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Formulation structure of the mass-flux convection parameterization

Jun-Ichi Yano *

CNRM, Météo-France and CNRS, 31057 Toulouse Cedex, France

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ABSTRACT

Structure of the mass-flux convection parameterization formulation is re-examined. Many of the equations associated with this formulation are derived in systematic manner with various intermediate steps explicitly presented. The nonhydrostatic anelastic model (NAM) is taken as a starting point of all the derivations.

Segmentally constant approximation (SCA) is a basic geometrical constraint imposed on a full system (e.g., NAM) as a first step for deriving the mass-flux formulation. The standard mass-flux convection parameterization, as originally formulated by Ooyama, Fraedrich, Arakawa and Schubert, is re-derived under the two additional hypotheses concerning entrainment–detrainment and environment, and an asymptotic limit of vanishing areas occupied by convection.

A model derived at each step of the deduction constitutes a stand-alone subgrid-scale representation by itself, leading to a hierarchy of subgrid-scale schemes. A backward tracing of this deduction process provides paths for generalizing mass-flux convection parameterization. Issues of the high-resolution limit for parameterization are also understood as those of relaxing various traditional constraints. The generalization presented herein can include various other subgrid-scale processes under a mass-flux framework.

* Correspondence to: GAME/CNRM, Météo-France, 42 av Coriolis, 31057 Toulouse Cedex, France. Tel.: +33 5 6107 9359.
E-mail address: jiy.gfder@gmail.com

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1. Introduction

The seminal works by Ooyama (1971), followed by Fraedrich (1973, 1974), Arakawa and Schubert (1974) lay foundations for the formulation of mass-flux convection parameterization (cf., Emanuel and Raymond, 1993). This formulation is currently adopted in the majority of atmospheric circulation models both global and regional, and those both for operational forecasts and climate projections (e.g., Tiedtke, 1989; Gregory and Rowntree, 1990; Emanuel, 1991; Moorthi and Suarez, 1992; Donner, 1993; Zhang and McFarlane, 1995; Bechtold et al., 2001). Thus, the importance of their original work is hardly overemphasized (cf., McFarlane, 2011).

The present paper calls the original, common formulation developed by Ooyama, Fraedrich, and Arakawa and Schubert the standard formulation, because all the subsequently-developed mass-flux convection parameterizations closely follow their original formulation. The goal here is to expose the structure of mass-flux convection parameterization formulation in lucid manner. The present paper does not intend any systematic review of existing mass-flux convection parameterizations.

There are several reasons for such an exposition. Most importantly, a systematic derivation of the mass-flux convection parameterization is missing in the literature. The original papers sited above only provide outlines of their derivations with many equations presented without derivations. Sketches for such systematic derivations found, for example, in Siebesma (1998), Yano et al. (2005a), are expanded by the present paper. Many of the equations presented herein are originally derived.

A lucid presentation of the mass-flux convection parameterization formulation also much facilitates its generalization. The original mass-flux parameterization was formulated by assuming that the atmospheric convective system consists solely of an ensemble of convective updrafts (cf., Yano, 2009). However, the observed atmospheric convective system is more complex: presence of downdrafts as well as an organization into mesoscale, intrinsic interactions of convective dynamics with boundary-layer processes (e.g., cold pools), cloud physics, radiative transfer processes, *etc*. Modifications of convection parameterization for incorporating these elements have rather been slow (cf., Randall et al., 2003; Arakawa, 2004). Its formulation structure must first be elucidated for this purpose.

For example, although the majority of current operational mass-flux convection parameterizations includes convective downdrafts in one way or another (e.g., Fritsch and Chappell, 1980; Tiedtke, 1989; Zhang and McFarlane, 1995; Bechtold et al., 2001), they are implemented in a rather *ad hoc* manner (Kerry Emanuel, personal communication, 1992: cf., Section 8.3.5) below). The operational versions of convection parameterizations are slow in including more convection-related processes. Presently, only the Donner (1993) scheme includes mesoscale downdraft as a part of a deep convection parameterization. In order to make these generalizations easier, needed first is a lucid exposition of the formulation structure.

More urgently, with a rapid increase of the model resolutions, especially for the regional forecasts, there is a need for relaxing the basic constraints of the standard mass-flux convection parameterization in order to adopt it in more general contexts (Yano et al., 2010a). Under the high-resolution limit, the scale of deep moist convection is no longer considered distinctively smaller than the grid size, but it begins to be resolved. As a result, the traditional assumption of scale separation is no longer applicable. Various exploratory attempts already exist towards the high-resolution limit (cf., Gerard and Geleyn, 2005; Gerard, 2007; Kuell et al., 2007; Gerard et al., 2009), however, without general consensus on a systematic procedure. The basic structure of the problem must be exposed in order to place it into a wider perspective.

