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The use of Chebyshev cardinal functions for the solution of a partial differential equation with an unknown time-dependent coefficient subject to an extra measurement

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ABSTRACT

A numerical technique is presented for the solution of a parabolic partial differential equation with a time-dependent coefficient subject to an extra measurement. The method is derived by expanding the required approximate solution as the elements of Chebyshev cardinal functions. Using the operational matrix of derivative, the problem can be reduced to a set of algebraic equations. From the computational point of view, the solution obtained by this method is in excellent agreement with those obtained by previous works and also it is efficient to use.

1. Introduction

The parameter determination in a parabolic partial differential equation from the overspecified data plays an important role in applied mathematics, physics and engineering. These problems are widely encountered in the modelling of physical phenomena [1–3].

In this paper we shall consider an inverse problem of finding an unknown parameter $a(t)$ in a parabolic partial differential equation.

The classical example is that one needs to find the temperature distribution $u(x, t)$ as well as the thermal coefficient $a(t)$ simultaneously that satisfy

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ 0 < t \le T, \tag{1.1}$$

with an initial condition

$$u(x, 0) = f(x), \quad 0 \le x \le 1, \tag{1.2}$$

and boundary conditions

$$u(0, t) = g_0(t), \quad 0 \le t \le T, \tag{1.3}$$

$$u(1, t) = g_1(t), \quad 0 \le t \le T, \tag{1.4}$$

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and subject to an extra measurement

$$u(x^*, t) = E(t), \quad 0 \le t \le T,\ x^* \in (0, 1), \tag{1.5}$$

where f, g_0, g_1, and E are known functions, while the functions $u(x, t)$ and $a(t)$ are unknown. It is worth pointing that, the problem (1.1)–(1.4) is under-determined and we are forced to impose an additional boundary condition, such that a unique solution pair (u, a) is obtained. Employing the condition (1.5), a recovery of the function $a(t)$ together with the solution u can be made possible.

The problem (1.1)–(1.5) represents a large class of parabolic inverse problems in which an unknown function $a(t)$ is to be determined as well as the solution itself. Here and through this paper "parabolic inverse problem" means that an unknown coefficient that is assumed to be a function of only the time variable and the solution of a parabolic partial differential equation subject to suitable initial-boundary conditions are to be determined [1–3].

1.1. A brief review of other methods existing in the literature

The numerical solution of the problem (1.1)–(1.5) is discussed by several authors. The problem of determining a conductivity $a(t)$ in Eq. (1.1) subject to the time-dependent boundary condition is investigated in [4–6]. The main idea behind their approach is to reduce the problem to an integral equation for the coefficient $a(t)$. It is worth pointing out that this idea does not easily extend to the problems with n space variables. Also, authors of [7] proved the determination of a time-dependent conductivity is possible for an arbitrary domain in R^n in a well-posed manner. Several local and global existence results for this problem are given in [8]. Cannon and Yin [9] studied the numerical solution of (1.1)–(1.5) and developed finite element techniques. In [2] several explicit and implicit finite difference procedures have been developed to find the numerical solution of the problem (1.1)–(1.5). Authors of [10] presented the backward Euler finite difference formula [11] for solving this problem. Their method is stable in the maximum norm and results in an error of $O\left((\Delta x)^2 + (\Delta t)\right)$. Authors of [12] used the pseudospectral Legendre method to solve this problem. Also, a method is proposed in [13] to solve this problem which is based on a semi-analytical approach. An unconditionally stable efficient fourth-order numerical algorithm based on the functional transformation, the Pade approximation and the Richardson extrapolation is proposed in [14] to compute the main function and the unknown coefficient in (1.1). We refer the interested reader to [15–21] for more research works on inverse problems.

1.2. A brief discussion on the existence and uniqueness

The existence and uniqueness of the solutions to this problem are discussed in [7,8]. It is shown [8] that the following conditions lead to [10] an existence and uniqueness theorem (also note that the solution $u \in C^{4,2}(Q_T)$, where $Q_T = \{(x, t) : 0 < x < 1, 0 < t < T\}$):

(i) $f(x) \in C^{4+\alpha}[0, 1], f_{xx}(x) > 0, f_{xxxx}(x^*) = \frac{\delta_0}{2} > 0,$ and $f_{xxxx}(x) > 0,$ on $[0, 1]$.

