the δ^+-rules are indeed smaller than the ones of the related δ^--rules, resulting in the intended liberalization. As explained in Note 1 of Wirth (2008) and in Note 3 of Wirth (2011), this liberalization becomes effective in all practically relevant cases also if we mix both kinds of variables and rules. All in all, the δ^--variables, on the one hand, have the advantage that they can be instantiated in the application of a sequent as a lemma or as an induction hypothesis; on the other hand, the δ^+-variables are introduced by the δ^+-rules with (in effect)

All in all, we have to extend our global variable-condition R every time we do a δ^-- or δ^+-step and every time we globally instantiate a rigid variable from $V_{\gamma\delta^+}$ by a substitution σ (to the σ-update, cf. Definition 1). Extension of R always preserves soundness and even the solutions of the rigid variables, because it can only *restrict* the ways in which open branches may be closed (by instantiation of rigid variables). As R has to remain well-founded,[9] however, it is advantageous to keep R as small as possible.

3.2. Lemma application between proof trees

The reason why we speak of a proof *forest* is that a proof may be spread over several trees that are connected by generative application of the root of one tree in the reductive proof of another tree, either as a lemma or as an induction hypothesis. While the application of lemmas must be well-founded, induction hypotheses may be applied to the proof of themselves and mutually. In this paper, we only need lemma application.

Lemma application works as follows. When a lemma $A_1, \ldots, A_m$ is a subsequent of a leaf sequent Γ to be proved (i.e. if, for all $i \in \{1, \ldots, m\}$, the formula A_i is listed in Γ), its application closes the branch of this sequent (*subsumption*). Otherwise, the conjugates of the missing formulas C_i are added to the child sequents (premises), one child per missing formula. This can be seen as cuts on C_i plus subsumption. More precisely, a sequent $A_1, \ldots, A_m, B_1, \ldots, B_n$ can be reduced by application of the lemma $A_1, \ldots, A_m, C_1, \ldots, C_p$ (modulo associativity, commutativity, and idempotency of the disjunctive "$,$") to the sequents

$$\overline{C_1}, A_1, \ldots, A_m, B_1, \ldots, B_n \qquad \cdots \qquad \overline{C_p}, A_1, \ldots, A_m, B_1, \ldots, B_n.$$

In addition, before every application of a lemma, we can instantiate its free δ^--variables locally and arbitrarily.[10] This instantiation of outermost δ^--variables mirrors mathematical practice,[11] saves repetition of initial δ-steps, and is essential for induction, where the weights depend on these free δ^--variables to guarantee well-foundedness. There will be a sufficient number of self-explanatory examples of application of *open lemmas* (i.e. yet unproved lemmas) in Section 4.

4. The proof of $(\lim +)$ (limit theorem on sums in R)

4.1. Explanation and initialization

Compared to the proof of $(\lim +)$ as presented in the lecture courses, the version we present here admits a more rigorous argumentation for non-permutability of β and δ^+ in the following sections.[12] By standard mathematical abuse of notation, we want to prove the theorem

smaller variable-conditions than the ones that the corresponding δ^--rules would introduce (i.e. more instantiations possible, more ways of closing a branch).

9 Because δ-rules introduce *fresh variables*, well-foundedness can only be a problem for the instantiation of rigid variables; cf. Definition 1.

10 Actually, there is an exception here, which, however, is irrelevant to this paper: We may instantiate all the free δ^--variables of a lemma *except* those that depend on rigid variables which (in rare cases) may already occur in an input lemma. More precisely, the set of free δ^--variables of a lemma Φ we may instantiate is

$$\left\{\ y^{\delta^-} \in \mathcal{V}_{\delta^-}(\Phi)\ \middle|\ \mathcal{V}_{\gamma\delta^+}(\Phi) \times \{y^{\delta^-}\} \subseteq R\ \right\}.$$

Typically, $\mathcal{V}_{\gamma\delta^+}(\Phi)$ is empty and no restrictions apply. This is the case for all lemmas occurring in this paper.

Note that we also may extend this set of free δ^--variables by extending the variable-condition R; indeed, an extension by $\mathcal{V}_{\gamma\delta^+}(\Phi) \times \mathcal{V}_{\delta^-}(\Phi)$ admits us to instantiate all δ^--variables of Φ.

11 It is standard mathematical practice to omit the outermost universal quantifiers in the notation of lemmas and to instantiate the resulting free variables with fresh instances tacitly every time a lemma is applied.

12 I did not succeed in finding a really satisfying definition of non-local permutability that fits the non-local situation of the failure of the $(\lim +)$ proof as presented in the lecture courses (Wirth et al., 2003/04; Autexier et al., 2004/05). The problem was to permute the critical β-step from below the critical δ^+-steps to a place far up above the δ^+-steps. And on this partial path from β down to δ^+ there were other inference steps which may or may not contribute to the non-permutability. Thus, instead of globalizing the notion of permutability I localized the example proof; although the original version had pedagogical advantages.

$$(\lim +) \qquad \lim_{x\to x_0^{\delta^-}} \left(f^{\delta^-}(x) + g^{\delta^-}(x) \right) = \lim_{x\to x_0^{\delta^-}} f^{\delta^-}(x) + \lim_{x\to x_0^{\delta^-}} g^{\delta^-}(x)$$

Before we start the formal proof, we expand $(\lim +)$ into a better notation:

$$(\mathbf{1}): \qquad \left(\begin{array}{ll} & \lim_{x\to x_0^{\delta^-}} f^{\delta^-}(x) = y_f^{\delta^-} \\ \wedge & \lim_{x\to x_0^{\delta^-}} g^{\delta^-}(x) = y_g^{\delta^-} \end{array} \right) \Rightarrow \lim_{x\to x_0^{\delta^-}} \left(f^{\delta^-}(x){+}g^{\delta^-}(x) \right) = y_f^{\delta^-}{+}y_g^{\delta^-}$$

The reader should be aware that, although the introduced implication symbol now makes implicit assumptions on the existence of limits clearer, the symbol "=" in Formula (1) is still no real equality symbol! In fact, the symbol "=" in *pseudo-equations* such as $\lim_{x\to 0} \left(x^2 \sin \frac{1}{x} \right) = 0$, or, more formally, say $\lim_{x\to z} t_x = t'$ (*definiendum*), is defined by the formula (*definiens*)

$$\forall\varepsilon{>}0.\ \exists\delta{>}0.\ \forall x \neq z.\ \left(\ |t_x{-}t'| < \varepsilon \ \Leftarrow \ |x{-}z| < \delta \ \right).$$

Note that $\forall\varepsilon{>}0.\ A$ and $\exists\delta{>}0.\ B$ and $\forall x \neq z.\ C$ (*definienda*) abbreviate $\forall\varepsilon.\ (0 < \varepsilon \Rightarrow A)$ and $\exists\delta.\ (0 < \delta \wedge B)$ and $\forall x.\ (x \neq z \Rightarrow C)$ (*definientia*), respectively. Thus, when we will speak of an *expansion of* "$\forall\varepsilon{>}0.\ \ldots$" (from *definiendum* to *definiens*) or simply of an *expansion of* $\forall$, we mean the replacement of $\forall\varepsilon{>}0.\ A$ with $\forall\varepsilon.\ (0 < \varepsilon \Rightarrow A)$ for some formula A in a reductive proof step. Analogous proof steps are meant by *expansion of* $\exists$ and *expansion of* lim, respectively. For convenience, we will often reorder the formulas listed in the sequents without mentioning this explicitly.

We initialize our global variable-condition R by $R := \emptyset$, and our global R-choice-condition C by $C := \emptyset$.

4.2. Expanding the Proof Tree with Root (1)

By two α-steps and expansion of lim from *definiendum* to *definiens*, we reduce (1) to its single child (1.1), writing (1^2) for (1.1):

$$(\mathbf{1^2}): \qquad \forall\varepsilon{>}0.\ \exists\delta{>}0.\ \forall x \neq x_0^{\delta^-}.\ \left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)) - (y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta \end{array} \right), \\ \lim_{x\to x_0^{\delta^-}} f^{\delta^-}(x) \neq y_f^{\delta^-},\ \ \lim_{x\to x_0^{\delta^-}} g^{\delta^-}(x) \neq y_g^{\delta^-}$$

By expansion of "$\forall\varepsilon{>}0.\ \ldots$" from *definiendum* to *definiens*, then by a δ^--step[13] (introducing ε^{δ^-}), by an α-step, and finally by an expansion of $\exists$, we reduce (1^2) to:

$$(\mathbf{1^3}):\ \exists\delta.\ \left(\ 0 < \delta \wedge \forall x \neq x_0^{\delta^-}.\ \left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)) - (y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta \end{array} \right) \right), \\ 0 \not< \varepsilon^{\delta^-},\ \ \lim_{x\to x_0^{\delta^-}} f^{\delta^-}(x) \neq y_f^{\delta^-},\ \ \lim_{x\to x_0^{\delta^-}} g^{\delta^-}(x) \neq y_g^{\delta^-}$$

A γ-step (introducing δ^γ) yields:

$$(\mathbf{1^4}): \qquad 0 < \delta^\gamma \wedge \forall x \neq x_0^{\delta^-}.\ \left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^\gamma \end{array} \right),\ (1^3)$$

Note that the "(1^3)" listed at the end of Sequent (1^4) is intended to mean that the whole parent sequent (1^3) is part of the child sequent (1^4).

Expanding lim and $\forall$, plus a γ-step, each twice, we get (cf. Fig. 2 for Ξ):

[13] We could just as well use the corresponding δ^+-step here and introduce ε^{δ^+} instead of ε^{δ^-}, without any relevant effect on the later discussion, although the δ^+-step would add $\{x_0^{\delta^-}, f^{\delta^-}, g^{\delta^-}, y_f^{\delta^-}, y_g^{\delta^-}\} \times \{\varepsilon^{\delta^+}\}$ to our current global variable-condition. The reason we prefer a δ^--step here is simply the following general heuristics: "Use δ^--steps exactly for the *outermost* (in the sense that no γ-quantifier precedes them) δ-quantifiers!" The justification behind this heuristics is that – in the special case of an outermost δ-quantifier – on the one hand, the liberalization achieved in general by the δ^+-steps does not apply, whereas, on the other hand, the δ^--variables enable lemma and induction-hypothesis application.