With these needs in mind, this paper presents the structure of the mass-flux convection parameterization formulation in mathematically lucid and general manner. This paper also suggests how to develop a parameterization with fewer constraints by generalizing the standard mass-flux parameterization.

As an important basic perspective, the paper regards the subgrid-scale parameterization as that of a systematic reduction from a full physical system, such as a cloud-resolving model (CRM) and a large-eddy simulation (LES). Yano et al. (2005a) propose the mode decomposition as such a general procedure. Under this perspective, the mass-flux convection parameterization is a special case based on a segmentally-constant mode decomposition: subdividing the grid-box domain into subdomains with constant values. This procedure at each vertical level is coined the segmentally-constant approximation (SCA) by Yano et al. (2010b). Thus, SCA is a chief geometrical constraint to be imposed onto a full physical system (e.g., CRM, LES) as a first step for deriving the standard mass flux parameterization formulation.

The nonhydrostatic anelastic model (NAM), as adopted for CRM and LES, is taken as a starting point of the analysis. A basic formulation of NAM is reviewed in the next section. The subgrid-scale parameterization problem is formally stated in Section 3. SCA is imposed on NAM in Section 4. The obtained system is coined NAM–SCA by Yano et al. (2010b). The standard mass-flux convection parameterization formulation is reconstructed by stepwise deductions from NAM–SCA in subsequent sections: introduction of entrainment–detrainment hypothesis (Section 5), the environment hypothesis (Section 6), and a limit of vanishing fractional area occupied by convection (Section 7). These steps are taken one by one so that the evolution of the formulation can explicitly be traced.

The present paper emphasizes that CRM/LES and parameterization are not the two dichotomic choices, but we can generate various intermediate models in between. This paper also objects to a traditional dichotomy between convection and environment, and emphasizes a possibility of considering various intermediate components, especially for the mesoscale.

Throughout the paper, frequent references are made to Arakawa and Schubert (1974, AS hereinafter) in order to point to an original source to the equations. We also refer to Ooyama (1971 O71), and Yanai et al. (1973, YEC) for fundamentals of AS and for observational applications, respectively. Reference to original sources is cited by abbreviated notations, e.g., (cf., Eq. AS.33), pointing to Eq. 33 of AS.

2. Full system: basic set of equations

By following a standard formulation for CRMs and LESs, nonhydrostatic anelastic model (NAM) is taken as a full physical system. This particular choice does not pose any fundamental limitation on the following analysis, because the repetition of the analysis under a different model is straightforward with only minor modifications. For example, Yano (2012a) applies the same procedure to the primitive equation system.

2.1. Basic set of equations

The NAM system is charactrized by a reference density profile $\rho = \rho(z)$ as a function of height z only. The Cartesian coordinates (x, y, z) are adopted here with (x, y) the two horizontal directions, and t designates the time. A basic set of prognostic equations consists of the momentum equations and those for the other prognostic variables, which include the temperature, the moisture, the condensed water, as well as various chemical species. The condensed water may further be categorized into various species, such as cloud and rain water.

We express the momentum equations in terms of the horizontal wind components, $\mathbf{u}$, and the vertical velocity, w:

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathbf{u}\mathbf{u} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho w \mathbf{u} = -\frac{1}{\rho}\nabla_H p', \tag{2.1a}$$

$$\frac{\partial}{\partial t}w + \nabla \cdot \mathbf{u}w + \frac{1}{\rho}\frac{\partial}{\partial z}\rho w^2 = -\frac{1}{\rho}\frac{\partial}{\partial z}p' + b. \tag{2.1b}$$

Here, p' is the *aerodynamic* pressure defined as a deviation from the hydrostatic pressure, b is the buoyancy. Buoyancy may be defined in various different manners depending on the approximations taken: see e.g., Appendix (b) of Yano et al. (2005b).

For considering the other prognostic variables, we designate a general prognostic variable by φ and an associated source term by F:

$$\frac{\partial}{\partial t}\varphi + \nabla \cdot \mathbf{u}\varphi + \frac{1}{\rho}\frac{\partial}{\partial z}\rho w \varphi = F. \tag{2.1c}$$

Note that the above prognostic equations are given in flux form for a later convenience. In the momentum equation, ∇ operates on all the variables to the right with $\mathbf{uu}$ representing a tensor. It may also be noted that the two momentum Eqs. (2.1a) and (2.1b) also reduce to a general prognostic Eq. (2.1c) once the source term F is defined accordingly. For this reason, the following discussions are focused on the generic prognostic Eq. (2.1c) except for discussing specific issues concerning the momentum equation.