(ii) $g_0(t), g_1(t)$ and $E(t) \in C^{l+\frac{\alpha}{2}}[0, T], E'(t) > 0,$ on $[0, T], 0 < \frac{g_0(t)}{E'(t)} < 1, 0 < \frac{g_1(t)}{E'(t)} < 1, \left(\frac{g_1(t)}{E'(t)}\right)' > 0,$ on $[0, T]$.

Remark. There is a fundamental difference between the direct and inverse problems. It is known that an inverse problem is not well posed in general, while the direct problem is well posed. Thus an important task is to formulate the problem properly and to find the conditions that ensure its well posedness. If the solution of the given problem exists and is unique but it does not depend continuously on the data, then in general the computed solution has nothing to do with the true solution. The ill-posedness may be a main difficulty for the inverse problems. Since it is hard to avoid some errors in the observation $E(t)$ which is obtained from experiments, a small perturbation in $E(t)$ may result in a big change in $a(t)$ which may make the obtained results meaningless [22]. In the current investigation we will not discuss on this issue. However, we refer the interested reader to [23,24].

1.3. A brief introduction to application

Certain types of physical problems can be modelled by (1.1)–(1.5). The problem (1.1)–(1.5) can be used to determine the unknown properties in a region by measuring only data on the boundary [1,3]. The coefficient $a(t)$ can represent some physical quantities [2]. Here, we briefly mention the conductivity of a medium.

As is said in [10] one application is in the determination of the unknown properties in a region by measuring only data on the boundary, and particular attention has been focused to coefficients that present physical meaning quantities. For example, the conductivity of a medium. The techniques used depend strongly on the type of equations and variables on which the unknown coefficient is assumed a priori to depend. An interesting case is when the unknown conductivity depends on the dependent variable of the solution u. When we study the heat flow problem, this has the physical interpretation of a temperature dependent on conductivity. Note that if the spatial change in the function $u(x, t)$ is small in comparison with the change in time, then a reasonable approximation to this state of affairs may be to consider the coefficient to be a function only of the time variable [10,2]. It can be seen that in the current paper we study an approximation method to an inverse problem of finding the function $u(x, t)$ and the unknown positive coefficient $a(t)$ in a parabolic initial-boundary value problem.

It is worth to note that some other types of inverse problems are studied in [25–27,22,28,29]. When an unknown coefficient appears in the lower order terms, some results have been obtained. We refer the interested reader to [1,3,30–32] and the references therein. Some applications are described in [3]. Also, very recently authors of [33,34] investigated some new types of inverse parabolic problems. Some research works on the theoretical solution of the inverse parabolic problems can be found in [35–41]. Also the references [42–45] contain recent investigation on the numerical solution of one-dimensional parabolic problems with non-classic boundary specifications.

The outline of this paper is as follows. In Section 2, we describe Chebyshev cardinal functions and its properties and construct its operational matrix of derivative. In Section 3 the presented technique is used to approximate the solution of problem (1.1)–(1.5). As a result a set of algebraic equations is formed and the solution of the considered problem is introduced. Some numerical illustrations are given in Section 4 to show the efficiency of the proposed method. Finally, a brief conclusion is drawn in Section 5.

2. Chebyshev cardinal functions

Chebyshev cardinal functions of order N in $[-1, 1]$ are defined as [46]

$$C_j(z) = \frac{T_{N+1}(z)}{T_{N+1,z}(z_j)(z - z_j)}, \quad j = 1, 2, \ldots, N + 1, \tag{2.6}$$

where $T_{N+1}(z)$ is the second kind Chebyshev function of order $N + 1$ in $[-1, 1]$, defined by

$$T_{N+1}(z) = \cos((N + 1) \arccos(z)),$$

the subscript z denotes z-differentiation and $z_j, j = 1, 2, \ldots, N + 1$ are the zeros of $T_{N+1}(z)$ defined by $\cos((2j - 1)/(2N + 2)), j = 1, 2, \ldots, N + 1$.