In the proof of Section 4, Formulas (2), (3), (4), (5), (6), (7), (8), (9) (where the boxes around the formulas just indicate the matching in the lemma application), the sequents Γ, Ξ, Θ, Ω, the substitution σ, and the term t abbreviate the following, respectively:

(2): $\min(y^{\delta^-}, z^{\delta^-}) \leq y^{\delta^-}$

(3): $z_4^{\delta^-} < z_6^{\delta^-},\ z_4^{\delta^-} \not< z_5^{\delta^-},\ z_5^{\delta^-} \not\leq z_6^{\delta^-}$

(4): $z_9^{\delta^-} < \min(z_{10}^{\delta^-}, z_{11}^{\delta^-}),\ z_9^{\delta^-} \not< z_{10}^{\delta^-},\ z_9^{\delta^-} \not< z_{11}^{\delta^-}$

(5): $|(z_0^{\delta^-}+z_1^{\delta^-})-(z_2^{\delta^-}+z_3^{\delta^-})| \leq |z_0^{\delta^-}-z_2^{\delta^-}| + |z_1^{\delta^-}-z_3^{\delta^-}|$

(6): $\boxed{z_4^{\delta^-} < z_6^{\delta^-}},\ \boxed{\boxed{z_4^{\delta^-} \not\leq z_5^{\delta^-}}},\ z_5^{\delta^-} \not< z_6^{\delta^-}$

(7): $z_{12}^{\delta^-}+z_{13}^{\delta^-} < z_{14}^{\delta^-}+z_{15}^{\delta^-},\ \boxed{z_{12}^{\delta^-} \not< z_{14}^{\delta^-}},\ \boxed{\boxed{z_{13}^{\delta^-} \not< z_{15}^{\delta^-}}}$

(8): $\frac{\varepsilon^{\delta^-}}{2} + \frac{\varepsilon^{\delta^-}}{2} \leq \varepsilon^{\delta^-}$

(9): $0 < \frac{\varepsilon^{\delta^-}}{2},\ 0 \not< \varepsilon^{\delta^-}$

Γ:

$$\neg\forall\varepsilon_f.\left(0 < \varepsilon_f \Rightarrow \exists\delta_f{>}0.\ \forall x_f \neq x_0^{\delta^-}.\left(\begin{array}{l} |f^{\delta^-}(x_f) - y_f^{\delta^-}| < \varepsilon_f \\ \Leftarrow\ |x_f - x_0^{\delta^-}| < \delta_f \end{array}\right)\right),$$
$$\neg\forall\varepsilon_g.\left(0 < \varepsilon_g \Rightarrow \exists\delta_g{>}0.\ \forall x_g \neq x_0^{\delta^-}.\left(\begin{array}{l} |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \varepsilon_g \\ \Leftarrow\ |x_g - x_0^{\delta^-}| < \delta_g \end{array}\right)\right),$$
$$\exists\delta.\left(0 < \delta \wedge \forall x \neq x_0^{\delta^-}.\left(\begin{array}{l} |(f^{\delta^-}(x)+g^{\delta^-}(x)) - (y_f^{\delta^-}+y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow\ |x - x_0^{\delta^-}| < \delta \end{array}\right)\right)$$

Ξ:

$$0 < \delta^{\gamma} \wedge \forall x {\neq} x_0^{\delta^-}.\left(\begin{array}{l} |(f^{\delta^-}(x)+g^{\delta^-}(x)) - (y_f^{\delta^-}+y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow\ |x - x_0^{\delta^-}| < \delta^{\gamma} \end{array}\right),$$
$$0 \not< \varepsilon^{\delta^-},\ \Gamma$$

Θ:

$$\neg\left(0 < \delta_f^{\delta^+} \wedge \forall x_f {\neq} x_0^{\delta^-}.\left(\begin{array}{l} |f^{\delta^-}(x_f) - y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow\ |x_f - x_0^{\delta^-}| < \delta_f^{\delta^+} \end{array}\right)\right),$$
$$\neg\exists\delta_g.\left(0 < \delta_g \wedge \forall x_g {\neq} x_0^{\delta^-}.\left(\begin{array}{l} |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow\ |x_g - x_0^{\delta^-}| < \delta_g \end{array}\right)\right),$$
$$0 \not< \varepsilon^{\delta^-},\ \Gamma$$

Ω:

$$0 \not< \delta_f^{\delta^+},\ \neg\forall x_f {\neq} x_0^{\delta^-}.\left(\begin{array}{l} |f^{\delta^-}(x_f) - y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow\ |x_f - x_0^{\delta^-}| < \delta_f^{\delta^+} \end{array}\right),$$
$$0 \not< \delta_g^{\delta^+},\ \neg\forall x_g {\neq} x_0^{\delta^-}.\left(\begin{array}{l} |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow\ |x_g - x_0^{\delta^-}| < \delta_g^{\delta^+} \end{array}\right),$$
$$0 \not< \varepsilon^{\delta^-},\ \Gamma$$

σ: $\{x_f^{\gamma} \mapsto x^{\delta^+},\ x_g^{\gamma} \mapsto x^{\delta^+},\ \delta^{\gamma} \mapsto \min(\delta_f^{\delta^+}, \delta_g^{\delta^+})\}$

t: $|\ (f^{\delta^-}(x^{\delta^+})+g^{\delta^-}(x^{\delta^+})) - (y_f^{\delta^-}+y_g^{\delta^-})\ |$

Fig. 2. Global abbreviations for the proof of Section 4.

(1^5):

$$\neg\left(0 < \varepsilon_f^{\gamma} \Rightarrow \exists\delta_f{>}0.\ \forall x_f \neq x_0^{\delta^-}.\left(\begin{array}{l} |f^{\delta^-}(x_f) - y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow\ |x_f - x_0^{\delta^-}| < \delta_f \end{array}\right)\right),$$
$$\neg\left(0 < \varepsilon_g^{\gamma} \Rightarrow \exists\delta_g{>}0.\ \forall x_g \neq x_0^{\delta^-}.\left(\begin{array}{l} |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow\ |x_g - x_0^{\delta^-}| < \delta_g \end{array}\right)\right),\ \Xi$$

Two β-steps and two expansions of $\exists$ yield the following three child sequents:

($1^5.1$):

$$0 < \varepsilon_f^{\gamma},\ \neg\left(0 < \varepsilon_g^{\gamma} \Rightarrow \exists\delta_g{>}0.\ \forall x_g \neq x_0^{\delta^-}.\left(\begin{array}{l} |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow\ |x_g - x_0^{\delta^-}| < \delta_g \end{array}\right)\right),\ \Xi$$

($1^5.2$):

$$0 < \varepsilon_g^{\gamma},\ \neg\exists\delta_f{>}0.\ \forall x_f \neq x_0^{\delta^-}.\left(\begin{array}{l} |f^{\delta^-}(x_f) - y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow\ |x_f - x_0^{\delta^-}| < \delta_f \end{array}\right),\ \Xi$$

$$(\mathbf{1^5.3}):\quad \begin{array}{l} \neg\exists\delta_f.\left(0<\delta_f\wedge\forall x_f{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f){-}y_f^{\delta-}|<\varepsilon_f^{\gamma}\\ \Leftarrow & |x_f{-}x_0^{\delta-}|<\delta_f\end{array}\right)\right),\\ \neg\exists\delta_g.\left(0<\delta_g\wedge\forall x_g{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g){-}y_g^{\delta-}|<\varepsilon_g^{\gamma}\\ \Leftarrow & |x_g{-}x_0^{\delta-}|<\delta_g\end{array}\right)\right),\ \Xi\end{array}$$

A δ^+-step (introducing $\delta_f^{\delta+}$) applied to the first formula at (1^5.3) yields

$$\begin{array}{l} \neg\left(0<\delta_f^{\delta+}\wedge\forall x_f{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f){-}y_f^{\delta-}|<\varepsilon_f^{\gamma}\\ \Leftarrow & |x_f{-}x_0^{\delta-}|<\delta_f^{\delta+}\end{array}\right)\right),\\ \neg\exists\delta_g.\left(0<\delta_g\wedge\forall x_g{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g){-}y_g^{\delta-}|<\varepsilon_g^{\gamma}\\ \Leftarrow & |x_g{-}x_0^{\delta-}|<\delta_g\end{array}\right)\right),\ \Xi\end{array}$$

where R is extended with $\{x_0^{\delta-}, f^{\delta-}, y_f^{\delta-}, \varepsilon_f^{\gamma}\}\times\{\delta_f^{\delta+}\}$, and the choice-condition C with:

$$\left\{\ \delta_f^{\delta+}\mapsto\left(0<\delta_f^{\delta+}\wedge\forall x_f{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f){-}y_f^{\delta-}|<\varepsilon_f^{\gamma}\\ \Leftarrow & |x_f{-}x_0^{\delta-}|<\delta_f^{\delta+}\end{array}\right)\right)\right\}.$$

For convenience, we prefer to refer to this sequent in the following reordered form:

$$(\mathbf{1^5.3.1}):\quad 0<\delta^{\gamma}\wedge\forall x\neq x_0^{\delta-}.\left(\begin{array}{ll} & |(f^{\delta-}(x){+}g^{\delta-}(x)){-}(y_f^{\delta-}{+}y_g^{\delta-})|<\varepsilon^{\delta-}\\ \Leftarrow & |x{-}x_0^{\delta-}|<\delta^{\gamma}\end{array}\right),\ \Theta$$

4.3. A bad turn

Now we apply a β-step to the first formula of the sequent (1^5.3.1).[14] Note that this β-step will make the whole following subproof fail! (A reader who is interested only in a successful example proof may continue reading with Section 4.6.)

$$(\mathbf{1^5.3.1.1}):\quad 0<\delta^{\gamma},\ \Theta$$

$$(\mathbf{1^5.3.1.2}):\quad \forall x\neq x_0^{\delta-}.\left(\begin{array}{ll} & |(f^{\delta-}(x){+}g^{\delta-}(x))-(y_f^{\delta-}{+}y_g^{\delta-})|<\varepsilon^{\delta-}\\ \Leftarrow & |x{-}x_0^{\delta-}|<\delta^{\gamma}\end{array}\right),\ \Theta$$

A δ^+-step (introducing $\delta_g^{\delta+}$), two α-steps, and expansion of $\forall$, applied to (1^5.3.1.2), yield (cf. Fig. 2 for Ω):

$$(\mathbf{1^5.3.1.2.1}):\quad \forall x.\left(x\neq x_0^{\delta-}\Rightarrow\left(\begin{array}{ll} & |(f^{\delta-}(x){+}g^{\delta-}(x))-(y_f^{\delta-}{+}y_g^{\delta-})|<\varepsilon^{\delta-}\\ \Leftarrow & |x{-}x_0^{\delta-}|<\delta^{\gamma}\end{array}\right)\right),\ \Omega$$

where R is extended with $\{x_0^{\delta-}, g^{\delta-}, y_g^{\delta-}, \varepsilon_g^{\gamma}\}\times\{\delta_g^{\delta+}\}$, and C with:

$$\left\{\ \delta_g^{\delta+}\mapsto\left(0<\delta_g^{\delta+}\wedge\forall x_g{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g){-}y_g^{\delta-}|<\varepsilon_g^{\gamma}\\ \Leftarrow & |x_g{-}x_0^{\delta-}|<\delta_g^{\delta+}\end{array}\right)\right)\right\}$$

A δ^+-step (introducing $x^{\delta+}$) and two α-steps yield (cf. Fig. 2 for t):

$$(\mathbf{1^5.3.1.2.1^2}):\quad x^{\delta+}{=}x_0^{\delta-},\ t<\varepsilon^{\delta-},\ |x^{\delta+}{-}x_0^{\delta-}|\not<\delta^{\gamma},\ \Omega$$

where R is extended with $\{x_0^{\delta-}, f^{\delta-}, g^{\delta-}, y_f^{\delta-}, y_g^{\delta-}, \varepsilon^{\delta-}, \delta^{\gamma}\}\times\{x^{\delta+}\}$
and our R-choice-condition C with

$$\left\{\ x^{\delta+}\mapsto\neg\left(x^{\delta+}\neq x_0^{\delta-}\Rightarrow\left(t<\varepsilon^{\delta-}\ \Leftarrow\ |x^{\delta+}{-}x_0^{\delta-}|<\delta^{\gamma}\right)\right)\right\}$$

Expansion of $\forall$ and a γ-step, each twice, namely to the second and fourth formulas of Ω, yield:

$$(\mathbf{1^5.3.1.2.1^3}):\quad \begin{array}{r} \neg\left(x_f^{\gamma}{\neq}x_0^{\delta-}\Rightarrow\left(\begin{array}{ll} & |f^{\delta-}(x_f^{\gamma}){-}y_f^{\delta-}|<\varepsilon_f^{\gamma}\\ \Leftarrow & |x_f^{\gamma}{-}x_0^{\delta-}|<\delta_f^{\delta+}\end{array}\right)\right),\\ \neg\left(x_g^{\gamma}{\neq}x_0^{\delta-}\Rightarrow\left(\begin{array}{ll} & |g^{\delta-}(x_g^{\gamma}){-}y_g^{\delta-}|<\varepsilon_g^{\gamma}\\ \Leftarrow & |x_g^{\gamma}{-}x_0^{\delta-}|<\delta_g^{\delta+}\end{array}\right)\right),\\ x^{\delta+}{=}x_0^{\delta-},\ t<\varepsilon^{\delta-},\ |x^{\delta+}{-}x_0^{\delta-}|\not<\delta^{\gamma},\ \Omega\end{array}$$

---

[14] Note that this is an unforced early application of a β-step to this β-formula, against the folklore heuristics presented in Section 2. We do this β-step right now, because (in Section 6) we want to prove the non-permutability of this β-step and the first subsequent δ^+-step (introducing $\delta_g^{\delta+}$). This position of application in our proof tree would be more natural (but would still not resulting in a successful proof), if the second subsequent δ^+-step (introducing $x^{\delta+}$) came right after this β-step (because it is enabled only by the β-step).