Mass continuity under the anelastic approximation is:

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho w = 0. \tag{2.2}$$

2.2. Pressure equation

The above set of Eqs. (2.1) and (2.2) is technically closed once appropriate thermodynamic variables required for defining the buoyancy are included as a part of the set (2.1c). However, as a practical problem for solving this set of system, there is no obvious way for evaluating the aerodynamic pressure p' directly.

Physically speaking, the aerodynamic pressure p' must be defined in such a way that the mass continuity (2.2) is kept satisfied at the next time step by integrating the momentum equation defined by Eqs. (2.1a) and (2.1b). A required constraint (e.g., Bernardet, 1995) is obtained by applying the divergence on the momentum equation multiplied by density ρ, which is constrained by mass continuity (2.1c). As a result, we obtain a Poisson equation for the aerodynamic pressure p':

$$\nabla^2 p' = -\nabla \cdot \nabla \cdot \rho \mathbf{vv} + \frac{\partial}{\partial z}\rho b, \tag{2.3}$$

where $\mathbf{v} \equiv \mathbf{u} + \hat{\mathbf{z}} w$ is a three-dimensional velocity.

3. Parameterization

3.1. Statement of the problem

A need for parameterization of the subgrid-scale physical processes arises because an atmospheric model has only a limited horizontal resolution. The easiest way of seeing an issue is to take an interpretation that a single grid-point value within a model represents a grid-box mean value, say, $\overline{\varphi}$. By taking an average over a grid box, the generic prognostic equation (2.1c) reduces to

$$\frac{\partial}{\partial t}\overline{\varphi} + \overline{\nabla} \cdot \overline{\mathbf{u}\varphi} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho\overline{w\varphi} = \mathcal{Q}. \tag{3.1}$$

Here, all the quantities with the overbar indicate the model-resolved variables defined as an average over the grid box.

The right hand side $\mathcal{Q}$ is called apparent source by Yanai et al. (1973: cf., Eqs. YEC. 8 and 9), which is defined in terms of the subgrid-scale variables as

$$\mathcal{Q} = -\overline{\nabla \cdot \mathbf{u}'\varphi'} - \frac{1}{\rho}\overline{\frac{\partial \rho w'\varphi'}{\partial z}} + \overline{F}. \tag{3.2}$$

Here, the prime $'$ designates a deviation from the grid-box mean (e.g., $\varphi' = \varphi - \overline{\varphi}$). This apparent source term, $\mathcal{Q}$, is not possible to evaluate directly in terms of the model-resolved variables, thus it requires a *parametric* representation. The goal of parameterization is to define a closed expression for the apparent source $\mathcal{Q}$.

3.2. Super-parameterization

A conceptual starting point for constructing a parameterization would be to try to explicitly simulate the evolution of the whole grid-box domain by a NAM (or CRM, LES). Then the apparent

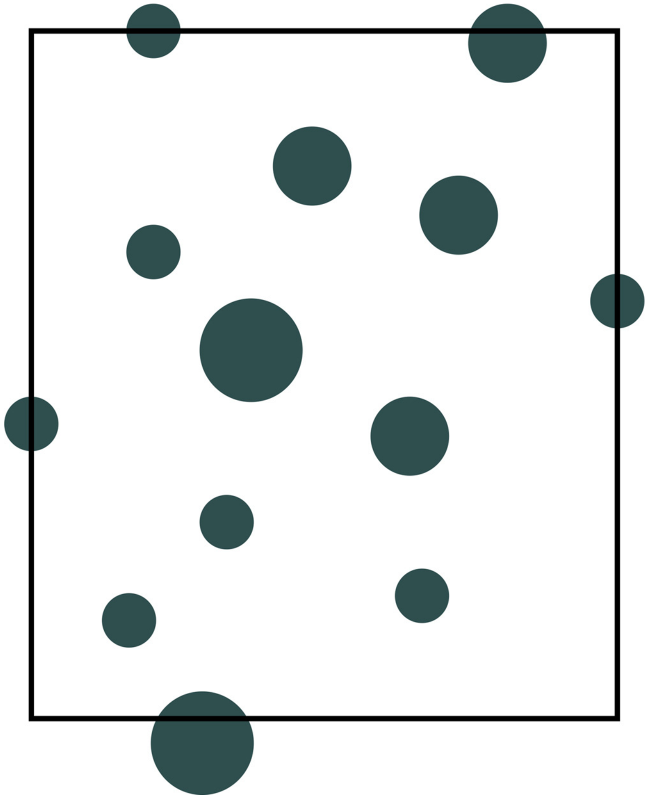

Fig. 1. A conceptual representation of Riehl and Malkus' hot-tower hypothesis from a top view. Here, the hot towers are represented by gray circles with a gird box boundary suggested by think lines. Note that some hot towers may cross the grid-box boundaries as indicated.

