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Note

On the number of minimal partitions of a box into boxes

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Abstract

The number of minimal partitions of a box into proper boxes is examined.

Keywords: Minimal partition; Box; Orbit; Stabilizer

1. Introduction

A non-empty subset A of the cartesian product $X := X_1 \times \cdots \times X_d$ of finite sets X_i, $i \in [d]$, is called a *box* if $A = A_1 \times \cdots \times A_d$ and $A_i \subseteq X_i$ for each $i \in [d]$. We say that A is *proper* if $A_i \neq X_i$ for each $i \in [d]$. In what follows, we shall always assume that X contains a proper box, which is equivalent to saying that $|X_i| \geqslant 2$ for each i.

Kearnes and Kiss [3] asked whether each partition of X into proper boxes has to contain at least 2^d boxes. This question has been answered affirmatively by Alon et al. [1]. Let us call any partition which consists of exactly 2^d proper boxes *minimal*. It would be of some interest to determine the number of the minimal partitions of X. At the present stage of investigations this problem seems to be out of reach, as we know little on the structure of the minimal partitions. Certain efforts made in order to understand this structure are reported in [2]. The objective of this note is to show the following

Theorem 1. *Let $X = X_1 \times \cdots \times X_d$ and let $|X_i| \geqslant 3$ for each $i \in [d]$. Then the number $\mu(X)$ of minimal partitions of X is congruent to 1 modulo 4, whenever $d \geqslant 2$.*

2. Proof of the theorem

For a proper box $A \subset X$ and $\varepsilon \in \mathbb{Z}_2^d$ let us define a new box $A^\varepsilon := A_1^{\varepsilon_1} \times \cdots \times A_d^{\varepsilon_d}$ by the formula

$$(A^\varepsilon)_i = \begin{cases} A_i & \text{if } \varepsilon_i = 0, \\ X_i \setminus A_i & \text{if } \varepsilon_i = 1. \end{cases}$$

As is easily seen, the family $\mathscr{F} := \{A^\varepsilon : \varepsilon \in \mathbb{Z}_2^d\}$ is a minimal partition of X. Such a partition is called *simple*.

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Now, let us suppose that $\mathscr{F}$ is a minimal partition of X. For $\varepsilon \in \mathbb{Z}_2^d$ we define

$$\mathscr{F}^\varepsilon :=\{A^\varepsilon : A \in \mathscr{F}\}.$$

$\mathscr{F}^\varepsilon$ is again a minimal partition [2, Lemma 3]. It is clear that for every ε and $\gamma \in \mathbb{Z}_2^d$ we have

$$(\mathscr{F}^\varepsilon)^\gamma = \mathscr{F}^{\varepsilon+\gamma},$$

which means that the group $G := \mathbb{Z}_2^d$ acts on the family $\mathscr{X}$ of all minimal partitions of X. In particular, each orbit $\mathrm{O}(\mathscr{F})$ has length equal to a certain power of 2.

Let

$$\mu_k(X) :=|\{\mathscr{F} \in \mathscr{X} : |\mathrm{O}(\mathscr{F})| = 2^k\}|.$$

(Further we abbreviate $\mu_k(X)$ to μ_k.) It should be clear that in order to show our theorem it suffices to calculate two numbers: μ_0, μ_1.

Proposition 2. (i) $|\mathrm{O}(\mathscr{F})| = 1$ *if and only if* $\mathscr{F}$ *is a simple partition.*
(ii) $\mu_0 = (2^{|X_1|-1} - 1) \cdots (2^{|X_d|-1} - 1)$.

Proof. (i) The part 'if' follows immediately from the definition of simple partitions. Suppose now that $A \in \mathscr{F}$. Then the simple partition $\mathscr{G} := \{A^\varepsilon : \varepsilon \in G\}$ is contained in $\mathscr{F}$. Since $\mathscr{F}$ is a partition, they have to coincide.

(ii) Fix $x \in X$. For each simple partition $\mathscr{F}$, let us pick $F \in \mathscr{F}$ so that $x \in F$. The mapping $\mathscr{F} \mapsto F$ defines a $1-1$ correspondence between the simple partitions and the proper boxes containing x. Clearly, the number of such boxes equals $(2^{|X_1|-1} - 1) \cdots (2^{|X_d|-1} - 1)$. □

Remark 3. The above proposition shows that

$$\mu_0(X) \equiv 1 \pmod 2.$$

Now, we shall calculate μ_1. Since saying that the orbit of a minimal partition $\mathscr{F}$ is of length 2 is equivalent to saying that the stabilizer $G(\mathscr{F})$ of $\mathscr{F}$ has index 2, it makes sense to begin with the description of all subgroups of index 2 in G.