4.4. Partial success

2 β-steps, each twice, yield:
$(\mathbf{1^5.3.1.2.1^3.1})$: $x_f^{\gamma} \neq x_0^{\delta-},\ x^{\delta+}{=}x_0^{\delta-},\ \ldots$
$(\mathbf{1^5.3.1.2.1^3.2})$: $x_g^{\gamma} \neq x_0^{\delta-},\ x^{\delta+}{=}x_0^{\delta-},\ \ldots$
$(\mathbf{1^5.3.1.2.1^3.3})$: $|x_f^{\gamma}{-}x_0^{\delta-}| < \delta_f^{\delta+},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \delta^{\gamma},\ \ldots$
$(\mathbf{1^5.3.1.2.1^3.4})$: $|x_g^{\gamma}{-}x_0^{\delta-}| < \delta_g^{\delta+},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \delta^{\gamma},\ \ldots$
$(\mathbf{1^5.3.1.2.1^3.5})$: $|f^{\delta-}(x_f^{\gamma}) - y_f^{\delta-}| \not< \varepsilon_f^{\gamma},\ |g^{\delta-}(x_g^{\gamma}) - y_g^{\delta-}| \not< \varepsilon_g^{\gamma},\ x^{\delta+}{=}x_0^{\delta-},\ t < \varepsilon^{\delta-},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \delta^{\gamma},\ \Omega$
By formula unification and some basic knowledge of the domain, we can now easily see that global application of the substitution σ from Section 4.1 admits to close the branches of the first four of these five sequents. According to Definition 1, the application of σ adds

$$\{(x^{\delta+}, x_f^{\gamma}),\ (x^{\delta+}, x_g^{\gamma}),\ (\delta_f^{\delta+}, \delta^{\gamma}),\ (\delta_g^{\delta+}, \delta^{\gamma})\}$$

to our variable-condition R, which, luckily, remains acyclic, cf. the acyclic graph of Fig. 5. In fact, Sequents $(1^5.3.1.2.1^3.1)$ and $(1^5.3.1.2.1^3.2)$ become logical axioms by application of σ, and applying Lemma (2) of Fig. 2 instantiated via $\{y^{\delta-}{\mapsto}\delta_f^{\delta+},\ z^{\delta-}{\mapsto}\delta_g^{\delta+}\}$, we reduce $(1^5.3.1.2.1^3.3)$ to:
$(\mathbf{1^5.3.1.2.1^3.3.1})$: $\min(\delta_f^{\delta+}, \delta_g^{\delta+}) \not\leq \delta_f^{\delta+},\ |x^{\delta+}{-}x_0^{\delta-}| < \delta_f^{\delta+},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \min(\delta_f^{\delta+}, \delta_g^{\delta+}),\ \ldots$
which is subsumed by the transitivity lemma (3) of Fig. 2. Moreover, Sequent $(1^5.3.1.2.1^3.4)$ can be closed analogously to $(1^5.3.1.2.1^3.3)$.

4.5. Failure

Abstractly, our proof tree looks as in Fig. 3 now. By application of σ, Sequent $(1^5.3.1.1)$ has become

$$0\ <\ \min(\delta_f^{\delta+}, \delta_g^{\delta+}),\ \Theta$$

If its first formula – which is the only new one as compared to its parent sequent – is irrelevant for the proof of $(1^5.3.1.1)$ (in the sense that it is not contributing as a principal formula, cf.(Gentzen, 1935; Schmidt-Samoa, 2006c,b)), then we had better prove $(1^5.3.1)$ instead, because this saves us the proof of the whole β_2-subtree of $(1^5.3.1)$, starting at $(1^5.3.1.2)$. We have to notice the following, however: $\delta_g^{\delta+}$ is not introduced before $(1^5.3.1.2.1)$, which in $(1^5.3.1.2.1^2)$ results in the context $0{\not<}\delta_f^{\delta+},\ 0{\not<}\delta_g^{\delta+}$ (as listed in Ω of Fig. 2) with which we could prove $0\ <\ \min(\delta_f^{\delta+}, \delta_g^{\delta+})$ by Lemma (4) of Fig. 2. Thus, the β-step applied to $(1^5.3.1)$ cannot have any benefit unless it is done *below* $(1^5.3.1.2.1)$.
Now, we have three possibilities in principle:

(1) We can backtrack to $(1^5.3.1)$, deleting all its sub-trees.
(2) We could try to use the choice-condition of $\delta_g^{\delta+}$ to find out that it is positive. $C(\delta_g^{\delta+})$ is

$$0\ <\ \delta_g^{\delta+}\ \wedge\ \forall x_g{\neq}x_0^{\delta-}.\ \big(\ |g^{\delta-}(x_g){-}y_g^{\delta-}| < \varepsilon_g^{\gamma}\ \Leftarrow\ |x_g{-}x_0^{\delta-}| < \delta_g^{\delta+}\ \big).$$

But this guarantees $0\ <\ \delta_g^{\delta+}$ only if also the second part of the conjunction can be shown to be satisfiable, for which we again lack the context.
(3) We can prove $(1^5.3.1.1)$ by proving its subsequent Θ. As Θ is already a subsequent of $(1^5.3.1)$, this means that we could prove already $(1^5.3.1)$ this way: Indeed, after finding this proof for Θ, we could backtrack to $(1^5.3.1)$ and replay the proof we found for Θ. In this way, the whole previous subproof below $(1^5.3.1.2)$ could be pruned.[15] Therefore, this third possibility has no advantage over our first one (i.e. all proofs we can find here result in proofs there). Moreover, this third possibility would result in a higher maximum of γ-multiplicity than necessary, because we would necessarily have to expand the principal γ-formula of (1^3) (i.e. the first formula of (1^3)) a second time.

All in all, by now it should have become plausible that the following lemma holds.

---

[15] This replay mechanism is an essential part of the powerful automation of proof construction in QuodLibet and was introduced in Schmidt-Samoa (2006b). To the best of our knowledge, it is the only such mechanism of proof re-use.

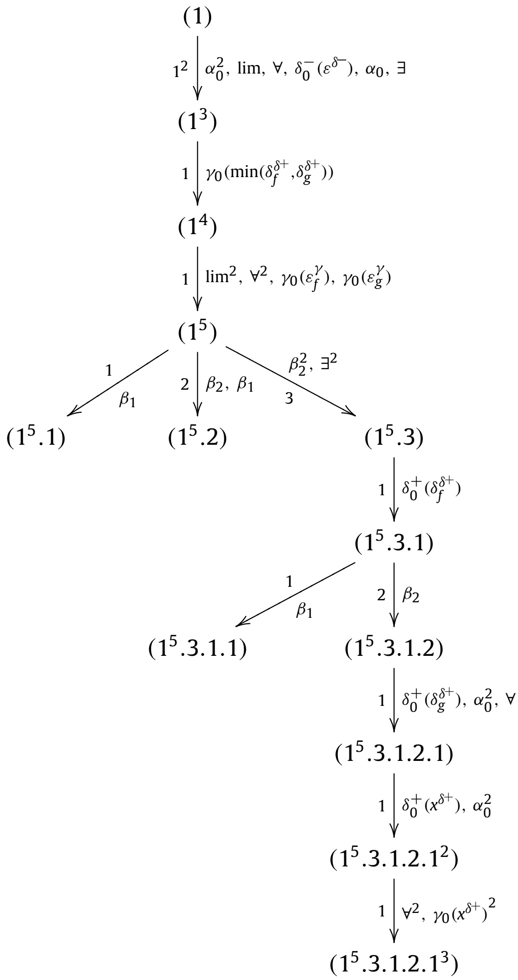
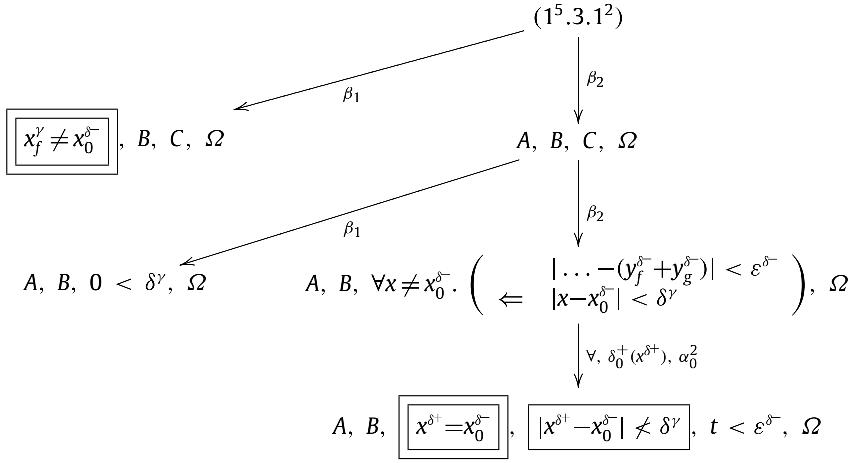


Fig. 3. Non-permutability of β at (1^5.3.1) and δ^+ at (1^5.3.1.2): no chance to prove $0 < \min(\delta_f^{\delta+}, \delta_g^{\delta+})$ at (1^5.3.1.1).

Lemma 2. *Using the reductive rules of Fig. 1 with a γ-multiplicity threshold of* 1*, the current proof tree (with the partial instantiation σ) cannot be expanded and simultaneously instantiated to a closed proof tree below its open nodes* (1^5.1)*,* (1^5.2)*, and* (1^5.3.1.1)*.* □

For a proof of Lemma 2 see Section 6.1. Note that the validity of Lemma 2 depends on the δ^-- and δ^+-rules being the only δ-rules available: With δ^{+^+}-rules the situation would be different, cf. Section 5.4. Moreover, as our proof trees are customary AND-trees,[16] Lemma 2 means that – given

[16] AND-trees are customary for sequent calculi. This means that – to close a proof tree – we have to close *all* its branches, or, equivalently, that the conjunction of the leaf sequents entails the root sequent. AND/OR-trees are standard in artificial intelligence and computer science. They are most useful in automated theorem proving for presenting several alternative possibilities in a proof simultaneously. To the best of our knowledge, QuodLibet (Avenhaus et al., 2003; Kühler, 2000; Wirth, 1997) was the first theorem prover where AND/OR-trees were implemented in the 1990s, with primarily interactive intentions. In the first decade of the 3rd millennium, they turned out to be most useful in automation as well (Schmidt-Samoa, 2006b). For the new generation of mathematics assistance systems, AND/OR-trees have now become an essential ingredient (Autexier, 2003; Autexier et al., 2006).



Here A denotes the formula $\neg\left(\ |f^{\delta-}(x_f^\gamma) - y_f^{\delta-}| < \varepsilon_f^\gamma \ \Leftarrow \ \boxed{|x_f^\gamma - x_0^{\delta-}| < \delta_f^{\delta+}} \ \right)$. B and C denote the second and third β-formula of the sequent $(1^5.3.1^2)$, respectively.

Fig. 4. Non-permutability of β at $(1^5.3.1^2)$ and β at the β_2-child of $(1^5.3.1^2)$: no chance to prove $x_f^\gamma \neq x_0^{\delta-}$ at leftmost leaf.

the current proof tree – the whole proof attempt is failed for a γ-multiplicity of 1 (unless we admit backtracking).

