Analysis of Heat and Mass Transfer on MHD Flow of a Nanofluid Past a Stretching Sheet

M. Chandrasekar*, M.S. Kasiviswanathan

Department of Mathematics, Anna University, Chennai 600 025, India

Abstract

In this work, a variational technique is applied to MHD, radiative nanofluid flow over a non-isothermal stretching sheet with Brownian motion and thermophoresis effects by Gyarmati’s principle. The heat and mass transfer effects have been investigated and analyzed by this technique. The flow fields inside the boundary layer are approximated as polynomial functions. Euler-Lagrange equations for the functional of variational principle are constructed. The non-linear boundary layer equations are simplified as simple polynomial equations in terms of momentum, thermal and concentration boundary layer thicknesses. The temperature, concentration profiles, local heat and mass transfer rates are analyzed and are compared with existing numerical results. The comparison shows remarkable accuracy.

Keywords: Gyarmati’s variational principle; stretching sheet; nanofluid; boundary layer; heat and mass transfer; Brownian motion; thermophoresis; thermal radiation

1. Introduction

The prime objective of this work is to analyse the heat transfer enhancement of MHD nanofluid flow over a non-isothermal stretching sheet with radiation effect by using the field of thermodynamics of irreversible processes and to obtain numerical solution to heat and mass transfer with the help of a variational technique based on the Governing Principle of Dissipative Processes (GPDP).

In many industrials, extrusion is an important process in manufacturing of products. The quality of these products solely depends on the heat transfer rate at the stretching sheets. Sakiadis [1] analysed the boundary layer...

The effects of magneto hydrodynamics and thermal radiation on convective heat transfer play vital role in the phenomena of electrically conducting fluid past a heated surface and thermal processes involving high temperatures such as power generators, nuclear power plants etc. Swati [3] analysed these effects on boundary layer flow over an exponentially stretching sheet.

In recent years, nanofluid which is a mixture of nano-sized particles suspended in a conventional fluid is used to enhance the heat transfer rate. The benefits of nanofluids are theoretically investigated by Choi [4]. The explanation for abnormal convective heat transfer enhancement in nanofluids was observed by Buongiorno [5].


As suggested in Buongiorno model the two important slip mechanisms Brownian motion and thermophoresis effects are considered in this boundary layer flow over a non-isothermal stretching sheet through quiescent nanofluid in the presence of radiation and constant magnetic flux density. Gyarmati’s variational technique has been employed and the results are given for the temperature profile, concentration profile, the local Nusselt number (heat transfer) and the Sherwood number (mass transfer) for various values of Prandtl number \( Pr \), magnetic parameter \( \xi \), wall temperature parameter \( n \), radiation parameter \( Nr \), the slip parameters \( Nb \) (Brownian effect), \( Nt \) (thermophoresis effect) and Lewis number \( Le \). The present results are compared with known numerical results and are found to be quite in agreement. The intention of this research work is to establish the fact that Gyarmati’s principle is one of the exact and most general variational techniques in solving heat and mass transfer problems. Chandrasekar [8], Chandrasekar and Kasiviswanathan [9] have already applied Gyarmati’s variational principle for steady and unsteady heat transfer and boundary layer flow problems.

2. The governing boundary layer equations

The system of steady, two dimensional and laminar boundary layer flow of a nanofluid over a non-isothermal stretching sheet with velocity \( U_0 \) in \( x \)-direction is considered. The leading edge of the sheet is at \( x = 0 \) and the sheet is parallel to the \( x \)-axis. It is assumed that as \( y \to \infty \), the quiescent nanofluid is with ambient temperature \( T_\infty \) and concentration \( C_\infty \). By Boundary layer-Boussinesq approximations and with the assumption that all fluid properties are constants, the boundary layer equations in the presence of thermal radiation are considered as follows,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{u}{x} \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \rho_f \frac{\kappa B_0^2}{\rho_f} u
\]

\[
\frac{u}{x} \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial y^2} \right) - \frac{1}{\rho c_f} \frac{\partial q_f}{\partial y} + \tau \left[ D_B \left( \frac{\partial C}{\partial y} \right) + D_T \left( \frac{\partial T}{T_\infty} \right) \right]
\]