We change the variable $x = (z + 1)L/2$ to use these functions on $[0, L]$. Now any function $g(x)$ on $[0, L]$ can be approximated as

$$g(x) \approx \sum_{j=1}^{N+1} g(x_j)C_j(x) = G^T \Phi_N(x), \tag{2.7}$$

where $x_j, j = 1, 2, \ldots, N + 1$ are the shifted points of $z_j, j = 1, 2, \ldots, N + 1$ by using the transformation $x = (z + 1)L/2$,

$$G = [g(x_1), g(x_2), \ldots, g(x_{N+1})]^T, \tag{2.8}$$

and

$$\Phi_N(x) = [C_1(x), C_2(x), \ldots, C_{N+1}(x)]^T. \tag{2.9}$$

2.1. The operational matrix of derivative

The differentiation of vectors Φ_N in (2.9) can be expressed as

$$\Phi_N' = D_N \Phi_N, \tag{2.10}$$

where D_N is $(N + 1) \times (N + 1)$ operational matrix of derivative for Chebyshev cardinal functions. The matrix D_N can be obtained by the following process. Let

$$\Phi_N'(x) = [C_1'(x), C_2'(x), \ldots, C_{N+1}'(x)]^T.$$

Using Eq. (2.7), any function $C_j'(x)$ can be approximated as

$$C_j'(x) = \sum_{k=1}^{N+1} C_j'(x_k)C_k(x). \tag{2.11}$$

Comparing Eqs. (2.10) and (2.11), we obtain

$$D_N = \begin{bmatrix} C_1'(x_1) & \ldots & C_1'(x_{N+1}) \\ \vdots & & \vdots \\ C_{N+1}'(x_1) & \ldots & C_{N+1}'(x_{N+1}) \end{bmatrix}. \tag{2.12}$$

To calculate the entries $C_j'(x_k), j, k = 1, 2, \ldots, N+1$, we have

$$\frac{T_{N+1}(x)}{x - x_j} = \alpha \times \prod_{\substack{k=1 \\ k \neq j}}^{N+1} (x - x_k), \tag{2.13}$$

where $\alpha = 2^{2N+1}/L^{N+1}$ is the coefficient of x^{N+1} in the shifted Chebyshev polynomial function $T_{N+1}(x)$. Using Eq. (2.13), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T_{N+1}(x)}{x - x_j}\right) = \alpha \times \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \prod_{\substack{k=1 \\ k \neq i,j}}^{N+1} (x - x_k) = \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{T_{N+1}(x)}{(x - x_j)(x - x_i)}, \tag{2.14}$$

so we have

$$C_j'(x) = \frac{1}{T_{N+1,x}(x_j)} \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{T_{N+1}(x)}{x - x_j}\right) = \frac{1}{T_{N+1,x}(x_j)} \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{T_{N+1}(x)}{(x - x_j)(x - x_i)} = C_j(x) \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{1}{x - x_i}. \tag{2.15}$$

For $k = j$ using Eq. (2.15), we obtain

$$C_j'(x_j) = \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{1}{x_j - x_i}. \tag{2.16}$$

For $k \neq j$ using Eq. (2.15) we have

$$C_j'(x_k) = \frac{\alpha}{T_{N+1,x}(x_j)} \prod_{\substack{\ell=1 \\ \ell \neq k,j}}^{N+1} (x_k - x_\ell). \tag{2.17}$$

So the entries of the matrix D_N can be found using Eqs. (2.16) and (2.17).

3. Description of the new computational technique

To use the Chebyshev cardinal functions for solving the problem (1.1)–(1.5), at first we use the following transformation.

3.1. The employed transformation

Using (1.1)–(1.4) and (1.5), we have

$$E'(t) = a(t) u_{xx}(x^*, t). \tag{3.18}$$

We assume that $u_{xx}(x^*, t) > 0$; hence

$$a(t) = \frac{E'(t)}{u_{xx}(x^*, t)}, \quad 0 \leq t \leq T. \tag{3.19}$$

Thus the inverse problem (1.1)–(1.5) is equivalent [2] to the following problem:

$$u_t = \frac{E'(t)}{u_{xx}(x^*, t)} u_{xx}, \quad 0 \leq x \leq 1,\ 0 < t \leq T, \tag{3.20}$$

$$\begin{aligned} &u(x, 0) = f(x), \quad 0 \leq x \leq 1, \\ &u(0, t) = g_0(t), \quad 0 \leq t \leq T, \\ &u(1, t) = g_1(t), \quad 0 \leq t \leq T. \end{aligned} \tag{3.21}$$