4.6. *Backtracking to a non-failed proof tree*

As Item 1 in the before-mentioned list of possibilities is obviously the only reasonable one, let us restart from $(1^5.3.1)$ — not without storing σ and its connections before.
Applied to $(1^5.3.1)$, a δ^+-step (introducing $\delta_g^{\delta+}$), two α-steps, two expansions of $\forall$, and two γ-steps yield as in Section 4.3 (and with the same extensions of R and C):

$(1^5.3.1^2)$:

$$\begin{array}{l}
\neg\left(\ x_f^\gamma \neq x_0^{\delta-} \ \Rightarrow \left(\begin{array}{l} \ \ |f^{\delta-}(x_f^\gamma) - y_f^{\delta-}| < \varepsilon_f^\gamma \\ \Leftarrow \ |x_f^\gamma - x_0^{\delta-}| < \delta_f^{\delta+} \end{array} \right) \right), \\
\neg\left(\ x_g^\gamma \neq x_0^{\delta-} \ \Rightarrow \left(\begin{array}{l} \ \ |g^{\delta-}(x_g^\gamma) - y_g^{\delta-}| < \varepsilon_g^\gamma \\ \Leftarrow \ |x_g^\gamma - x_0^{\delta-}| < \delta_g^{\delta+} \end{array} \right) \right), \\
0 < \delta^\gamma \wedge \forall x \neq x_0^{\delta-}. \left(\begin{array}{l} \ \ |(f^{\delta-}(x) + g^{\delta-}(x)) - (y_f^{\delta-} + y_g^{\delta-})| < \varepsilon^{\delta-} \\ \Leftarrow \ |x - x_0^{\delta-}| < \delta^\gamma \end{array} \right), \ \Omega
\end{array}$$

Now we *have to* expand one of the three first β-formulas of $(1^5.3.1^2)$. Note that the third one is the one whose expansion made our proof fail before. We should have learned that it is difficult to avoid failed proof trees.

4.7. *A Non-permutability of two β-steps*

The discussion in this section is technically involved. The reader may skip this section on a first reading and continue with Section 4.8.

If we expand the first β-formula of Sequent $(1^5.3.1^2)$ before the third, this will result in the subtree depicted in Fig. 4. Its first β-step can represent progress only if the first (β_1-) child is easier to prove than the root itself. But the only reasonable connection of its only new formula $\boxed{\boxed{x_f^\gamma \neq x_0^{\delta-}}}$ is to the third formula $\boxed{\boxed{x^{\delta^+}=x_0^{\delta^-}}}$ of the rightmost leaf; via our substitution σ. Thus, we would have to copy the proof starting below the second (β_2-) child of the root to its first (β_1-) child. But, if we do so, this proof will fail again, for the following reason:

The sequent of the β_2-child of a β-step to the third formula of the leftmost leaf of Fig. 4, after – just as on the way to the rightmost leaf of Fig. 4 – expansion of $\forall$, another δ^+-step (introducing $x_2^{\delta^+}$), and two α-steps, reads as follows:

$$\boxed{\boxed{x_f^{\gamma}\neq x_0^{\delta^-}}},\ B,\ \boxed{\boxed{x_2^{\delta^+}=x_0^{\delta^-}}},\ \boxed{|x_2^{\delta^+}-x_0^{\delta^-}|\not<\delta^{\gamma}},\ |\ (f^{\delta^-}(x_2^{\delta^+})+g^{\delta^-}(x_2^{\delta^+}))-(y_f^{\delta^-}+y_g^{\delta^-})\ |<\varepsilon^{\delta^-},\ \Omega.$$

But if we now instantiated x_f^{γ} with $x_2^{\delta^+}$, we would also have to instantiate the occurrences of x_f^{γ} in the rightmost leaf of Fig. 4 in the same way, where, however, we have to instantiate x_f^{γ} with x^{δ^+} (as can be seen in from the Sequents ($1^5.3.1^2.2.1.3$) and ($1^5.3.1^2.2.1.5$) in Section 4.8). This means that this subproof fails at a γ-multiplicity threshold of 1.[17]

Note that – with the δ^{++}-rule (Beckert et al., 1993) instead of the δ^+-rule – we could have introduced the variable x^{δ^+} in the copied proof a second time, resulting in a non-failed proof attempt.

This non-permutability of two β-steps will be further discussed in Section 5.1.

4.8. *Continuing with the previously fatal β-step*

The discussion of Section 4.7 shows that expanding the first β-formula of ($1^5.3.1^2$) leads to a failure of the proof on the current threshold for γ-multiplicity again. By symmetry, the same holds for the second. Thus, we take the third. Note that the β-step we *have to* do now is the one whose too early application forced us to backtrack before.

A β-step to the third β-formula of ($1^5.3.1^2$), and expansion of $\forall$ yield:

$(\mathbf{1^5.3.1^2.1})$: $\ 0<\delta^{\gamma},\ 0\not<\delta_f^{\delta^+},\ 0\not<\delta_g^{\delta^+},\ \ldots$

$(\mathbf{1^5.3.1^2.2})$:

$$\begin{array}{l}\neg\left(x_f^{\gamma}\neq x_0^{\delta^-}\Rightarrow\left(\begin{array}{ll}&|f^{\delta^-}(x_f^{\gamma})-y_f^{\delta^-}|<\varepsilon_f^{\gamma}\\ \Leftarrow&|x_f^{\gamma}-x_0^{\delta^-}|<\delta_f^{\delta^+}\end{array}\right)\right),\\ \neg\left(x_g^{\gamma}\neq x_0^{\delta^-}\Rightarrow\left(\begin{array}{ll}&|g^{\delta^-}(x_g^{\gamma})-y_g^{\delta^-}|<\varepsilon_g^{\gamma}\\ \Leftarrow&|x_g^{\gamma}-x_0^{\delta^-}|<\delta_g^{\delta^+}\end{array}\right)\right),\\ \forall x.\left(x\neq x_0^{\delta^-}\Rightarrow\left(\begin{array}{ll}&|(f^{\delta^-}(x)+g^{\delta^-}(x))-(y_f^{\delta^-}+y_g^{\delta^-})|<\varepsilon^{\delta^-}\\ \Leftarrow&|x-x_0^{\delta^-}|<\delta^{\gamma}\end{array}\right)\right),\ \Omega\end{array}$$

As a δ^--step with the first formula of the last line of ($1^5.3.1^2.2$) as principal formula would block the later instantiation of x_f^{γ} and x_g^{γ} with the newly introduced free δ-variable, for the proof to succeed on the current threshold for γ-multiplicity, we have to take a δ^+-step instead. Note that this problem did not occur for Sequent ($1^5.3.1.2.1$) of Section 4.3, in which x_f^{γ} and x_g^{γ} did not occur yet. Beside the δ^+-step extending R and C as in Section 4.3, we do two α-steps. This results exactly in what was seen before at the end of Section 4.3, with the exception of a different label:

$(\mathbf{1^5.3.1^2.2.1})$:

$$\begin{array}{r}\neg\left(x_f^{\gamma}\neq x_0^{\delta^-}\Rightarrow\left(\begin{array}{ll}&|f^{\delta^-}(x_f^{\gamma})-y_f^{\delta^-}|<\varepsilon_f^{\gamma}\\ \Leftarrow&|x_f^{\gamma}-x_0^{\delta^-}|<\delta_f^{\delta^+}\end{array}\right)\right),\\ \neg\left(x_g^{\gamma}\neq x_0^{\delta^-}\Rightarrow\left(\begin{array}{ll}&|g^{\delta^-}(x_g^{\gamma})-y_g^{\delta^-}|<\varepsilon_g^{\gamma}\\ \Leftarrow&|x_g^{\gamma}-x_0^{\delta^-}|<\delta_g^{\delta^+}\end{array}\right)\right),\\ x^{\delta^+}=x_0^{\delta^-},\ t<\varepsilon^{\delta^-},\ |x^{\delta^+}-x_0^{\delta^-}|\not<\delta^{\gamma},\ \Omega\end{array}$$

[17] Contrary to the rightmost leaf of Fig. 4, we cannot close this branch via the connection between the fourth formula $\boxed{|x^{\delta^+}-x_0^{\delta^-}|\not<\delta^{\gamma}}$ and the positive subformula $\boxed{|x_f^{\gamma}-x_0^{\delta^-}|<\delta_f^{\delta^+}}$ of the formula A (via σ, (2), and (3) as at the end of Section 4.4), because this connection is only available at the original position, but not at the position the subproof is copied to, simply because the positive subformula is not present at the latter position (it is part of the β_2-side formula A of the β-step at the root of the subtree of Fig. 4).

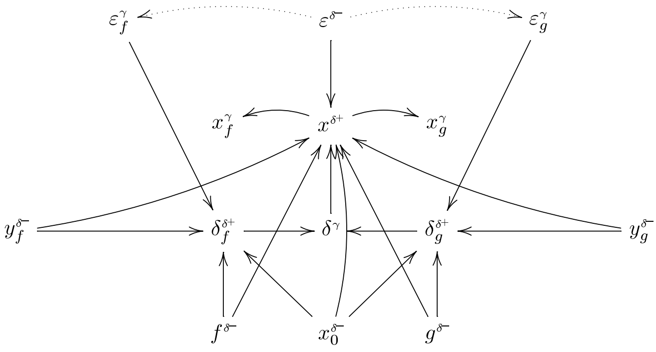


Fig. 5. (Acyclic) variable-condition R.
With dotted edges: final state in Section 4.10.
Without dotted edges:
state after application of σ, both in Section 4.4 and in Section 4.8.

Again, two β-steps to each of the first two formulas, yield:
$(\mathbf{1^5.3.1^2.2.1.1})$: $\ x_f^\gamma \neq x_0^{\delta-},\ x^{\delta+}{=}x_0^{\delta-},\ \ldots$
$(\mathbf{1^5.3.1^2.2.1.2})$: $\ x_g^\gamma \neq x_0^{\delta-},\ x^{\delta+}{=}x_0^{\delta-},\ \ldots$
$(\mathbf{1^5.3.1^2.2.1.3})$: $\ |x_f^\gamma{-}x_0^{\delta-}| < \delta_f^{\delta+},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \delta^\gamma,\ \ldots$
$(\mathbf{1^5.3.1^2.2.1.4})$: $\ |x_g^\gamma{-}x_0^{\delta-}| < \delta_g^{\delta+},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \delta^\gamma,\ \ldots$
$(\mathbf{1^5.3.1^2.2.1.5})$: $\ |f^{\delta-}(x_f^\gamma) - y_f^{\delta-}| \not< \varepsilon_f^\gamma,\ |g^{\delta-}(x_g^\gamma) - y_g^{\delta-}| \not< \varepsilon_g^\gamma,\ x^{\delta+}{=}x_0^{\delta-},\ t < \varepsilon^{\delta-},\ |x^{\delta+}{-}x_0^{\delta-}| \not< \delta^\gamma,\ \Omega$
As before in Section 4.4, application of σ admits the closure of the four branches of $(1^5.3.1^2.2.1.[1–4])$. But now, contrary to what made us backtrack before, $(1^5.3.1^2.1)$ becomes

$$0\ <\ \min(\delta_f^{\delta+}, \delta_g^{\delta+}),\ 0{\not<}\delta_f^{\delta+},\ 0{\not<}\delta_g^{\delta+},\ \ldots,$$

which is subsumed by an instance of Lemma (4) of Fig. 2.