\[
\frac{u}{x} \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \left( \frac{\partial^2 C}{\partial y^2} \right) + D_T \left( \frac{\partial^2 T}{T_\infty} \right)
\]

subject to the boundary conditions
\[ \begin{align*}
\frac{\partial u}{\partial y} & \to 0 \Rightarrow u = U_0 = ax, v = 0, T = T_0 = T_\infty + Ax, C = C_0 \\
\frac{\partial u}{\partial y} & \to \infty \Rightarrow u = 0, v = 0, T = T_\infty, C = C_\infty
\end{align*} \] 

(5)

where \( u, v, T, C \) are the velocity of the fluid in the longitudinal direction, transverse direction, temperature and concentration of the nanofluid respectively. The symbols \( \nu, \kappa, B_0, \rho_f, c_f, \tau, D_b, D_T, T_0 \) are respectively the kinematic viscosity, electrical conductivity, externally imposed magnetic field in the \( y \)-direction, density and specific heat of the fluid, \( \tau = (\rho c)_p / (\rho c)_f \) is the ratio of nanoparticle heat capacity and base fluid heat capacity, Brownian diffusion coefficient, thermophoresis diffusion coefficient and temperature of the stretching sheet. It is assumed that the temperature of the sheet \( T_0 \) is greater than the ambient temperature \( T_\infty \).

Using Rosseland approximation, the radiative heat flux is described by

\[ q_r = \frac{4\sigma^*}{3k} \frac{\partial T^4}{\partial y}, \]

where \( \sigma^* \), \( k^* \) are the Stefan-Boltzmann constant and mean absorption coefficient respectively. Linearization of the radiation \( T^4 \) in terms of temperature difference between the atmospheric level and the main flow as follows,

By Taylor series, we obtain

\[ T^4 \approx T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + \cdots . \]

We assume that the lowest temperature differences within the main flow. Therefore the higher order terms are to be neglected, hence \( T^4 \equiv 4T_\infty^3T - 3T_\infty^4 \). Thus, we have

\[ q_r = \frac{-16T_\infty^3\sigma^*}{3k} \frac{\partial T}{\partial y}. \]

3. Gyarmati’s variational principle

On the basis of irreversible thermodynamics, Gyarmati’s “Governing Principle of Dissipative Processes” is given in its energy picture (Gyarmati [10,11]) as

\[ \delta \int V (T\sigma - T_\Psi - T\Phi) dV = 0 \]

(6)

Here the energy dissipation \( T\sigma \) and dissipation potentials \( T_\Psi, T\Phi \) are given by

\[ T\sigma = -P_{12} \frac{\partial u}{\partial y} - J_q \frac{\partial \ln T}{\partial y} - J_c \frac{\partial C}{\partial y}, \]

\[ T_\Psi = \frac{1}{2} \left[ L_1 \left( \frac{\partial u}{\partial y} \right)^2 + L_4 \left( \frac{\partial \ln T}{\partial y} \right)^2 + L_5 \left( \frac{\partial C}{\partial y} \right)^2 \right] \]

and

\[ T\Phi = \frac{1}{2} \left[ R_q J_q^2 + R_c J_c^2 + R_{12} J_{12}^2 \right], \]

where \( P_{12} = -L_5 \frac{\partial u}{\partial y} \),

\( J_q \left( = -L_4 \frac{\partial T}{\partial y} \right) \) and \( J_c \left( = -L_5 \frac{\partial C}{\partial y} \right) \) are heat, momentum and concentration fluxes respectively. The constants \( L \)'s,

\( R \)'s represent conductivities and resistances. It is well known that ‘\( \ln T \)’ is the proper state variable instead of \( T \) when the governing principle assumes energy picture.