Lemma 4. *Let H be a subgroup of G. Then $|G : H| = 2$ if and only if there is a non-empty set $I \subseteq [d]$ such that*

$$H = H(I) := \left\{\varepsilon \in G : \sum_{i\in I} \varepsilon_i = 0\right\}.$$

Lemma 5. *If $H(I)$ is the stabilizer of a minimal partition of X, then I is a proper subset of* $[d]$.

Proof. If $I = [d]$, then H consists of all elements $\varepsilon \in G$ such that $\varepsilon_1 + \cdots + \varepsilon_d = 0$. Suppose that for some minimal partition $\mathscr{F}$ we have $H = G(\mathscr{F})$. Fix any $A \in \mathscr{F}$ and define $\mathscr{G} :=\{A^\varepsilon : \varepsilon \in H\}$. Clearly, $\mathscr{G} \subseteq \mathscr{F}$ and $|\mathscr{G}| = 2^{d-1}$. Let $B \in \mathscr{F}\backslash\mathscr{G}$. First, we show that there is $\alpha \in G\backslash H$ such that $B \subseteq A^\alpha$. If it were not so, then there would exist two elements $\beta, \gamma \in G\backslash H$ such that $B \cap A^\beta \neq \emptyset$ and $B \cap A^\gamma \neq \emptyset$. Since they are different, there is i for which $\beta_i \neq \gamma_i$. Let $\delta \in G$ be defined by the equations $\delta_j = \beta_j$ for $j \neq i$ and $\delta_i = \gamma_i$. We have $\delta \in H$. Moreover, $B \cap A^\delta \neq \emptyset$. Hence, by the fact that $\mathscr{F}$ is a partition we get $B = A^\delta$, which is a contradiction. Now, since $\mathscr{F}$ and $\mathscr{E} :=\{A^\varepsilon : \varepsilon \in G\}$ are both partitions and each element of $\mathscr{F}$ is contained in an element of $\mathscr{E}$, we deduce that $\mathscr{F} = \mathscr{E}$. Thus, $\mathscr{F}$ is a simple partition and $G(\mathscr{F}) = G$, which is again a contradiction. □

To formulate our next lemma, we need some additional notation.

Let $\mathscr{G}$ be a simple partition of X. Boxes A and B that belong to $\mathscr{G}$ are said to be of the *same parity* if there exists $\varepsilon \in H([d])$ such that $A^\varepsilon = B$. It is clear that $\mathscr{G}$ splits up into two disjoint subfamilies $\mathscr{G}^-$ and $\mathscr{G}^+$ of equal cardinality that consist of boxes of the same parity.

If A is a box in X and $I := \{i_1 < \cdots < i_k\}$ is a subset of $[d]$, then we define $A_I := A_{i_1} \times \cdots \times A_{i_k}$. (If $I = \emptyset$, then we let $A_\emptyset = \{\emptyset\}$). Let I' be the complement of I in $[d]$. To simplify our notation, we shall always identify $A_I \times A_{I'}$ with A in an obvious manner.

Lemma 6. *If I is a non-empty and proper subset of $[d]$ and $|X_i| \geqslant 3$ for each $i \in [d]$, then $H(I)$ is the stabilizer of a minimal partition. Moreover, if $\mathscr{F}$ is a minimal partition of X, then $G(\mathscr{F}) = H(I)$ if and only if there is a simple partition $\mathscr{G}$ of X_I and two different simple partitions $\mathscr{K}_1$ and $\mathscr{K}_2$ of $X_{I'}$ such that each box A belonging to $\mathscr{F}$ satisfies one of the following two relations*:

$$A = C \times D \quad \text{where } C \in \mathscr{G}^- \text{and } D \in \mathscr{K}_1,$$
$$A = C \times D \quad \text{where } C \in \mathscr{G}^+ \text{and } D \in \mathscr{K}_2.$$

Each triple $\mathscr{G}$, $\mathscr{K}_1$, $\mathscr{K}_2$ determines exactly two partitions whose stabilizers are equal to $H(I)$, and to each such a partition corresponds exactly one such a triple.

Proof. Observe first that if $\mathscr{G}$ is any simple partition of X_I and $\mathscr{K}_1, \mathscr{K}_2$ are two different simple partitions of $X_{I'}$ (they always exist as I' is non-empty and $|X_i| \geqslant 3$ for every $i \in I'$), then $\mathscr{F}$, which is built from these partitions as described above, has its stabilizer equal to $H(I)$.