4.9. *A working mathematician's immediate focus*

Note that $(1^5.3.1^2.2.1.5)$ would have been the immediate focus of a working mathematician. He would have sequenced all the β-steps *after* doing the crucial steps of the proof which we can do in our formal sequent calculus only now. Note that the matrix (or indexed formula tree) versions of our calculus enable us to support this human behavior. Let us repeat Sequent $(1^5.3.1^2.2.1.5)$ here with some omissions and some reordering:

$$t < \varepsilon^{\delta-},\ |f^{\delta-}(x^{\delta+}){-}y_f^{\delta-}| \not< \varepsilon_f^\gamma,\ |g^{\delta-}(x^{\delta+}){-}y_g^{\delta-}| \not< \varepsilon_g^\gamma,\ \ldots$$

where $t < \varepsilon^{\delta-}$ actually reads (with some added wave-front annotation to be used in Section 4.10)

$$|\ \boxed{\boxed{(\underline{f^{\delta-}(x^{\delta+})}{+}\underline{g^{\delta-}(x^{\delta+})})}} - \boxed{\boxed{(\underline{y_f^{\delta-}}{+}\underline{y_g^{\delta-}})}}\ |\ <\ \lfloor\varepsilon^{\delta-}\rfloor$$

Now the essential idea of the whole proof is to apply Lemma (5) of Fig. 2 via $\{z_0^{\delta-} \mapsto f^{\delta-}(x^{\delta+}),\ z_1^{\delta-} \mapsto g^{\delta-}(x^{\delta+}),\ z_2^{\delta-} \mapsto y_f^{\delta-},\ z_3^{\delta-} \mapsto y_g^{\delta-}\}$, by which we get:

$(\mathbf{1^5.3.1^2.2.1.5.1})$: $\ \boxed{\boxed{t \not\leq |f^{\delta-}(x^{\delta+}){-}y_f^{\delta-}| + |g^{\delta-}(x^{\delta+}){-}y_g^{\delta-}|}},\ \boxed{t < \varepsilon^{\delta-}},$
$|f^{\delta-}(x^{\delta+}){-}y_f^{\delta-}| \not< \varepsilon_f^\gamma,\ |g^{\delta-}(x^{\delta+}){-}y_g^{\delta-}| \not< \varepsilon_g^\gamma,\ \ldots$

4.10. *Automatic clean-up*

The rest of the proof is perfectly within the scope of automatic proof search today. When we apply the other transitivity lemma (6) of Fig. 2 to $(1^5.3.1^2.2.1.5.1)$ as indicated by the single and double boxes in the goal and the lemma, via $\{\ z_4^{\delta^-} \mapsto t,\ z_6^{\delta^-} \mapsto \varepsilon^{\delta^-},\ z_5^{\delta^-} \mapsto |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}|\ \}$, we get:

$(\mathbf{1^5.3.1^2.2.1.5.1^2})$: $\quad |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| < \varepsilon^{\delta^-},$

$$\boxed{|f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| \not< \varepsilon_f^{\gamma}},\ \boxed{\boxed{|g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| \not< \varepsilon_g^{\gamma}}},\ \ldots$$

In Yoshida et al. (1994), not only this step, but even the two steps from $(1^5.3.1^2.2.1.5)$ to $(1^5.3.1^2.2.1.5.1^2)$ are automated with the wave-front annotation of $t < \varepsilon^{\delta^-}$ as given in Section 4.9 (which is generated by the givens of $|f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| < \varepsilon_f^{\gamma}$ and $|g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| < \varepsilon_g^{\gamma}$ in the context of $t < \varepsilon^{\delta^-}$ in $(1^5.3.1^2.2.1.5)$), provided that the following two lemmas (annotated as wave-rules) are in the rippling system:

$$\boxed{(\underline{z_0^{\delta^-}} + \underline{z_1^{\delta^-}})} - \boxed{(\underline{z_2^{\delta^-}} + \underline{z_3^{\delta^-}})} = \boxed{\underline{(z_0^{\delta^-} - z_2^{\delta^-})} + \underline{(z_1^{\delta^-} - z_3^{\delta^-})}}$$

$$|\boxed{\underline{z_4^{\delta^-}} + \underline{z_5^{\delta^-}}}| < z_6^{\delta^-}, \quad \boxed{\underline{|z_4^{\delta^-}|} + \underline{|z_5^{\delta^-}|}} \not< z_6^{\delta^-}$$

Applying Lemma (7) of Fig. 2 (monotonicity of $+$) in the obvious way, we get:

$(\mathbf{1^5.3.1^2.2.1.5.1^3})$: $\quad |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| \not< \varepsilon_f^{\gamma} + \varepsilon_g^{\gamma},$
$\quad |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| < \varepsilon^{\delta^-},\ \ldots$

The R-substitution $\{\varepsilon_f^{\gamma} \mapsto \frac{\varepsilon^{\delta^-}}{2},\ \varepsilon_g^{\gamma} \mapsto \frac{\varepsilon^{\delta^-}}{2}\}$ closes the remaining open branches of $(1^5.3.1^2.2.1.5.1^3)$ and $(1^5.[1–2])$ with Lemmas (3), (8), and (9), respectively. The final variable-condition is acyclic indeed. Its graph is depicted in Fig. 5. The whole proof tree with a minor permutation of the critical β-step is depicted in Fig. 7.

5. Discussion of the non-permutabilities

The non-permutability of β and δ^+ at the nodes $(1^5.3.1)$ and $(1^5.3.1.2)$, respectively (cf. Fig. 3), as well as the non-permutability of β and β at the node $(1^5.3.1^2)$ and its β_2-child node, respectively (cf. Fig. 4), have become practically evident by the proof of (lim $+$) in Section 4. Now we have to answer the question why the non-permutabilities of β-steps have not been realized before.

It was well-known that the only problem with the sequencing of β-steps that occurs either with the δ^--rules or else with the δ^{+^+}-rules (Beckert et al., 1993) is that a bad choice makes the proofs suffer from the repetition of common sub-proofs, which is an optimization problem not subsumed under the notion of non-permutability, cf. Section 2.

Thus, we should make it even clearer why – contrary to the δ^-- and δ^{+^+}-rules – just the δ^+-rules show the non-permutability with the β-steps.

5.1. Non-permutability of β and β is only a secondary problem

The non-permutability of β and δ^+ is the primary problem, and the only one we have to explain. It causes the non-permutability of β and β we have seen in Fig. 4 as a secondary problem as follows:

The 2nd β-step in Fig. 4 must come before the 1st β-step, simply because the 2nd β-step generates the principal δ-formula of the δ^+-step resulting in the rightmost leaf, and this δ^+-step (introducing x^{δ^+}) must come before the 1st β-step; namely for the leftmost leaf's first formula $x_f^{\gamma} \neq x_0^{\delta^-}$ to be of any use in the proof. Writing "$S_0 <_{\mathrm{REASON}} S_1$" for "step S_0 has to precede step S_1 because of REASON", this means that

$$2^{\mathrm{nd}}\beta \ <_{\text{superformula}}\ \delta_0^+(x^{\delta^+})\ <_{\beta\text{-}\delta^+\text{-non-permutability}}\ 1^{\mathrm{st}}\beta$$

causes the non-permutability of $1^{\mathrm{st}}\beta$ and $2^{\mathrm{nd}}\beta$ by the transitivity of $<$.

As already discussed in Section 4.7, the non-permutability of β and β disappears if we replace the δ^+-rule with the δ^{+^+}-rule.

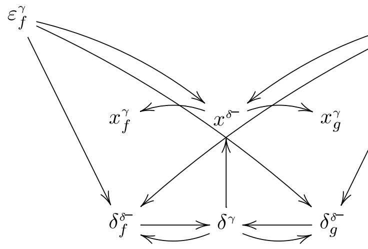


Fig. 6. (Cyclic) state of variable-condition R for alternative proof of Section 5.2 with δ^--rules only.

5.2. δ^- *instead of* δ^+

Let us see how the proof of (lim +) would look like with the δ^--rules as the only δ-rules available. Roughly speaking, in the proof of Section 4, we have to replace each free δ^+-variable $v_n^{\delta+}$ with a free δ^--variable $v_n^{\delta-}$ and check how the variable-condition changes: $\delta_0^-(\delta_f^{\delta-})$ and $\delta_0^-(\delta_g^{\delta-})$ applied to $(1^5.3)$ of Section 4.2 and $(1^5.3.1.2)$ of Section 4.3 (cf. Fig. 3) add $\{\varepsilon_f^\gamma, \varepsilon_g^\gamma, \delta^\gamma\} \times \{\delta_f^{\delta-}\}$ and $\{\varepsilon_f^\gamma, \varepsilon_g^\gamma, \delta^\gamma\} \times \{\delta_g^{\delta-}\}$ to the initially empty variable-condition R, respectively. $\delta_0^-(x^{\delta-})$ applied roughly at $(1^5.3.1.2.1)$ adds $\{\varepsilon_f^\gamma, \varepsilon_g^\gamma, \delta^\gamma\} \times \{x^{\delta-}\}$ later.
Thus, after applying

$$\sigma^- := \{x_f^\gamma \mapsto x^{\delta-},\ x_g^\gamma \mapsto x^{\delta-},\ \delta^\gamma \mapsto \min(\delta_f^{\delta-}, \delta_g^{\delta-})\}$$

the σ^--updated variable-condition is extended with

$$\{(x^{\delta-}, x_f^\gamma),\ (x^{\delta-}, x_g^\gamma),\ (\delta_f^{\delta-}, \delta^\gamma),\ (\delta_g^{\delta-}, \delta^\gamma)\}$$

and looks as in Fig. 6. Compared to the graph of Fig. 5, it is small but cyclic: The two curved edges at the very bottom are new (among others), and they cause the cycles. Thus, σ^- is not an R-substitution and cannot be applied.

Therefore, in our example proof of Section 4 as depicted in Fig. 3, we have to move the γ-step applied to (1^3) down below $(1^5.3.1.2.1)$.[18] A fortiori, this movement of the γ-step forces the problematic β-step at $(1^5.3.1)$ to be moved below $(1^5.3.1.2.1)$ as well; simply because its principal β-formula is the side formula of the γ-step.

Indeed, if we replace the δ^+-rules with δ^--rules, the non-permutability of the β- and the δ^+-steps is hidden behind the well-known non-permutability of the γ- and the δ^--steps, cf. Note 4 in Section 2. Only when the latter non-permutability is removed by replacing the δ^--rules with δ^+-rules, the former becomes visible.

5.3. *Free* δ^+*-variables can escape their quantifiers' scopes*

The non-permutability of the β- and δ^+-steps is closely related to the following peculiar liberality of the δ^+-rules, which they share with the δ^{+^+}-rules (Beckert et al., 1993), the δ^*-rules (Baaz and Fermüller, 1995), and the δ^{*^*}-rules (Cantone and Nicolosi-Asmundo, 2000), but not with the δ^ε-rules (Giese and Ahrendt, 1999) and the δ^--rules. While soundness of both the δ^-- and δ^+-rules and preservation of solutions of the δ^--rules are immediate, the preservation of solutions of the δ^+-rules requires the restriction of the values of the free δ^+-variables by choice-conditions (Wirth, 2004, Theorem 2.49). Even without introducing the semantics of the several kinds of free variables of (Wirth, 2004) here, the reader may grasp the idea of the following example, namely that a solution for x^γ that makes the lower sequent true, may make the upper sequent false:

[18] Note that we cannot move it deeper because it has to precede $\delta_0^-(x^{\delta-})$ (i.e. the former $\delta_0^+(x^{\delta+})$): Indeed, the principal formula of this δ^--step is a subformula of the side formula of the γ-step.

Example 3 (*Reduction & Liberalized* δ)**.**
A δ^+-step can reduce $(\forall y.\ \neg \mathsf{P}(y)),\ \mathsf{P}(x^\gamma)$
to $\neg \mathsf{P}(y^{\delta^+}),\ \mathsf{P}(x^\gamma)$ with empty variable-condition $R = \emptyset$. □

Let us argue semantically at first: In the intuitively straightforward notion of validity (formalized in (Wirth, 2004, Sections 2.2.3 and 2.2.5)), the lower sequent is valid if we solve the free γ-variable x^γ by assigning the value of y^{δ^+} to it. This solution is admissible because the empty variable-condition is not putting any restrictions on such a solution. The upper sequent, however, is not true with respect to this solution in a structure $\mathcal{S}$ in that $\mathsf{P}^{\mathcal{S}}(a)$ is TRUE and $\mathsf{P}^{\mathcal{S}}(b)$ is FALSE for some values a, b from the universe of $\mathcal{S}$. To see this, suppose that y^{δ^+} has the value b, which is admissible unless a choice-condition restricts the value of y^{δ^+} in the way indicated to the upper right of the δ^+-rule in Fig. 1. Then, for the solution given above, x^γ and y^{δ^+} both have the value b. Thus, in $\mathcal{S}$, the upper sequent evaluates to FALSE (as a result of the intermediate $\mathsf{FALSE}, \mathsf{FALSE}$), whereas the lower sequent evaluates to TRUE (as a result of $\mathsf{TRUE}, \mathsf{FALSE}$ (the comma denoting disjunction)).