The variational principle (6) for the present problem takes the form

\[ \delta \int \left[ \int_{0}^{l} \left( -J_q \frac{\partial \ln T}{\partial y} - P_{12} \frac{\partial u}{\partial y} - J_c \frac{\partial C}{\partial y} - \frac{1}{2} \left[ L_1 \left( \frac{\partial \ln T}{\partial y} \right)^2 + L_4 \left( \frac{\partial u}{\partial y} \right)^2 + L_5 \left( \frac{\partial C}{\partial y} \right)^2 \right] - \frac{1}{2} \left[ R_q J_q^2 + R_c J_c^2 + R_{12} J_{12}^2 \right] \right) dy \right] dx = 0 \]

(7)

in which ‘\( l \)’ is the representative length of the stretching sheet.
4. Method of solution

The velocity, temperature and concentration fields inside the respective boundary layers are assumed as the following trial polynomials,

\[
\begin{align*}
    u &= 1 - \frac{2y}{d_1} + \frac{2y^3}{d_1^3} - \frac{y^4}{d_1^4} \quad \text{for } y < d_1, \\
    T &= T_\infty + \frac{2y}{d_2} + \frac{2y^3}{d_2^3} - \frac{y^4}{d_2^4} \quad \text{for } y < d_2, \\
    C &= C_\infty + \frac{3y}{d_3} + \frac{3y^2}{d_3^2} - \frac{y^3}{d_3^3} \quad \text{for } y < d_3,
\end{align*}
\]

These profiles satisfy the following conditions

\[
\begin{align*}
    y \to 0 &\implies u = U_0 = ax, v = 0, T = T_0 = T_\infty + Ax^n, C = C_0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 C}{\partial y^2} = 0, \\
    y \to \infty &\implies u = 0, T = T_\infty, C = C_\infty, \quad \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial C}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 C}{\partial y^2} = 0.
\end{align*}
\]

The unknown parameters \(d_1, d_2\) and \(d_3\) are momentum, thermal and concentration boundary layer thicknesses and are to be determined by the variational principle. The trial functions (8) are substituted in the governing boundary layer equations (1-4) and on integration with respect to \(y\) and with the help of boundary conditions (9), the fluxes \(P_{12}, J_q\), and \(J_c\) are obtained. The expressions \(P_{12}\) and \(J_c\) remain same for any Prandtl number \(Pr\). But the energy flux \(J_q\) assumes different expressions for \(Pr \leq 1\) and \(Pr \geq 1\) respectively. When \(Pr \leq 1\), the expression for \(J_q\) in the range \(d_1 \leq y \leq d_2\) is determined first and the expression for \(J_q\) in the range \(0 \leq y \leq d_1\) is obtained subsequently by matching the expression \(J_q\) of the two regions at the interface. Using the expressions \(P_{12}\), \(J_q\), and \(J_c\) along with the trial functions (8), the variational principle (7) is formulated. On integration with respect to \(y\), the variational principle becomes as

\[
\begin{align*}
    \delta \int_0^l L_1(d_1, d_2, d_3, d_1', d_2', d_3') dx = 0 \quad (Pr \leq 1) \quad (10) \\
    \delta \int_0^l L_2(d_1, d_2, d_3, d_1', d_2', d_3') dx = 0 \quad (Pr \geq 1)
\end{align*}
\]

where the notation \((\,')\) indicates the differentiation with respect to \(x\). These variational principles (10) are found identical when \(d_1 = d_2\). Accordingly, the Euler-Lagrange equations are

\[
\begin{align*}
    \frac{\partial L_{1,2}}{\partial d_1} - d \left( \frac{\partial L_{1,2}}{\partial d_1'} \right) &= 0 \quad (Pr \leq 1) \quad \text{and} \quad \frac{\partial L_{1,2}}{\partial d_2} - d \left( \frac{\partial L_{1,2}}{\partial d_2'} \right) = 0 \quad (Pr \geq 1), \\
    \frac{\partial L_{1,2}}{\partial d_3} - d \left( \frac{\partial L_{1,2}}{\partial d_3'} \right) &= 0 \quad (Pr \geq 1)
\end{align*}
\]