Our main idea is based on utilization of the following transformation: Employing [1–3]

$$v(x, t) = u_{xx}(x, t), \tag{3.22}$$

we will have

$$v_t(x, t) = \frac{E'(t)}{v(x^*, t)} v_{xx}(x, t), \quad 0 \leq x \leq 1,\ 0 < t \leq T, \tag{3.23}$$

$$v(x, 0) = f''(x), \quad 0 \leq x \leq 1, \tag{3.24}$$

$$v(0, t) = \frac{g_0'(t)}{E'(t)} v(x^*, t), \quad 0 \leq t \leq T, \tag{3.25}$$

$$v(1, t) = \frac{g_1'(t)}{E'(t)} v(x^*, t), \quad 0 \leq t \leq T. \tag{3.26}$$

Note that

$$\begin{aligned} v(0,t) &= u_{xx}(0,t) = \frac{u_t(0,t)}{a(t)} = u_t(0,t)\frac{u_{xx}(x^*,t)}{E'(t)} \\ &= \frac{u_t(0,t)}{E'(t)}v(x^*,t) = \frac{g_0'(t)}{E'(t)}v(x^*,t). \end{aligned} \tag{3.27}$$

3.2. The computational framework

Suppose $\Phi_N(x)$ is the vector of Chebyshev cardinal functions on [0,1] defined in (2.9) and $\Phi_M(t)$ is the vector of cardinal functions on [0,T]. Now the unknown function $v(x,t)$ in (3.23) can be approximated as

$$v(x,t) \simeq \sum_{i=1}^{N+1}\sum_{j=1}^{M+1} V_{i,j}C_i(x)C_j(t) = \Phi_N^T(x)V\Phi_M(t), \tag{3.28}$$

where the unknown matrix V is $(N+1)\times(M+1)$ and can be shown as

$$V = \begin{bmatrix} V_{1,1} & \dots & V_{1,M+1} \\ \vdots & & \vdots \\ V_{N+1,1} & \dots & V_{N+1,M+1} \end{bmatrix}.$$

Employing Eq. (2.10), we can write

$$\begin{aligned} v_t(x,t) &= \frac{\partial}{\partial t}\Phi_N^T(x)V\Phi_M(t) = \Phi_N^T(x)V\Phi_M'(t) \\ &= \Phi_N^T(x)VD_M\Phi_M(t). \end{aligned} \tag{3.29}$$

Also, we have

$$\begin{aligned} v_{xx}(x,t) &= \frac{\partial^2}{\partial x^2}\Phi_N^T(x)V\Phi_M(t) = \Phi''^T_N(x)V\Phi_M(t) \\ &= \Phi_N^T(x)(D_N^2)^TV\Phi_M(t), \end{aligned} \tag{3.30}$$

and

$$v(x^*,t) = \Phi_N^T(x^*)V\Phi_M(t). \tag{3.31}$$

Using Eqs. (3.29)–(3.31) in Eq. (3.23), we obtain

$$\Phi_N^T(x)VD_M\Phi_M(t) = \frac{E'(t)}{\Phi_N^T(x^*)V\Phi_M(t)}\Phi_N^T(x)(D_N^2)^TV\Phi_M(t). \tag{3.32}$$

By collocating Eq. (3.32) in $(N-1)\times M$ points (x_i,t_j), $i=2,\dots,N$, $j=2,\dots,M+1$, where x_i, $i=1,\dots,N+1$, are the shifted points of z_j on $[0,1]$ and t_j, $j=1,\dots,M+1$, are the shifted points of z_j on $[0,T]$, we obtain

$$R(x_i,t_j) = \Phi_N^T(x_i)VD_M\Phi_M(t_j) - \frac{E'(t_j)}{\Phi_N^T(x^*)V\Phi_M(t_j)}\Phi_N^T(x_i)(D_N^2)^TV\Phi_M(t_j) = 0, \quad i=2,\dots,N,\ j=2,\dots,M+1. \tag{3.33}$$