Let us now argue syntactically: After applying the R-substitution

$$\mu^+ := \{x^\gamma \mapsto y^{\delta^+}\},$$

the lower sequent is a tautology, whereas the upper sequent is not.

To the contrary, in case of δ^--rules, solutions to free γ-variables are always preserved. This can be seen as follows: If we apply a δ^--rule instead of the δ^+-rule in our given example (resulting in the new lower sequent $\neg \mathsf{P}(y^{\delta^-}),\ \mathsf{P}(x^\gamma)$), then this application adds $\{(x^\gamma, y^{\delta^-})\}$ to the variable-condition, thereby blocking the analogous solution

$$\mu^- := \{x^\gamma \mapsto y^{\delta^-}\},$$

simply because μ^- is no $\{(x^\gamma, y^{\delta^-})\}$-substitution, cf. Definition 1.

Roughly speaking, via μ^+, the δ^+-variable y^{δ^+} escapes the scope of the quantifier $\forall y$ on the bound variable y which was eliminated by the introduction of y^{δ^+}; indeed, in the upper sequent of Example 3, the variable x^γ does not occur in the scope of the quantifier $\forall y$. At least with matrix calculi and indexed formulas trees (Autexier, 2003; Wallen, 1990), this "escaping" is a natural way to talk about this peculiar liberality of the δ^+-rule.

Note that this kind of escaping also happens in Fig. 3 of the proof of (lim +): Taking the tree of Fig. 3 to be an indexed formula tree, roughly speaking, the quantifier for $\delta_g^{\delta^+}$ is situated at the term position $(1^5.3.1.2)$, but, via σ, it escapes to term position $(1^5.3.1.1)$.

5.4. δ^{+^+} instead of δ^+

Let us see how the proof of (lim +) would look like with the δ^{+^+}-rules (Beckert et al., 1993) as the only δ-rules available. This does not change anything in the proof as given in Section 4, but allows us to use the identical free δ^+-variable $\delta_g^{\delta^+}$ again when repeating the δ-step which introduced it. Thus, starting from $(1^5.3.1.1)$ of Section 4.3, we can repeat some of the steps done in proof of $(1^5.3.1.2)$, namely " $\delta_0^+(\delta_g^{\delta^+}), \alpha_0^2$ " of Fig. 3, but now as " $\delta_0^{+^+}(\delta_g^{\delta^+}), \alpha_0^2$ ". Note that the δ^+-rules would allow $\delta_0^+(\delta_G^{\delta^+})$ only, with a fresh variable $\delta_G^{\delta^+}$. The resulting sequent is
$(1^5.3.1.1.1)$: $0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}),\ \Omega$
It is like $(1^5.3.1.2.1)$ of Section 4.3, but with the β_2-side formula of the critical β-step replaced with the β_1-side formula $0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+})$. This formula admits to close this branch with the formulas $0 \not< \delta_f^{\delta^+}$ and $0 \not< \delta_g^{\delta^+}$ (as listed in Ω of Fig. 2), applying Lemma (4) of Fig. 2 as at the end of Section 4.6.

Note that this proof with the δ^{+^+}-rules does not have a higher number of γ-steps than the proof attempt failing in Section 4.5. Also the maximum number of δ-steps per formula and *per path* is still 1. Nevertheless, the multiple expansion of the same δ-formula in different paths is somehow counter-intuitive, especially in the sense that working mathematicians interacting with a computing system and supporting it in the construction of closed proof trees (as indicated in Section 1) would not expect such steps from their experience with natural language proof construction. Luckily, in indexed formula trees based on the δ^{+^+}-rules, δ-formulas have to be expanded only once. This again means that these matrix versions are more human-oriented than the tableau or sequent versions.

6. Proof of the non-permutability of β and δ^+

As we have seen in Section 5.2, the non-permutable β-step necessarily follows a γ-step that would be non-permutable without the liberalization from δ^- to δ^+. It follows indeed *necessarily*, because the principal formula of the β-step is the side formula of the γ-step.

- The γ-step $\gamma_0(\min(\delta_f^{\delta+}, \delta_g^{\delta+}))$ is permutable with the liberalized δ^+-step $\delta_0^+(\delta_g^{\delta+})$.
- The γ-step $\gamma_0(\min(\delta_f^{\delta-}, \delta_g^{\delta-}))$, however, is non-permutable with the δ^--step $\delta_0^-(\delta_g^{\delta-})$.

And even with the liberalization:

- The β-step is still non-permutable with the δ^+-step $\delta_0^+(\delta_g^{\delta+})$.

As the principal formula of the β-step can be regenerated by a second expansion of the principal formula of the γ-step, we cannot prove the non-permutability unless we restrict the γ-multiplicity. But, according to the description of the notion of non-permutability in Section 2, we may indeed restrict the γ-multiplicity, in which case the crucial step, namely Lemma 2, admits the following semantical proof.

6.1. *Proof of Lemma 2 (cf. the end of Section 4.5)*

Let us remove the three γ-formulas which form the sequent Γ (cf. Fig. 2) from the sequents $(1^5.1)$, $(1^5.2)$ (cf. Section 4.2), and $(1^5.3.1.1)$ (cf. Section 4.3). As these γ-formulas were already once expanded at (1^3) and (1^4) (cf. Fig. 3), this removal represents a restriction of the γ-multiplicity of the removed γ-formulas to 1, and results in the following sequents (after some reordering):

$$(\mathbf{1^5.1}\backslash\Gamma+):\quad 0 < \varepsilon_f^\gamma,\ 0 \not< \varepsilon^{\delta-},\ \neg\left(0 < \varepsilon_g^\gamma \Rightarrow \exists\delta_g{>}0.\ \forall x_g{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g)-y_g^{\delta-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g-x_0^{\delta-}| < \delta_g\end{array}\right)\right),$$
$$0 < \min(\delta_f^{\delta+}, \delta_g^{\delta+}) \wedge \forall x{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |(f^{\delta-}(x)+g^{\delta-}(x))-(y_f^{\delta-}+y_g^{\delta-})| < \varepsilon^{\delta-} \\ \Leftarrow & |x-x_0^{\delta-}| < \min(\delta_f^{\delta+}, \delta_g^{\delta+})\end{array}\right)$$

$$(\mathbf{1^5.2}\backslash\Gamma+):\quad 0 < \varepsilon_g^\gamma,\ 0 \not< \varepsilon^{\delta-},\ \neg\exists\delta_f{>}0.\ \forall x_f{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f)-y_f^{\delta-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f-x_0^{\delta-}| < \delta_f\end{array}\right),$$
$$0 < \min(\delta_f^{\delta+}, \delta_g^{\delta+}) \wedge \forall x{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |(f^{\delta-}(x)+g^{\delta-}(x))-(y_f^{\delta-}+y_g^{\delta-})| < \varepsilon^{\delta-} \\ \Leftarrow & |x-x_0^{\delta-}| < \min(\delta_f^{\delta+}, \delta_g^{\delta+})\end{array}\right)$$

$$(\mathbf{1^5.3.1.1}\backslash\Gamma+):\quad 0 < \min(\delta_f^{\delta+}, \delta_g^{\delta+}),\ 0 \not< \varepsilon^{\delta-},$$
$$\neg\left(0 < \delta_f^{\delta+} \wedge \forall x_f{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f)-y_f^{\delta-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f-x_0^{\delta-}| < \delta_f^{\delta+}\end{array}\right)\right),$$
$$\neg\exists\delta_g.\left(0 < \delta_g \wedge \forall x_g{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g)-y_g^{\delta-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g-x_0^{\delta-}| < \delta_g\end{array}\right)\right)$$

The related variable-condition R is shown in Fig. 5 (without the dotted edges) and the current R-choice-condition C is given as

$$\left\{\begin{array}{l} x^{\delta+} \mapsto \neg\left(x^{\delta+} \neq x_0^{\delta-} \Rightarrow \left(\begin{array}{ll} & |(f^{\delta-}(x^{\delta+})+g^{\delta-}(x^{\delta+}))-(y_f^{\delta-}+y_g^{\delta-})| < \varepsilon^{\delta-} \\ \Leftarrow & |x^{\delta+}-x_0^{\delta-}| < \min(\delta_f^{\delta+}, \delta_g^{\delta+})\end{array}\right)\right), \\ \delta_f^{\delta+} \mapsto \left(0 < \delta_f^{\delta+} \wedge \forall x_f \neq x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f)-y_f^{\delta-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f-x_0^{\delta-}| < \delta_f^{\delta+}\end{array}\right)\right), \\ \delta_g^{\delta+} \mapsto \left(0 < \delta_g^{\delta+} \wedge \forall x_g \neq x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g)-y_g^{\delta-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g-x_0^{\delta-}| < \delta_g^{\delta+}\end{array}\right)\right)\end{array}\right\}$$

It now suffices to show that there is no proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ with the δ^-- and δ^+-rules as the only δ-rules available.

We do this with a trivial transformation given by the substitution

$$\nu := \{\delta_f^{\delta+} \mapsto \delta_f^{\delta-},\ \delta_g^{\delta+} \mapsto \delta_g^{\delta-}\}$$

of an assumed proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ on the one hand, and with a deviation over invalidity and soundness on the other hand, as follows:

Instantiating the sequents $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ by ν we get the sequents

$$(\mathbf{1^5.1}\backslash\Gamma-)\colon\ 0<\varepsilon_f^{\gamma},\ 0\not<\varepsilon^{\delta-},\ \neg\left(0<\varepsilon_g^{\gamma}\Rightarrow\exists\delta_g{>}0.\ \forall x_g\neq x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g)-y_g^{\delta-}|<\varepsilon_g^{\gamma}\\ \Leftarrow & |x_g-x_0^{\delta-}|<\delta_g\end{array}\right)\right),$$

$$0<\min(\delta_f^{\delta-},\delta_g^{\delta-})\wedge\forall x\neq x_0^{\delta-}.\left(\begin{array}{ll} & |(f^{\delta-}(x)+g^{\delta-}(x))-(y_f^{\delta-}+y_g^{\delta-})|<\varepsilon^{\delta-}\\ \Leftarrow & |x-x_0^{\delta-}|<\min(\delta_f^{\delta-},\delta_g^{\delta-})\end{array}\right)$$

$$(\mathbf{1^5.2}\backslash\Gamma-)\colon\ 0<\varepsilon_g^{\gamma},\ 0\not<\varepsilon^{\delta-},\ \neg\exists\delta_f{>}0.\ \forall x_f\neq x_0^{\delta-}.\left(\begin{array}{ll} & |f^{\delta-}(x_f)-y_f^{\delta-}|<\varepsilon_f^{\gamma}\\ \Leftarrow & |x_f-x_0^{\delta-}|<\delta_f\end{array}\right),$$

$$0<\min(\delta_f^{\delta-},\delta_g^{\delta-})\wedge\forall x\neq x_0^{\delta-}.\left(\begin{array}{ll} & |(f^{\delta-}(x)+g^{\delta-}(x))-(y_f^{\delta-}+y_g^{\delta-})|<\varepsilon^{\delta-}\\ \Leftarrow & |x-x_0^{\delta-}|<\min(\delta_f^{\delta-},\delta_g^{\delta-})\end{array}\right)$$

$$(\mathbf{1^5.3.1.1}\backslash\Gamma-)\colon\ \ 0<\min(\delta_f^{\delta-},\delta_g^{\delta-}),\ 0\not<\varepsilon^{\delta-},$$

$$\neg\left(\ 0<\delta_f^{\delta-}\ \wedge\ \forall x_f{\neq}x_0^{\delta-}.\ \left(\ |f^{\delta-}(x_f)-y_f^{\delta-}|<\varepsilon_f^{\gamma}\ \Leftarrow\ |x_f-x_0^{\delta-}|<\delta_f^{\delta-}\ \right)\ \right),$$

$$\neg\exists\delta_g.\left(\ 0<\delta_g\ \wedge\ \forall x_g{\neq}x_0^{\delta-}.\left(\begin{array}{ll} & |g^{\delta-}(x_g)-y_g^{\delta-}|<\varepsilon_g^{\gamma}\\ \Leftarrow & |x_g-x_0^{\delta-}|<\delta_g\end{array}\right)\right)$$