where \(L_{1,2}\) represents the Lagrangian densities \(L_1\) and \(L_2\) respectively. Equations (11) are second order ordinary differential equations in terms of \(d_1, d_2\) and \(d_3\). The procedure for solving (11) can be considerably simplified by introducing the non-dimensional boundary layer thicknesses \(d_1^*, d_2^*\) and \(d_3^*\) given by \(d_1 = d_1' \sqrt{\nu/a}, \ d_2 = d_2' \sqrt{\nu/a} \) and
The Euler-Lagrange equations to the variational principles (10) subject to this transformation are obtained as simple polynomial equations

\[
\frac{\partial L_{1,2}}{\partial d_1} = 0, \quad \frac{\partial L_{1,2}}{\partial d_2} = 0 \text{ (Pr } \geq 1\text{)} \quad \text{and} \quad \frac{\partial L_{1,2}}{\partial d_3} = 0
\]

(12)

The coefficients of these equations (12) depend on the independent parameters \(Pr\), \(\xi\), \(n\), \(Nr\), \(Nb\), \(Nt\) and \(Le\), where \(Pr = \frac{\nu}{\alpha}\) (Prandtl number), \(\xi = kB_0^2 / \rho_f a\) (magnetic parameter), wall temperature parameter \(n\), \(Nr = 16\sigma^* T_\infty^3 / 3k^* k\) (radiation parameter), \(Nb = \tau D_b (C_0 - C_\infty) / \nu\) (Brownian motion parameter), \(Nt = \tau D_s (T_0 - T_\infty) / \nu T_\infty\) (thermophoresis parameter) and \(Le = \nu D_b / \nu\) (Lewis number).

Equation (12)1 is a simple polynomial equation in terms of momentum boundary layer thickness whose effects depend on the magnetic parameter \(\xi\). And equations (12)2,3 are coupled equations in terms of thermal and concentration boundary layer thicknesses whose coefficients depend on \(d_1^*, Pr, n, Nr, Nb, Nt\) and \(Le\). After obtaining the values of \(d_1^*, d_2^*\) and \(d_3^*\), the quantities of physical interest skin friction (shear stress), heat transfer (Nusselt number) and mass transfer (Sherwood number) are calculated using the following expressions,

\[
\eta = \gamma \sqrt{\frac{a}{\nu}}, \quad \tau_w = \sqrt{\frac{\nu x}{U_0^3} \left( -\frac{P_{12}}{L_x} \right)_{y=0}}
\]

(13)

\[
Nu_l = \sqrt{\frac{\nu x}{U_0 (T_0 - T_\infty^2)} \left( \frac{J_\xi}{L_\xi} \right)_{y=0}} \quad \text{and} \quad Sh_l = \sqrt{\frac{\nu x}{U_0 (C_0 - C_\infty) \left( \frac{J_\xi}{L_\xi} \right)_{y=0}}}
\]

5. Results and discussion

Whenever a new mathematical method is applied to a problem, the obtained results are compared with the available solution in order to establish the accuracy of the results involved in the present technique. Prasad and Vajravelu [12] numerically obtained the non-dimensional skin friction values for \(\xi = 0\) and \(\xi = 0.5\) as 1.00029111 and 1.22475886 respectively whereas the present computed values are 1.067727442 and 1.284711702 respectively.