The property of cardinal functions yields

$$\Phi_N(x_i) = \mathrm{e}_i^N, i=1,\dots,N+1 \quad \text{and} \quad \Phi_M(t_j) = \mathrm{e}_j^M, j=1,\dots,M+1, \tag{3.34}$$

where e_j^ℓ is the jth column of $(\ell+1)\times(\ell+1)$ identity matrix I. Using (3.34) in (3.33), we have

$$\begin{aligned} R(x_i,t_j) &= (\mathrm{e}_i^N)^TVD_M\mathrm{e}_j^M - \frac{E'(t_j)}{\Phi_N^T(x^*)V\mathrm{e}_j^M}(\mathrm{e}_i^N)^T(D_N^2)^TV\mathrm{e}_j^M \\ &= (VD_M)_{i,j} - \frac{E'(t_j)}{[\Phi_N^T(x^*)V]_j}((D_N^2)^TV)_{i,j} = 0, \quad i=2,\dots,N, j=2,\dots,M+1, \end{aligned} \tag{3.35}$$

where $(A)_{i,j}$ and $[W]_j$ show (i,j)th entry of matrix A and the jth entry of vector W, respectively. Using Eq. (3.28) in Eqs. (3.24)–(3.26) yields

$$\begin{aligned} &\Phi_N^T(x)V\Phi_M(0) = f''(x), \\ &\Phi_N^T(0)V\Phi_M(t) = g_0'(t)/E'(t)\Phi_N^T(x^*)V\Phi_M(t), \\ &\Phi_N^T(1)V\Phi_M(t) = g_1'(t)/E'(t)\Phi_N^T(x^*)V\Phi_M(t). \end{aligned} \tag{3.36}$$

Collocating Eq. (3.36) in $N+1$ points $x_i, i = 1, \ldots, N+1$ and M points $t_j, j = 1, \ldots, M$ and using the property of (3.34), we have

$$\begin{aligned}
&[V\Phi_M(0)]_i = f''(x_i), && i = 1, 2, \ldots N+1,\\
&[\Phi_N^T(0)V]_j = g_0'(t_j)/E'(t_j)[\Phi_N^T(x^*)V]_j, && j = 1, \ldots, M,\\
&[\Phi_N^T(1)V]_j = g_1'(t_j)/E'(t_j)[\Phi_N^T(x^*)V]_j, && j = 1, \ldots, M.
\end{aligned} \tag{3.37}$$

Eq. (3.35) together with Eq. (3.37) give a $(M+1) \times (N+1)$ system of nonlinear equations, which can be solved for $V_{i,j}, i = 1, 2, \ldots, N+1, j = 1, 2, \ldots, M+1$, using Newton's iterative method. So the unknown function of $v(x, t)$ can be found.

In order to recover u from v, we need to solve the following boundary value problem:

$$u_{xx}(x, t) = v(x, t), \quad 0 < x < 1, \tag{3.38}$$

$$u(0, t) = g_0(t), \quad 0 < t \le T, \tag{3.39}$$

$$u(1, t) = g_1(t), \quad 0 < t \le T. \tag{3.40}$$

By integration both sides of (3.38) from 0 to x, we obtain

$$u_x(x, t) - u_x(0, t) = \int_0^x v(x_1, t)\mathrm{d}x_1. \tag{3.41}$$

Again by integration both sides of (3.38) from 0 to x, we can write

$$u(x, t) - u(0, t) - u_x(0, t)x = \int_0^x \int_0^{x_2} v(x_1, t)\mathrm{d}x_1\mathrm{d}x_2. \tag{3.42}$$

Putting $x = 1$ in (3.42) and using (3.39), we have

$$u_x(0, t) = u(1, t) - g_0(t) - \int_0^1 \int_0^{x_2} v(x_1, t)\mathrm{d}x_1\mathrm{d}x_2. \tag{3.43}$$

Putting (3.43) in (3.42) and using (3.39) and (3.42), we obtain

$$u(x, t) = g_0(t) + \left(g_1(t) - g_0(t) - \int_0^1 \int_0^{x_2} v(x_1, t)\mathrm{d}x_1\mathrm{d}x_2\right)x + \int_0^x \int_0^{x_2} v(x_1, t)\mathrm{d}x_1\mathrm{d}x_2. \tag{3.44}$$

All functions on the right-hand side of (3.44) are known, so the unknown function u can be found.

4. Numerical tests

To give a clear overview of the procedure, the following test problems will be investigated.