Suppose now that we have an $\mathscr{F}$ such that $G(\mathscr{F}) = H(I)$. By the definition of $H(I)$, it can be decomposed into the direct sum $H_1 \oplus \mathbb{Z}_2^{I'}$, where H_1 consists of even elements of $\mathbb{Z}_2^I$, that is, $\varepsilon \in H_1$ if and only if $\varepsilon \in \mathbb{Z}_2^I$ and $\sum_{i \in I} \varepsilon_i = 0$. For each $A \in \mathscr{F}$ let us consider its orbit $\mathrm{O}'(A)$ with respect to the action of $\mathbb{Z}_2^{I'}$. Since $\mathscr{F}$ is a partition, the family $\{\bigcup \mathrm{O}'(A) : A \in \mathscr{F}\}$ is also a partition of X. Clearly, $\bigcup \mathrm{O}'(A) = A_I \times X_{I'}$. Thus, $\{A_I \times X_{I'} : A \in \mathscr{F}\}$ is a partition of X. Consequently, $\mathscr{G} := \{A_I : A \in \mathscr{F}\}$ is a partition of X_I. Obviously, H_1 is contained in the stabilizer of $\mathscr{G}$. It follows from Lemma 5 that $G(\mathscr{G}) = \mathbb{Z}_2^I$. Thus, $\mathscr{G}$ is a simple partition of X_I. Let us decompose $\mathscr{G}$ into the classes of parity $\mathscr{G}^-$ and $\mathscr{G}^+$. Let us pick $B \in \mathscr{F}$ so that $B_I \in \mathscr{G}^-$. Let us define

$$\mathscr{K}_1 := \{E_{I'} : E \in \mathscr{F}, \quad E_I = B_I\}.$$

Let $\varepsilon \in H(I)$ be odd on I, that is, $\sum_{i \in I} \varepsilon_i = 1$. Define

$$\mathscr{K}_2 := \{E_{I'} : E \in \mathscr{F}, \ E_I = B_I^\varepsilon\}.$$

It is easily seen that $\mathscr{G}$, $\mathscr{K}_1$ and $\mathscr{K}_2$ as defined above give us the desired decomposition of $\mathscr{F}$. The proof of the remaining part of our lemma is rather obvious. □

Lemma 7. *If X satisfies the assumptions of the preceding lemma and $d \geqslant 2$, then*

$$\mu_1 = \mu_0 \sum_{\emptyset \neq I \neq [d]} (\mu_0(X_I) - 1).$$

Proof. If $\emptyset \neq I \neq [d]$, then by Lemma 6 the number μ_I of minimal partitions of X whose stabilizers are equal to $H(I)$ is twice the number of simple partitions of X_I times the number of two-element subsets of the family of all simple partitions of $X_{I'}$. Since $\mu_0(X_I)\mu_0(X_{I'}) = \mu_0$, we obtain $\mu_I = \mu_0 \cdot (\mu_0(X_{I'}) - 1)$. Summing up the last equation with respect to I gives us the expected formula for μ_1. □

Lemma 8. *Let α_i, $i \in [d]$, be a sequence of reals. Then*

$$\sum_{\emptyset \neq I \neq [d]} \prod_{i \in I} (\alpha_i - 1) = -1 - (-1)^d + \sum_{\emptyset \neq J \neq [d]} (-1)^{d - |J| - 1} \prod_{j \in J} \alpha_j.$$

For a non-empty set $I \subseteq [d]$, let $\lambda_1(X_I) := \sum_{i \in I} |X_i|$. Moreover, let $\lambda_1(X_\emptyset) := 0$.

Lemma 9.

$$\sum_{\emptyset \neq I \neq [d]} \mu_0(X_I) = -1 + \sum_{J \neq [d]} (-1)^{d - |J| - 1} 2^{\lambda_1(X_J) - |J|}.$$

Proof. It suffices to put $\alpha_i = 2^{|X_i|-1}$ in the previous lemma. $\square$

Combining this lemma with Lemma 7 we obtain

Proposition 10.

$$\mu_1 = \mu_0 \left(-2^d + 1 + \sum_{J \neq [d]} (-1)^{d-|J|-1} 2^{\lambda_1(X_J)-|J|} \right).$$

Proof of the theorem. It follows from Proposition 2 that μ_0 is congruent to $(-1)^d$ modulo 4. By this fact and the preceding proposition μ_1 is congruent to $(-1)^d - 1$ modulo 4. Hence

$$\mu_0 + \mu_1 \equiv 2(-1)^d + 3 \pmod 4. \qquad \square$$

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