The conjunction of these sequents is invalid according to the standard semantics for parameters as well as the semantics of Wirth (2004). This can be seen by

$$\{\ \delta_f^{\delta-}\mapsto 1,\quad \delta_g^{\delta-}\mapsto 0,\quad \varepsilon^{\delta-}\mapsto 1,\quad x_0^{\delta-}\mapsto 0,\quad y_f^{\delta-}\mapsto 0,\quad y_g^{\delta-}\mapsto 0,\quad f^{\delta-}\mapsto\lambda x.0,\quad g^{\delta-}\mapsto\lambda x.0\ \}.$$

Indeed, if we instantiate $(1^5.1\backslash\Gamma-)$, $(1^5.2\backslash\Gamma-)$, and $(1^5.3.1.1\backslash\Gamma-)$ with this substitution and then $\lambda\beta$-normalize and simplify these sequents by equivalence transformations in the model of the real numbers **R**, we get the three sequents

$$0<\varepsilon_f^{\gamma},\ \mathsf{false},\ \neg\left(\ 0<\varepsilon_g^{\gamma}\Rightarrow\left(\begin{array}{ll} & 0<\varepsilon_g^{\gamma}\\ \Leftarrow & \forall\delta_g{>}0.\ \exists x_g\neq 0.\ |x_g|<\delta_g\end{array}\right)\right),\ \mathsf{false}$$

$$0<\varepsilon_g^{\gamma},\ \mathsf{false},\ \neg\left(\begin{array}{ll} & 0<\varepsilon_f^{\gamma}\\ \Leftarrow & \forall\delta_f{>}0.\ \exists x_f\neq 0.\ |x_f|<\delta_f\end{array}\right),\ \mathsf{false}$$

$$\mathsf{false},\ \mathsf{false},\ \neg(0<\varepsilon_f^{\gamma}\Leftarrow\exists x_f\neq 0.\ |x_f|<1),\ \neg\left(\begin{array}{ll} & 0<\varepsilon_g^{\gamma}\\ \Leftarrow & \forall\delta_g{>}0.\ \exists x_g\neq 0.\ |x_g|<\delta_g\end{array}\right)$$

Further equivalence transformation in **R** results in the three contradictory sequents

$$0<\varepsilon_f^{\gamma}$$
$$0<\varepsilon_g^{\gamma},\ 0\not<\varepsilon_f^{\gamma}$$
$$0\not<\varepsilon_f^{\gamma},\ 0\not<\varepsilon_g^{\gamma}$$

Thus, as our calculus is sound, it cannot prove $(1^5.1\backslash\Gamma-)$, $(1^5.2\backslash\Gamma-)$, and $(1^5.3.1.1\backslash\Gamma-)$ simultaneously.

Regarding free δ-variables that occur already in the upper sequents of our rules (i.e. in the conclusions), the following holds: The δ^+-rules treat free δ^-- and free δ^+-variables alike; and, for free δ^--variables, the δ^--rules generate a smaller variable-condition than for free δ^+-variables (cf. $\mathcal{V}_{\gamma\delta^+}(\ldots)$ in Fig. 1; cf. also Note 8). Therefore, a proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ would immediately translate into a proof of $(1^5.1\backslash\Gamma-)$, $(1^5.2\backslash\Gamma-)$, and $(1^5.3.1.1\backslash\Gamma-)$ with – after application of the substitution ν – unchanged inference steps, and with a possibly smaller variable-condition.[19]

Thus, we conclude that there is no proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$. q.e.d.

Finally note that the above trivial proof transformation does not result in a sound proof if we replace the δ^+-rules with the δ^{+^+}-rules: Indeed, the δ^{+^+}-rules may re-use $\delta_g^{\delta+}$, but not $\delta_g^{\delta-}$.

19 Note that we do not replace δ^+-*rules* with δ^--*rules* here; all we do is to replace some δ^+-*variables* in the sequents with δ^--*variables*.

6.2. Defining permutability

A reader with a good mathematical intuition can and should directly consider the non-permutability of β- and δ^+-steps as a corollary of Lemma 2 proved above. A formalist, however, may well require some rigorous definition of permutability. There were good reasons not to present a formal definition of permutability earlier in this paper:

The logically weakest reasonable definitions of permutability I can think of, still result in the non-permutability we want to show. Indeed, we may choose any definition of permutability that contradicts Lemma 2. For instance, as it strengthens our non-permutability result, we should (and will) use a notion that is weaker than the following standard one: Two inference steps S_1 and S_0 are *locally directly permutable* if replacing an occurrence of $\frac{S_0}{S_l \quad S_1 \quad S_r}$ in a closed proof tree (where S_1 is also applicable instead of S_0) with $\frac{S_1}{\frac{S_0}{S_l} \quad S_0 \quad \frac{S_0}{S_r}}$ results (*mutatis mutandis*) in a closed proof tree.

In fact, there is no definition of permutability or non-permutability in Wallen's whole book (Wallen, 1990), although the avoidance of non-permutability is one of its main subjects, cf. Section 2.

My formalization of the notion of permutability will depend on the notions of a *principal meta-variable* of an *inference rule* and is somewhat technical and difficult, even in the rudimentary form we will present below.

To avoid clutter, we define permutability only for sequent calculi. The definition for tableau calculi is analogous. Formally, for each inference rule, we have to define which meta-variables are principal and which are not. On the one hand, the meta-variables of the principal formulas have to be principal, and an instantiation of all principal meta-variables must determine the existence of an instantiation of the other meta-variables such that the inference rule becomes applicable. On the other hand, it is not appropriate to define all meta-variables of an inference rule to be principal, because this results in a general non-permutability of inference steps.

Definition 4 (*Principal Meta-Variables*). In our inference rules of Fig. 1 in Section 3.1 exactly the meta-variables A, B, x, t, x^{δ^-}, and x^{δ^+} are *principal*; and the other meta-variables (i.e. Γ, Π) are not principal. In lemma-application steps as explained in Section 3.2, the A_k and C_i are principal, whereas the B_j are not. For technical simplicity, we ignore our definitional expansion steps on ∀, ∃, lim, assuming a complete expansion of these definitions right from the start. □

Definition 5 (*Inference Step*). A *proof tree* is a labeled tree whose root is labeled with a sequent and whose paths are labeled with sequents and inference steps alternately, such that there is a proof history of applicable inference steps (expansion steps) and global applications of R-substitutions on free γ-variables (which instantiate the free γ-variables of their domains in all occurrences in all labels of the proof tree, i.e. in all sequents *and in all inference steps*), starting from a proof tree consisting only of a root node. (Of course, the parent and child nodes of a node labeled with an inference step must be labeled with the conclusion and the premises of this inference step, respectively.)
A proof tree is *closed* if all its leaves that are not labeled with inference steps are labeled with axioms.
An *inference step* is a triple (I, π, ϱ) labeling a node in a proof tree where I is an inference rule and π and ϱ are substitutions of the principal and non-principal meta-variables of I, respectively; such that $I(\pi \uplus \varrho)$ describes the inference step with parent (conclusion) and child (premise) nodes as an instance of the inference rule I. □

Note that in Definition 5 we indeed have to refer to the proof history because the δ^+-step $\delta^+_0(\delta^{\delta+}_g)$ applied to $(1^5.3.1)$ at the beginning of Section 4.6 would not be admitted if we applied the R-substitution σ before expanding the proof tree by the δ^+-step. This is because δ^+-steps have to introduce *new* free δ^+-variables, and σ would have introduced the variable $\delta^{\delta+}_g$ already before.
Roughly speaking, permutability of two steps S_1 and S_0 simply means the following: *In a closed proof tree where S_0 precedes S_1 and where S_1 was already applicable before S_0, we can do the step S_1 before S_0 and expand the resulting new subtree into a closed proof tree nevertheless.*

Definition 6 (*Permutability*). Let (I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) be two inference steps. (I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) are *permutable for a given threshold m for γ-multiplicity* if for any closed proof tree T with γ-multiplicity m satisfying that

(1) n_i is an inference node in T labeled with (I_i, π_i, ϱ_i), for $i \in \{0, 1\}$,
(2) n_0, n_1 are, in this order and with only a sequent node in between, on the same path in T from the root to a leaf, and
(3) there is a substitution ϕ such that the parent sequents (conclusions) of $I_0(\pi_0 \uplus \varrho_0)$ and of $I_1(\pi_1 \uplus \phi)$ are identical;

there is a closed proof tree with γ-multiplicity m which differs from T only in the subtree starting with n_0 and the root label of this subtree is (I_1, π_1, ϕ).
(I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) are *permutable* if they are permutable for any given threshold $m \in \mathbf{N}$ of γ-multiplicity.
I_1 and I_0 are *generally permutable* if all inference steps of the forms (I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) are permutable. □

Example 7. For inferring the non-permutability of β and δ^+ from Lemma 2, we have to instantiate Definition 6 as follows:

$n_0 \approx (1^5.3.1) \longrightarrow (1^5.3.1^2)$ (cf. Section 4.6)
I_0 is $(\delta^+, \neg\exists)$ of Fig. 1 in Section 3.1

$$\pi_0 = \left\{ \begin{array}{lll} x & \mapsto & \delta_g; \\ x^{\delta^+} & \mapsto & \delta_g^{\delta^+}; \\ A & \mapsto & \left(0 < \delta_g \wedge \exists x_g \neq x_0^{\delta^-}. \left(\begin{array}{ll} & |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \frac{\varepsilon^{\delta^-}}{2} \\ \Leftarrow & |x_g - x_0^{\delta^-}| < \delta_g \end{array} \right) \right) \end{array} \right\}$$

$$\varrho_0 = \left\{ \begin{array}{ll} \Gamma \mapsto & \left(\begin{array}{l} 0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \\ \wedge \quad \forall x \neq x_0^{\delta^-}. \left(\begin{array}{ll} & \left| \begin{array}{l} (f^{\delta^-}(x) + g^{\delta^-}(x)) \\ -(y_f^{\delta^-} + y_g^{\delta^-}) \end{array} \right| < \varepsilon^{\delta^-} \\ \Leftarrow & |x - x_0^{\delta^-}| < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \end{array} \right) \end{array} \right), \quad \ldots; \\ \Pi \mapsto \ldots & \end{array} \right\}$$

$n_1 \approx$ "a new step of an alternative closed proof tree that results from the closed proof tree of Section 4.6 by permuting the β-step at $(1^5.3.1^2)$ and the steps α^2, $\gamma_0(x^{\delta^+})^2$ applied to $(1^5.3.1)$. This alternative proof tree is depicted in Fig. 7. (Only for pedagogical reasons, we delayed the β-step with its failure-potential until we were forced to do it.)"