Example 1. Consider (1.1)–(1.5) with [2]

$$\begin{aligned}
f(x) &= \exp\left(\frac{x}{2}\right),\\
g_0(t) &= \frac{1+2t^3}{1+t^3} + \sin\left(\frac{t}{2}\right),\\
g_1(t) &= \frac{\sqrt{\exp(1)}(1+2t^3)}{1+t^3} + \sqrt{\exp(1)}\ \sin\left(\frac{t}{2}\right),\\
E(t) &= \frac{1.13315(1+2t^3)}{1+t^3} + 1.13315\ \sin\left(\frac{t}{2}\right),
\end{aligned}$$

with $x^* = 0.25$, for which the exact solution is

$$u(x, t) = \frac{\exp\left(\frac{x}{2}\right)(1+2t^3)}{1+t^3} + \exp\left(\frac{x}{2}\right)\ \sin\left(\frac{t}{2}\right),$$

and

$$a(t) = \frac{2\left[6t^2 + (1+t^3)^2\cos\left(\frac{t}{2}\right)\right]}{(1+t)^3\left[1+2t^3+(1+t^3)\ \sin\left(\frac{t}{2}\right)\right]}.$$

Table 1
Absolute values of error for u from Example 1 with $T = 1.0$.

x	Method [2]	$N = 6, M = 5$	$N = 4, M = 9$	$N = 8, M = 8$	$N = 9, M = 9$
0.1	1.9×10^{-3}	5.9×10^{-5}	1.8×10^{-5}	1.6×10^{-6}	1.0×10^{-7}
0.2	1.9×10^{-3}	1.1×10^{-4}	3.2×10^{-5}	2.9×10^{-6}	1.9×10^{-7}
0.3	1.5×10^{-3}	1.4×10^{-4}	4.3×10^{-5}	3.9×10^{-6}	2.6×10^{-7}
0.4	1.8×10^{-3}	1.6×10^{-4}	5.0×10^{-5}	4.5×10^{-6}	3.0×10^{-7}
0.5	1.4×10^{-3}	1.7×10^{-4}	5.3×10^{-5}	4.8×10^{-6}	3.2×10^{-7}
0.6	1.2×10^{-3}	1.6×10^{-4}	5.1×10^{-5}	4.7×10^{-6}	3.1×10^{-7}
0.7	1.6×10^{-3}	1.4×10^{-4}	4.6×10^{-5}	4.2×10^{-6}	2.7×10^{-7}
0.8	1.8×10^{-3}	1.1×10^{-4}	3.5×10^{-5}	3.2×10^{-6}	2.1×10^{-7}
0.9	1.5×10^{-3}	6.1×10^{-5}	2.0×10^{-5}	1.8×10^{-6}	1.2×10^{-7}
1.0	1.7×10^{-3}	0.0	0.0	0.0	0.0

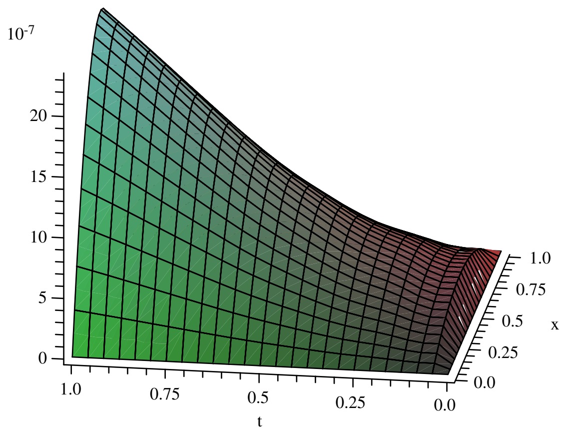

Fig. 1. Plot of error for u with $N = 7, M = 11$.

Note that

$$v(x, t) = \frac{\exp\left(\frac{x}{2}\right)(1 + 2t^3)}{4(1 + t^3)} + \frac{\exp\left(\frac{x}{2}\right)\sin\left(\frac{t}{2}\right)}{4}.$$

Table 1 shows the absolute values of Error for $T = 1$ and different values of M and N, using the method proposed in Section 3 and compares the result with the result obtained using the technique of [2]. Fig. 1 shows the plot of error for u with $N = 7, M = 11$ on the interval $x \in [0, 1]$ and $t \in [0, 1]$.