source term, $\mathcal{Q}$, is readily available from a NAM run. This is the basic idea of the so-called super-parameterization (cf., Grabowski and Smolarkiewicz, 1999; Randall et al., 2003). However, such a methodology would be extremely prohibitive, especially when a full grid-box domain is adopted for a NAM run. In standard super-parameterization implementations, much smaller domain sizes often with a two-dimensional configuration are adopted.

In the following, we take NAM (i.e., super-parameterization) as a starting point for re-constructing the standard mass-flux parameterization formulation in stepwise manner.

4. Segmentally-constant approximation (SCA)

4.1. The basic concept

The original conceptualization for the mass-flux convection parameterization may be traced back to Riehl and Malkus (1958: see also Yano, 2009). They propose that an ascending branch of the Hadley–Walker circulation consists of two distinctive sub-components: an ensemble of well-localized fast-ascending elements called "hot towers" or deep moist convective towers and a slowly descending background environment (Fig. 1). These "hot towers" can be interpreted as a prototype for the convective plumes, as studied extensively in laboratory during the same period (e.g., Morton et al., 1956; Scorer, 1957; Turner, 1962: see also Turner, 1969, 1986 as a review).

Ooyama (1971), Fraedrich (1973, 1974), and Arakawa and Schubert (1974) construct their mass-flux based convection parameterization with such a schematic picture in mind (see Fig. 1 of Arakawa and

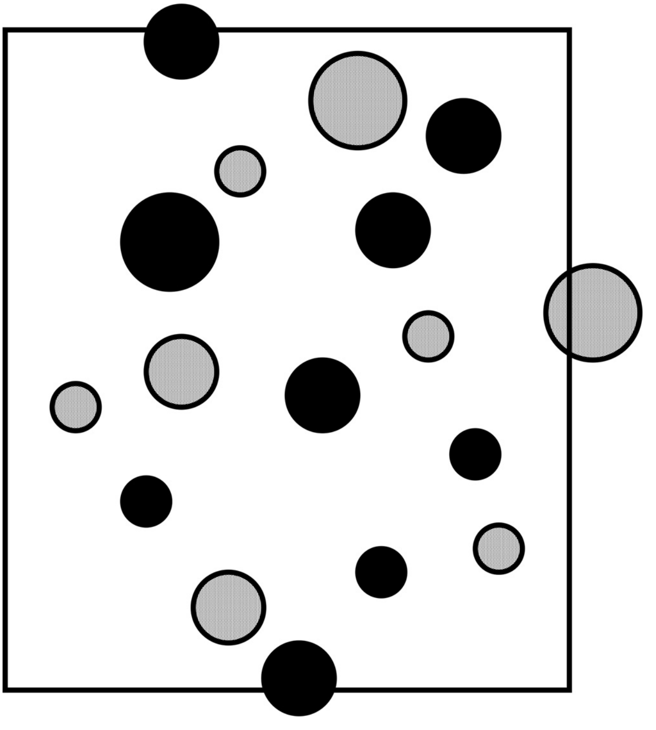

Fig. 2. A generalization of Riehl and Malkus' hot-tower hypothesis in Fig. 1 into two convective-scale components: dark circles and gray circles representing updrafts and downdrafts, respectively. This corresponds to a special case of segmentally-constant approximation (SCA) over subgrid-scale processes.

Schubert, 1974). Note that though Ooyama (1971) called his convective elements bubbles (thermals), his formulation can easily be re-interpreted in terms of convective plumes (see also Ooyama, 1972).

In standard formulation, the physical variables associated with each plume element are considered functions of height only. This treatment is some times called "top hat" approximation as originally coined by Morton et al. (1956). This approximation consists of assuming a *horizontal homogeneity* within a plume as originally assumed by Stommel (1947) in his observational analysis. In this paper, we call this approximation SCA in order to make its geometrical implication more explicit.