I_1 is $(\beta, \wedge)$ of Fig. 1 in Section 3.1

$$\pi_1 = \left\{ \begin{array}{lll} A & \mapsto & 0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}); \\ B & \mapsto & \forall x \neq x_0^{\delta^-}. \left(\begin{array}{ll} & \left| \begin{array}{l} (f^{\delta^-}(x) + g^{\delta^-}(x)) \\ -(y_f^{\delta^-} + y_g^{\delta^-}) \end{array} \right| < \varepsilon^{\delta^-} \\ \Leftarrow & |x - x_0^{\delta^-}| < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \end{array} \right) \end{array} \right\}$$ □

Now, the non-permutability of the critical β- and δ^+-steps of Example 7 follows from Lemma 2, because there is no alternative proof tree which differs only in the subtree starting at n_0 and having a new subtree there starting with the critical β-step. The deeper reason for this is that the instantiated free γ-variables occur outside the subtree of the δ^+-step, cf. Section 5.3. According to Lemma 2, there is no proof of $(1^5.1)$, $(1^5.2)$ and $(1^5.3.1.1)$ with the instantiation by σ. Since the partial instantiation by σ agrees with the full instantiation in the closed proof tree of the successful proof of Fig. 7, we have the required witness for the non-permutability of β and δ^+, indeed. Thus, as corollaries we get:

Corollary 8. *On a threshold for γ-multiplicity of* 1, *the inference steps*

$$((\beta, \wedge), \pi_1, \varrho_1) \quad \textit{and} \quad ((\delta^+, \neg\exists), \pi_0, \varrho_0)$$

(as labels of the nodes n_1 and n_0, resp.) as given in Example 7 are not permutable. □

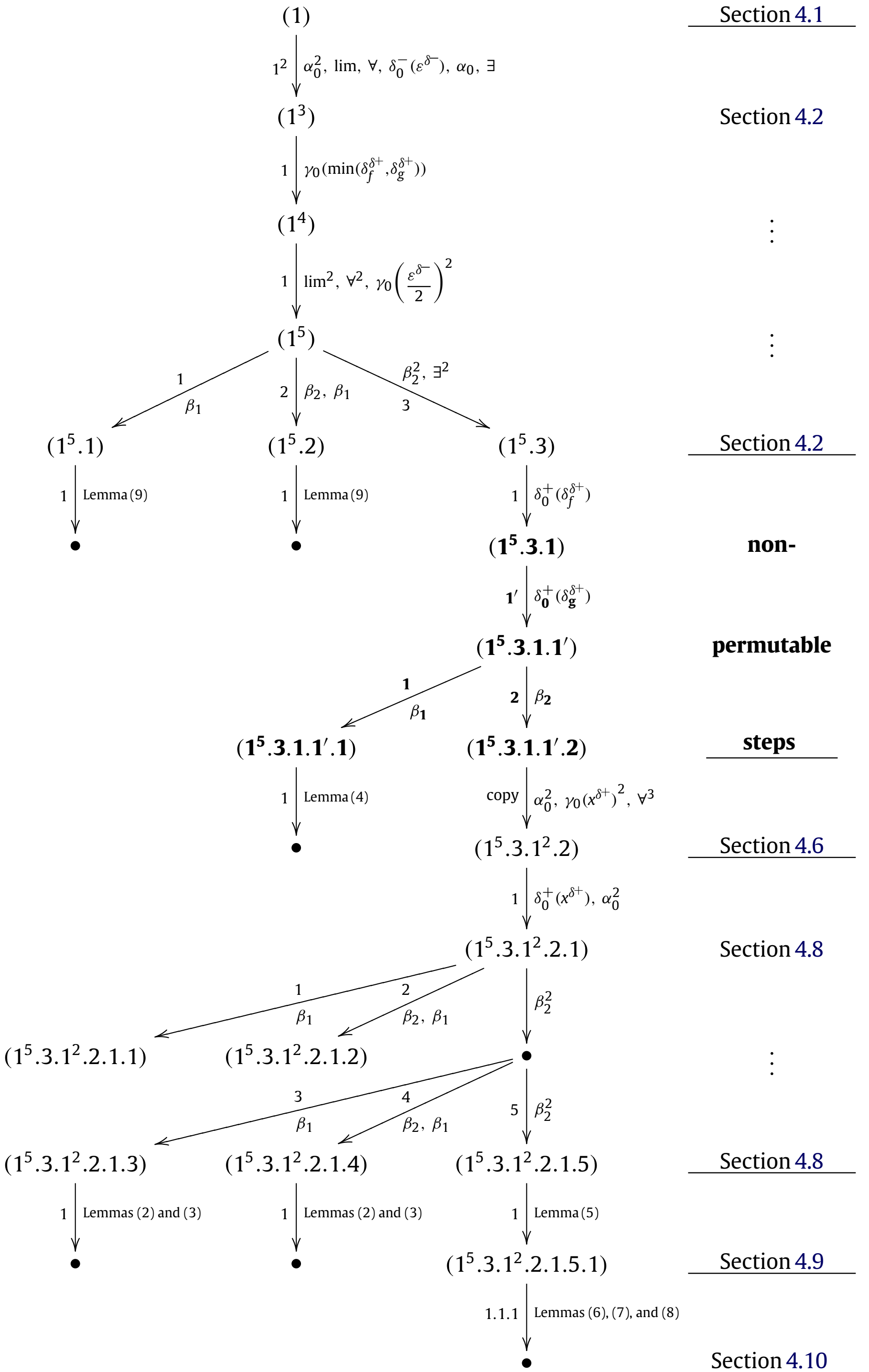


Fig. 7. Closed proof tree with non-permutable β- and δ^+-steps.

Theorem 9. *β- and δ^+-steps are not generally permutable,*

- *neither in the sequent calculus of Wirth (2004) (cf. our Fig. 1 in Section 3.1),*
- *nor in standard free-variable tableau calculi with δ^+-rules as the only δ-rules, such as the ones in Fitting (1996); Hähnle and Schmitt (1992).* □

7. Further discussion of related subjects

When the δ^+-rules occurred first in (Hähnle and Schmitt, 1992) (where their whole treatment takes only four pages actually), they seemed so simple and straightforward. Today, a dozen years later, they are still not completely understood. We have shown that the δ^+-rules still have unrealized properties, such as the non-permutability of β- and δ^+-steps. Indeed, there are several *open problems*, such as, from theoretical to practical:

7.1. Complexity?

Does the non-elementary reduction in proof size (Baaz and Fermüller, 1995) from the δ^-- to the δ^{+^+}-rules mean a non-elementary reduction in proof size from δ^- to δ^+, or from δ^+ to δ^{+^+} (exponential at least Beckert et al. (1993)), or both?

7.2. More non-permutabilities

Why were the non-permutabilities of β-steps presented in this paper not noticed before? May there be others around?

7.3. Is soundness sufficient in practice?

The notion of *safeness* (soundness of the reverse inference step, for failure detection after generalization, e.g. for induction) seems to become standard (Autexier, 2005; Nipkow et al., 2002; Wirth, 1997, 2004). And in Wirth (1998, 2004, 2008, 2011) we have also added the notion of *preservation of solutions*. This means that the closing substitutions on the rigid variables of the sub-goals must solve the input theorem's rigid variables, which make sense as placeholders for concrete bounds and side conditions of the theorem which only a proof can tell.

7.4. Are the known notions of completeness relevant in practice?

The mere existence of a proof is not sufficient for mathematics assistance systems, where we need the existence of a proof that closely mirrors the proof the mathematician interacting with the system has in mind, searches for, or plans. (Readers who think that the δ^--rules would admit human-oriented proof construction should try to do the proof of (lim +) with the δ^--rules as the only δ-rules!)

I must admit, however, that I do not know how to grasp a practically relevant notion of completeness. The sequent calculus of our inductive theorem prover QuodLibet (Avenhaus et al., 2003; Kühler, 2000; Schmidt-Samoa, 2004, 2006a,c,b; Wirth, 1997, 2005, 2009) has been improved over a dozen years of practical application to admit our proofs; and it still needs some further improvement.

7.5. Calculi for automation plus interaction

The automatic generation of a non-trivial proof for a given input conjecture is typically not possible today and probably will never be. Thus, beside some rare exceptions – as the automation of proof search will always fail on the lowest logic level from time to time – the only chance for automatic theorem proving to become useful for mathematicians seems to be a synergetic interplay between the mathematician and the machine. For this interplay – to give the human user a chance to interact – the calculus *itself* must be human-oriented. Indeed, it does not suffice to compute human-oriented representations; not in the end, and – as the syntactical problems have to be presented accurately – also not intermediately in a user interface.

8. Conclusion

We have exhibited unknown non-permutabilities of β-steps that surprised experts of the field: in Sections 4.3–4.5 a non-permutability of a β-step with a δ^+-step; and in Section 4.7 a non-permutability of a β-step with a β-step. In Section 5.1 we have explained why the latter is a consequence of the former, and in Sections 5.2–5.4 we have made clear how it comes to the former non-permutability and why it is so surprising. In Section 6.1 we proved Lemma 2, according to which the proof attempt in Section 4.5 is indeed a failed one for a γ-multiplicity threshold of 1. In Section 6.2 we have formalized a local notion of non-permutability and showed that Lemma 2 implies the existence of such a non-permutability of a β- with a δ^+-step indeed.

Although the non-permutability of β- and δ^+-steps is not visible with (non-liberalized) δ^--rules and dissolves into a problem of mere inefficiency with further liberalized δ-rules, the optimization of the sequencing of the β-steps is always of practical importance, both for efficiency of proof search and for human-orientedness of proof presentation. The same holds for the optimization problem of finding a good order of application for the β-steps.

Even with more liberalized δ-rules available today (such as δ^{+^+}-, δ^*-, δ^{*^*}-, and δ^ε-rules, cf. Section 5.3), the δ^+-rules remain important, both conceptually and for stepwise presentation and limitation of complexity in teaching, research, and publication. For instance, the δ^+-rules are the free-variable tableau rules used in the current edition of Melvin Fitting's excellent textbook (Fitting, 1996).

The δ^+-rules may also serve as a sound fallback in case that further liberalized δ-rules turn out to be unsound. For instance, until very recently (Cantone and Nicolosi-Asmundo, 2005), nobody realized that the δ^*- and δ^{*^*}-rules were unsound in their original publications (incl. their corrigenda!).[20]

Section 4 contains what seems to be the first publication of a more or less readable, complete, and human-oriented proof of a mathematical standard theorem in a standard general-purpose formal calculus. This paradigmatic example may be beneficial for the future discussion of human–computer interaction in proof construction.

Although more useful for proof search in classical logic than Hilbert (Hilbert and Bernays, 1968/70) and Natural Deduction calculi (Gentzen, 1935), sequent (Gentzen, 1935) and tableau calculi (Fitting, 1996) are still not adequate for a synergetic interplay of human proof guidance and automatic proof search (Wirth, 2004), which we hope to achieve with matrix calculi such as CoRe (Autexier, 2003).

In Section 5.4, we have described why we consider the possibility to overcome the non-permutability of β and δ^+ by replacing the δ^+-rules with the δ^{+^+}-rules not to be adequate for human-oriented reasoning yet.[21] We hope that it has become clear from our presentation that not only automated

[20] The δ^*-rule was unsound as printed in the lecture notes (Baaz and Fermüller, 1995). It was first corrected by the authors right at the 5th Int. Conf. on Tableaus and Related Methods, St. Goar (Germany), 1995. It had to be corrected once more in the presentation of (Cantone and Nicolosi-Asmundo, 2005) by Marianna Nicolosi-Asmundo, where also the δ^{*^*}-rule had to be corrected as compared to (Cantone and Nicolosi-Asmundo, 2000).

[21] An anonymous referee of a previous version of this paper wrote:

> "The arguments against the use of δ^{+^+} (that the proofs found this way are not human-oriented) are not convincing. It is well-known that improved Skolemization rules can be simulated with applications of the cut rule. So one could proceed as follows. Use δ^{+^+} for proof generation, for presentation insert the respective cut steps. This way any forms of sophisticated Skolemization could be replaced by case distinctions, which are easily understandable by any human user."

The point that was missed in this critique is described in Section 7.5.

theorem proving, but also human-oriented reasoning requires matrix calculi and indexed formula trees (Autexier, 2003; Wallen, 1990). Both for human- and machine-oriented theorem proving, we need these calculi also to admit a lazy sequencing of β-steps (so that the connection-driven path construction may tell us in the end, which sequencing of the β-steps we need).

As the automation of proof search will always fail on the lowest logic level from time to time, be aware: *The fine structure and human-orientedness of a calculus does matter in practice!*

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