Example 2. As the second example, consider (1.1)–(1.5) with [2]

$$f(x) = 2\exp(x),$$

$$g_0(t) = 1 + \frac{1 + 2t^3}{1 + t^3},$$

$$g_1(t) = \exp(1) + \frac{\exp(1)(1 + 2t^3)}{1 + t^3},$$

$$E(t) = 1.28403 + \frac{1.28403(1 + 2t^3)}{1 + t^3},$$

with $x^* = 0.25$, for which the exact solution is

$$u(x, t) = \exp(x) + \frac{\exp(x)(1 + 2t^3)}{1 + t^3},$$

and

$$a(t) = \frac{3t^2}{2 + 5t^3 + 3t^6}.$$

Table 2 shows the absolute values of Error for $T = 1$ and different values of M and N, using the method proposed in Section 3 and compares the result with the result obtained using the scheme introduced in [2]. Fig. 2. shows the plot of error $a(t)$ for $N = 9, M = 9$ on the interval $t \in [0, 1]$.

Table 2
Absolute values of error for u from Example 2 with $T = 1.0$.

x	Method [2]	$N = 5, M = 5$	$N = 7, M = 7$	$N = 6, M = 11$
0.1	5.1×10^{-3}	4.8×10^{-4}	4.6×10^{-5}	2.8×10^{-6}
0.2	4.9×10^{-3}	8.8×10^{-4}	8.5×10^{-5}	4.9×10^{-6}
0.3	4.9×10^{-3}	1.2×10^{-3}	1.2×10^{-4}	6.5×10^{-6}
0.4	5.1×10^{-3}	1.4×10^{-3}	1.4×10^{-4}	7.6×10^{-6}
0.5	4.9×10^{-3}	1.5×10^{-3}	1.5×10^{-4}	8.1×10^{-6}
0.6	5.0×10^{-3}	1.5×10^{-3}	1.5×10^{-4}	8.1×10^{-6}
0.7	5.0×10^{-3}	1.4×10^{-3}	1.3×10^{-4}	7.4×10^{-6}
0.8	5.1×10^{-3}	1.1×10^{-3}	1.0×10^{-4}	5.9×10^{-6}
0.9	5.3×10^{-3}	6.2×10^{-4}	6.1×10^{-5}	3.5×10^{-6}
1.0	5.3×10^{-3}	0.0	0.0	0.0

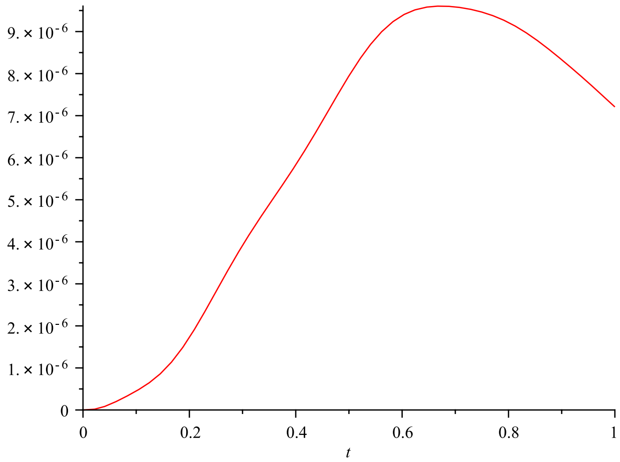

Fig. 2. Plot of error for $a(t)$ with $N = 9, M = 9$.

Table 3
Absolute values of error for u from Example 3 with $T = 1.0$.

x	$N = 4, M = 4$	$N = 5, M = 5$	$N = 7, M = 9$	$N = 8, M = 10$
0.1	3.1×10^{-3}	4.7×10^{-4}	3.7×10^{-7}	3.0×10^{-8}
0.2	5.7×10^{-3}	8.5×10^{-4}	6.8×10^{-7}	5.5×10^{-8}
0.3	7.7×10^{-3}	1.2×10^{-3}	9.2×10^{-7}	7.4×10^{-8}
0.4	9.0×10^{-3}	1.4×10^{-3}	1.1×10^{-6}	8.7×10^{-8}
0.5	9.7×10^{-3}	1.5×10^{-3}	1.2×10^{-6}	9.3×10^{-8}
0.6	9.6×10^{-3}	1.5×10^{-3}	1.1×10^{-6}	9.2×10^{-8}
0.7	8.7×10^{-3}	1.3×10^{-3}	1.0×10^{-6}	8.3×10^{-8}
0.8	6.8×10^{-3}	1.0×10^{-3}	8.2×10^{-7}	6.6×10^{-8}
0.9	4.0×10^{-3}	6.0×10^{-4}	4.8×10^{-7}	3.8×10^{-8}
1.0	0.0	0.0	0.0	0.0