Though the original conceptualization of the hot-tower hypothesis (Fig. 1) consists only of the two components, hot towers and the environment, the observed convective systems contain other components, especially the downdrafts both in convective and meso-scales (Zipser, 1969, 1977; Houze and Betts, 1981). The schematic may be generalized to include the downdrafts as shown in Fig. 2, in which outside of those circles still constitute a homogeneous environment. Here, being consistent with a standard picture in mass-flux parameterizations, the downdrafts are also assumed to be surrounded by the environment (cf., Section 6).

However, the real convective elements are not always surrounded by the environment, but rather they directly interact with each other, as schematically shown in Fig. 3 as a side view. Clearly such a picture is a leap from the standard mass-flux formulation. In this section, we are going to show how such a generalization is possible by purely introducing a geometrical constraint of SCA into the NAM system.

The obtained formulation may be considered a fully-prognostic prototype of the mass-flux parameterization. Though this approximation is not explicitly stated in those original papers by Ooyama (1971), Fraedrich (1973, 1984), and Arakawa and Schubert (1974), their formulation is understood most lucidly with SCA.

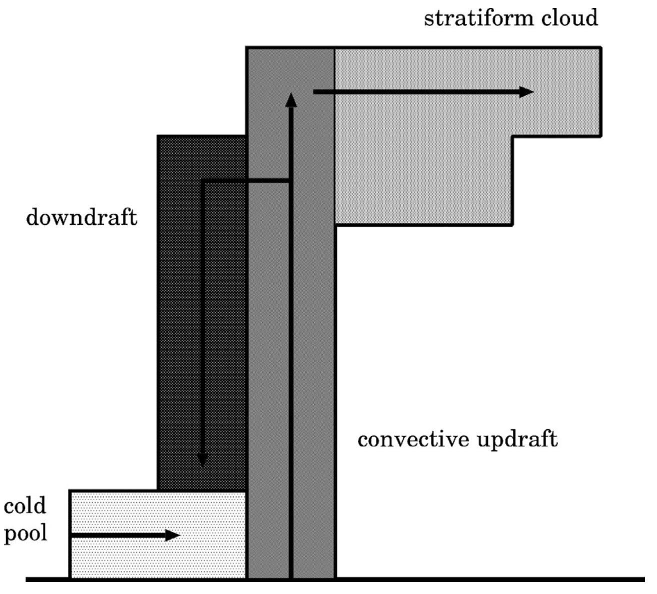

Fig. 3. A side view for a further generalization of SCA. Unlike the case of Fig. 2, the subgrid-scale components are no longer exclusively interacting with the environment, but with various other components: convective updraft, downdraft, cold pool, stratiform cloud.

4.2. SCA–NAM: NAM–SCA

SCA approximates a full system by a set of constant segments designated by areas S_j and corresponding boundaries ∂S_j with the indices $j = 1, 2, \ldots n$ at each vertical level. These areas may be enclosed (as updrafts and downdrafts) or open (as for the environment), but the whole grid-box domain is subdivided into those segments so that the sum of the areas for all segments recover the total grid-box area, $i.e.$, $S = \sum_{j=1}^{n} S_j$. The basic concept is already schematically depicted by Figs. 1–3. Alternatively, each segment may be considered in analogous manner as cloud types as shown by Fig. 2 of Yano et al. (2005a), and Fig. 1 of Yano (2012a).

All the physical variables under SCA may be defined by

$$\varphi = \sum_{j=1}^{n} \mathcal{I}_j(x, y, z)\varphi_j, \tag{4.1}$$

where $\mathcal{I}_j(x, y, z)$ is an indicator for the jth segment, which is defined by

$$\mathcal{I}_j(x, y, z) = \begin{cases} 1, & \text{if}\,(x, y, z) \in S_j, \\ 0, & \text{if}\,(x, y, z) \notin S_j. \end{cases} \tag{4.2}$$

In vertical, each segment can extend for any finite extent, as schematically suggested in Fig. 3: vertical extents of plumes may differ; a single SCA plume may ramify into several, and several SCA plumes may merge into a single one with height. At the top and the bottom ends of each segment, the segmentally-constant area may exchange physical quantities vertically, directly with segmentally-constant areas with different sizes located just above and below. The details of these vertical-transport treatments are discussed in Yano et al. (2010b).

4.3. Prognostic equation

The prognostic equation for each constant segment under SCA is obtained by averaging the full system (2.1c) over a segment S_j. An average of a variable φ over the jth segment is designated by a subscript j, and defined by

$$\varphi_j = \frac{1}{S_j}\int_{S_j} \varphi dxdy, \tag{4.3}$$

where $\int_{S_j} dxdy$ is a horizontal integral over the jth segment.