| Table 1. Local Nusselt number for various values of \(Pr\) when \(n = \xi = Nr = Nb = Nt = 0\). | Table 2. Local Nusselt number for various values of \(Nb\) and \(Nt\) for \(Pr = Le = 10\) when \(n = \xi = Nr = 0\). |
|---|---|---|---|---|---|
| 0.0001 | 0.003740181 | - | - | 0.1 | 0.977406861 | 2.191652072 | 0.9524 | 2.1293 |
| 0.001 | 0.009820938 | - | - | 0.2 | 0.500699120 | 2.684563469 | 0.5057 | 2.3818 |
| 0.07 | 0.06974396 | 0.0663 | - | 0.3 | 0.253888331 | 2.727608866 | 0.2524 | 2.4100 |
| 0.2 | 0.11452259 | 0.1691 | 0.1691 | 0.4 | 0.163034746 | 2.740152595 | 0.1199 | 2.3996 |
| 0.7 | 0.449085177 | 0.4539 | 0.4539 | 0.5 | 0.91943878 | 0.9113 | 0.9114 | 0.1 | 0.010234914 | 2.749951585 | 0.0550 | 2.3835 |
| 2 | 1.259986067 | 1.8954 | 1.8954 | 0.2 | 0.797704861 | 2.191652072 | 0.9524 | 2.1293 |
| 7 | 3.38772736 | 3.3539 | 3.3540 | 0.1 | 0.611192596 | 2.23257122 | 0.6932 | 2.2739 |
| 20 | 6.500527527 | 6.4621 | - | 0.1 | 0.546697708 | 2.337538842 | 0.5201 | 2.5284 |
| 100 | 7.804404847 | - | - | 0.1 | 0.431699137 | 2.67119287 | 0.4026 | 2.7951 |
| 1000 | 25.05246371 | - | - | 0.1 | 0.351810422 | 3.115183857 | 0.3211 | 3.0350 |
In table 1, the heat transfer values of regular fluid for various values of Pr, when \( n = \xi = Nr = Nb = Nt = 0 \) are obtained by the present variational technique. It is noted that when \( Nb = Nt = 0 \) (regular fluid), the concentration equation is of no physical significance. From this table, it is evident that the present results are in good agreement with Khan and Pop [6] and Wubshet and Shanker [7].

Table 2 displays the heat and mass transfer values of nanofluid for various values of Brownian motion parameter \( Nb \) and thermophoresis parameter \( Nt \) with \( Pr = Le = 10 \) when \( n = \xi = Nr = 0 \). From this table, the values of local Nusselt number and Sherwood number obtained from the Governing Principle of Dissipative Processes are quite in agreement with numerical results of Wubshet and Shanker [7]. It is noted that the local Nusselt number decreases while the Sherwood number increases as both Brownian motion and thermophoresis parameter increases.

Figures 1-5 present the temperature distributions for the wall temperature parameter \( n \), Prandtl number \( Pr \), Radiation parameter \( Nr \), diffusion parameters \( (Nb, Nt) \) and magnetic parameter \( \xi \) respectively.

From figure 1, it is observed that the temperature profile decreases and thinning the thermal boundary layer thickness as wall temperature parameter \( n \) increases. Figure 2 represents the temperature profile for different values of \( Pr \). Since the higher Prandtl number has lower thermal conductivity, temperature profiles are steeper due to the rapid heat transfer. Figure 3 shows the effects of radiation parameter \( Nr \) on the temperature profile. It is found that an increase in the radiation parameter \( Nr \) increases the temperature profile but not in significant level.

Figure 4 illustrates, the influence of Brownian, thermophoresis diffusion parameters \( Nb, Nt \) (taken as \( Nb = Nt \)) on the temperature distribution. One can observe that temperature profile is an increasing function of these diffusion parameters.
parameters. This is because of the fact that these parameters are directly influenced by the thermal enhancing fields. From figure 5, it can be easily observed that as magnetic parameter $\xi$ increases accordingly, the temperature profile increases. Figures 6-8 represent the concentration profile for Lewis number $Le$, Brownian motion parameter $Nb$ and thermophoresis parameter $Nt$ respectively.

Figure 6 indicates that as $Le$ increases the concentration within the boundary layer decreases and also the concentration boundary layer thickness decreases. From figures 7 and 8, it is observed that Brownian motion decreases the concentration profile while thermophoresis parameter increases the concentration profile.

6. Conclusion

By GPDP, governing partial differential equations are simplified as polynomial equations whose coefficients are of independent parameters $Pr$, $\xi$, $n$, $Nr$, $Nb$, $Nt$ and $Le$. This variational technique offers a practicing engineer a rapid way of obtaining heat and mass transfer rates for any combination of these parameters. The advantage involved in this technique is that the results are obtained with the high order of accuracy and the time taken to solve the problem is certainly less when compared with more conventional methods. Hence the practicing engineers and scientists can apply this unique approximate technique as a powerful tool for solving boundary layer flow, heat and mass transfer problems.
References

[3] M. Swati, Slip effects on MHD boundary layer flow over an exponentially stretching sheet with suction/blowing and thermal radiation, Ain