Example 3. As another test problem, we use (1.1)–(1.5) with

$$f(x) = 2\exp(x),$$
$$g_0(t) = \exp(t^2),$$
$$g_1(t) = \exp(1 + t^2),$$
$$E(t) = \exp(t^2 + 0.25),$$

with $x^* = 0.25$ for which the exact solution is

$$u(x, t) = \exp(x + t^2), \quad \text{and} \quad a(t) = 2t.$$

Table 3 shows the absolute values of Error for $T = 1$ and different values of M and N, using the method presented in Section 3.

Example 4. As the last example, we consider (1.1)–(1.5) with

$$f(x) = \cos(\pi(x - 1/2)),$$

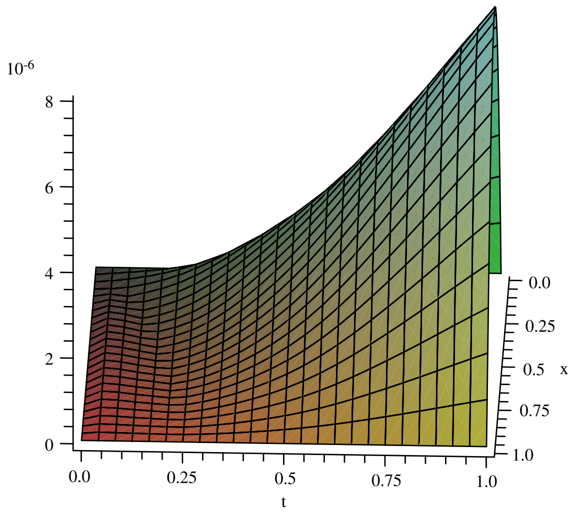

Fig. 3. Plot of error for u with $N = 9$, $M = 9$.

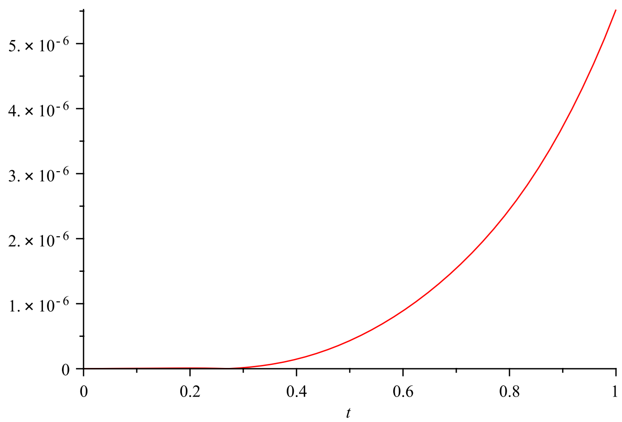

Fig. 4. Plot of error for $a(t)$ with $N = 9$, $M = 9$.

$$g_0(t) = 0,$$
$$g_1(t) = 0,$$
$$E(t) = \cos(0.2\pi)\ \exp(-t^2),$$

with $x^* = 0.3$, which has the following exact solution:

$$u(x, t) = \exp(-t^2)\ \cos(\pi(x - 1/2)), \quad \text{and} \quad a(t) = 2t/\pi^2.$$

Figs. 3 and 4 show the plot of error $u(x, t)$ and $a(t)$ respectively for $N = 9$, $M = 9$ on the interval $x \in [0, 1]$ and $t \in [0, 1]$.

5. Conclusion

The inverse problems not only have intrinsic mathematical interests but also have a variety of applications in industry and engineering sciences [1–3]. This paper focused on the inverse parabolic problem of determination of the leading coefficient in the heat equation with an extra condition. In this paper we presented a numerical scheme for solving a parabolic partial differential equation with a time-dependent coefficient subject to an extra measurement. The shifted Chebyshev cardinal functions on interval [0, 1] and [0, T] were employed. The new algorithm proposed in the current paper was tested on several examples from the literature. The obtained results showed that this approach can solve the problem effectively and needs few computations.

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