In deriving the following equations, we frequently invoke the Leibniz's theorem, which states that for any function $f(x, y, t)$ depending on the two spatial coordinates x, y, and the time t, a time derivative of its integral over a spatial area defined by S is divided into two parts:

$$\frac{d}{dt}\int_S f(x, y, t)dxdy = \int_S \frac{\partial f(x, y, t)}{\partial t}dxdy + \oint_{\partial S} f(x, y, t)\dot{\mathbf{r}}_b \cdot d\mathbf{r} \tag{4.4}$$

where the second term on the right hand side refers to a line integral over the boundary ∂S of the area S, and $\mathbf{r}_b$ is a position vector of the boundary. With a help of Eq. (4.4), we obtain

$$\frac{\partial}{\partial t}\sigma_j\varphi_j + \sigma_j\overline{(\nabla \cdot \mathbf{u}\varphi)}_j^* + \frac{1}{\rho}\frac{\partial}{\partial z}\rho\sigma_j(w\varphi)_j = \sigma_j F_j. \tag{4.5}$$

Here, $\sigma_j = S_j/S$ is the fractional area occupied by the jth segment. The second term on the left-hand side represents an effective horizontal divergence, including those due to the displacement $\dot{\mathbf{r}}_b$ of the segment boundary ∂S_j:

$$\sigma_j\overline{(\nabla \cdot \mathbf{u}\varphi)}_j^* \equiv \frac{1}{S}\oint_{\partial S_j} \varphi(\mathbf{u}^* - \dot{\mathbf{r}}_b)\cdot d\mathbf{r}. \tag{4.6}$$

Here, an *effective* horizontal velocity $\mathbf{u}^*$ that takes into account of an inclination $\partial\mathbf{r}_{j,b}/\partial z$ of the boundary is defined by

$$\mathbf{u}^* = \mathbf{u} - w\frac{\partial \mathbf{r}_{j,b}}{\partial z}. \tag{4.7}$$

4.4. Mass continuity

Mass continuity under SCA is obtained by directly averaging the mass continuity Eq. (2.2) over a segment S_j. By applying Leibniz's theorem (4.4) again, we obtain

$$\frac{1}{S}\oint_{\partial S_j} \mathbf{u}^* \cdot d\mathbf{r} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho\sigma_j w_j = 0. \tag{4.8}$$

Alternatively, the mass continuity is obtained by setting $\varphi_j = 1$, $F_j = 0$ in Eq. (4.5):

$$\frac{\partial}{\partial t}\sigma_j + \frac{1}{S}\oint_{\partial S_j} (\mathbf{u}^* - \dot{\mathrm{r}}_b)\cdot d\mathbf{r} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho\sigma_j w_j = 0. \tag{4.9}$$

Furthermore, a difference between Eqs. (4.8) and (4.9) leads to

$$\frac{\partial}{\partial t}\sigma_j = \frac{1}{S}\oint_{\partial S_j} \dot{\mathbf{r}}_b \cdot d\mathbf{r}. \tag{4.10}$$

Eq. (4.10) states how a fractional area, σ_j, occupied by a subgrid-scale component changes with time by a shift, $\dot{\mathbf{r}}_b$, of the component segment boundary.

4.5. Vertical flux: standard mass-flux approximation

A key ingredient of SCA is to introduce an approximation

$$(w\varphi)_j = w_j\varphi_j \tag{4.11}$$

to the vertical flux terms. The approximation (4.12) may be considered a core of mass-flux formulation (cf., Yano et al., 2004).

The vertical flux term can be, more generally, given by

$$(w\varphi)_j = w_j\varphi_j + (w_j''\varphi_j'')_j \tag{4.12}$$

with the last term representing a contribution of a deviation from the segmentally constant approximation (SCA). Here, the deviations are defined by $w_j'' = w - w_j$ and $\varphi_j'' = \varphi - \varphi_j$.

These deviations from SCA can easily be taken into account by following the procedures of the moment expansion in the boundary-layer turbulence (cf., Mellor and Yamada, 1974). A contribution of such effect in the environment component is heuristically taken into account by e.g., Soares et al. (2004). On the other hand, a simple scale analysis would show that the fluctuations of sub-plume scale is as important as the environmental-scale fluctuations, with further analyses left for future study.

4.6. Short summary

Eqs. (4.5), (4.8), (4.9), (4.10), along with the Poisson problem (2.3), constitute a basic set of equations describing the evolution of NAM–SCA. Under SCA, the problem of evaluating continuous physical fields is replaced by that of evaluating segmentally constant values φ_j.

A full formulation for NAM–SCA is presented for a two-dimensional case by Yano et al. (2010b). A full description of a three-dimensional SCA system under the primitive approximation is presented by Yano (2012a). Leaving the full details to the above two references, the next two subsections discuss a few issues to be sorted out in order to close the system.

4.7. Winds and pressure

In the NAM–SCA system, information on the positions of the subcomponent-segment boundaries, ∂S_j, is explicitly required for solving the wind (horizontal velocity) and pressure fields.

SCA may be applied to the vertical velocity as in two-dimensional implementation by Yano et al. (2010b). Then, the divergent-wind field is evaluated from the mass continuity (2.2) with the vertical divergence term given under SCA. It leads to a segmentally linear distribution of divergent winds. The rotational winds can be evaluated by applying SCA to the vorticity. The resulting rotational winds are also segmentally linear. In evaluating the divergent and the rotational winds from the given divergence and vorticity, the procedure of the contour dynamics (cf., Dritschel and Ambaum, 1997; Dritschel, 1989) may be adopted. Recall that a key to the contour dynamics is to perform a contour integral along the segment boundary, thus information on the positions of a segment becomes crucial.

The pressure problem is harder to handle. It must be diagnosed by inverting the Laplacian in Eq. (2.3). However, the horizontal distribution of the associated source term (right-hand side), which can be known only by specifying the segment distribution explicitly, is far from segmentally constant, so is the inverted pressure field. While Yano et al. (2010b) simply solve this Poisson problem in a standard grid space with additional numerical expenses, a numerically more efficient method would likely be developed under a finite element approach (cf., Elman et al., 2005).

4.8. Segment boundaries

The segment-boundary values for the physical variables, φ, must be specified in order to evaluate the horizontal fluxes crossing the segment boundaries. The simplest choice is to take an upstream (upwind) value, by following the earlier studies (e.g., Asai and Kasahara, 1967; Arakawa and Schubert, 1974). The choice also corresponds to the simplest stable scheme under a finite volume approach (cf., LeVeque, 2002: see more in Section 5.2).

A more serious issue is a determination of the displacement rate $\dot{\mathbf{r}}_b$ of the boundaries. There is no obvious principle to define this quantity because SCA is an approximation imposed by our choice to a full physical system: the original full system does not evolve consistently under SCA. In a point of view of the finite volume method, we expect $\dot{\mathbf{r}}_b = \mathbf{u}^*$ from the Rankine–Hugoniot condition (cf., Section 5.1.1 in Durran, 1999). Yano and Baizig (2012) consider a generalized condition $\dot{\mathbf{r}}_b = (1 - \mu)\mathbf{u}^*$ with a free parameter μ, and find that overall behavior of a single SCA-plume is rather insensitive to a choice of the parameter μ. Yano et al. (2010b) simply assume $\dot{\mathbf{r}}_b = 0$.

A further issue is when the segment boundary coincides with (or crosses) the grid-box boundary (cf., Figs. 1 and 2). In this case, certain communication between the neighboring grid boxes must be introduced in order to enable proper horizontal flux calculations. Yano et al. (2010b), and Yano and Baizig (2012) avoid this issue by assuming a periodic boundary condition. A procedure for incorporating interactions with the neighboring grid boxes is outlined in Yano (2012a), and further discussed in the next section.

4.9. *Discussions*

From a purely numerical algorithmic point of view, NAM–SCA is nothing other than a finite volume approach (cf., LeVeque, 2002). This correspondence becomes more evident when a sufficiently large number of segments are introduced, as the case in Yano et al. (2010b). However, under SCA, the number of elements may be radically reduced in an analogous manner as an image *compression* by wavelet (cf., Mallat, 1998): retaining a high local resolution only where a high variability is found, but keeping a much lower resolution where less variability is found. As a result, NAM–SCA provides a *compressed*-CRM (NAM), and introduction of NAM–SCA in place of a standard CRM (NAM) into GCM leads to *compressed* super-parameterization (cf., Yano et al., 2012).

The number and distribution of the segments are first prescribed by an initial condition in time integrating the NAM–SCA. However, as in other finite-volume approaches, time-dependent adoptive mesh-refinement may be introduced with time (Yano et al., 2010b): some segments may be removed when they become inactive, and some new segments may be added by following the evolution of disturbances.

On the other hand, when a limit to lower numbers of segments is taken, a philosophical departure from the traditional finite volume approach becomes more evident. Under the strongest truncation, only a single constant segment representing a single convective plume may be placed within a grid box (Yano and Baizig, 2012). This configuration is geometrically identical to that of the bulk mass-flux parameterization.

5. Entrainment–detrainment hypothesis

5.1. *Lateral exchange with the other subgrid-scale components*

This section considers exchange of physical variables by lateral air-mass transport between the different subgrid-scale components (segments). This process is described by the horizontal divergence term (defined by Eq. (4.6)) in Eq. (4.5). Under the standard formulation, a concept of entrainment and detrainment is introduced in order to describe this process.

We begin by categorizing the boundary, ∂S_j, for a given subgrid component segment into the two major parts (cf., Figs. 1 and 2): the part crossing the grid-box boundary, $\partial S_{G,j}$, and the part that shares its boundary with the other subgrid components, $i(\neq j)$. Here, a part of the boundary for the jth segment sharing with the ith segment may be designated as $\partial S_{j,i}$. Thus,

$$\partial S_j = \partial S_{G,j} + \sum_{i=1, i\neq j}^{n} \partial S_{j,i}. \tag{5.1}$$

As a result, the divergence term may be re-written as

$$\sigma_j\overline{(\nabla\cdot\mathbf{u}\varphi)}_j^* = \frac{1}{S}\oint_{\partial S_{G,j}} \varphi(\mathbf{u}^* - \dot{\mathbf{r}}_b)\cdot d\mathbf{r}. + \sum_{i\neq j}^{n}\frac{1}{S}\oint_{\partial S_{j,i}} \varphi(\mathbf{u}^* - \dot{\mathbf{r}}_b)\cdot d\mathbf{r}. \tag{5.2}$$

Note that not all the subgrid components may be adjacent to each other, thus $\partial S_{j,i} = 0$ for some i in practice.

The first term on the right hand side is simplified by noting that the grid-box boundary is normally perpendicular, and also fixed with time, thus $\mathbf{u}^* - \dot{\mathbf{r}}_b = \mathbf{u}$. Furthermore, it is statistically most likely that the fraction, σ_j, of the whole grid-box boundary is crossed by the jth segment, replacing the integral over the whole grid-box boundary weighted by σ_j. Finally, the resulting line integral can be re-interpreted as a divergence evaluation for a large-scale model. These considerations lead to:

$$\frac{1}{S}\oint_{\partial S_{G,j}} \varphi(\mathbf{u}^* - \dot{\mathbf{r}}_b)\cdot d\mathbf{r}. = \frac{1}{S}\oint_{\partial S_G} \sigma_j\varphi_j\mathbf{u}_j\cdot d\mathbf{r}. = \overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j. \tag{5.3}$$

Here, the bar is added to ∇ in order to make it clear that this divergence term concerns only the large scales. In derivation, we assume that each segment component j is cut by a grid-box boundary in a manner as suggested by both Figs. 1 and 2. Keep in mind that Eq. (5.3) implicitly assumes a scale separation.

5.2. Lateral boundary condition over $\partial S_{j,i}$

In order to reduce the second term on the right hand side of Eq. (5.2), we first note that the value of φ at the segment boundary, $\partial S_{j,i}$ may be given under an upstream (upwind) approximation, as already discussed in Section 4.8:

$$\varphi_{b,j} = \begin{cases} \varphi_j & \text{if} \oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r} > 0 \\ \varphi_i & \text{if} \oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r} < 0 \end{cases} \tag{5.4}$$

Note that Eq. (5.4) implicitly assumes that φ is conserved by crossing a discontinuous segment boundary. When the transported variable is not conservative, a jump value arising from a source term must be added (cf., Section 3 of Arakawa and Schubert, 1974).

The following formulation may be generalized by considering a local flow condition, $(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r}$, rather than an integral effect over the whole boundary. This generalization is straightforward except for the expressions in what follows will be slightly more involved, as shown by Yano, 2012a.

5.3. Entrainment–detrainment formulation

Under the upstream formulation just introduced, a converging segment *entrains* the physical quantities from a surrounding segment, whereas a divergent segment *detrains* its own physical quantities into a surrounding segment. The rates of entrainment and detrainment, $E_{j,i}$ and $D_{j,i}$, respectively, may be defined by

$$E_{j,i} = \begin{cases} -(\rho/S)\oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_b)\cdot d\mathbf{r}, & \oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r} < 0 \\ 0 & \oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r} > 0 \end{cases} \tag{5.5a}$$

$$D_{j,i} = \begin{cases} (\rho/S)\oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_b)\cdot d\mathbf{r}, & \oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r} > 0 \\ 0 & \oint_{\partial S_{j,i}}(\mathbf{u}^* - \dot{\mathbf{r}}_{b,j})\cdot d\mathbf{r} < 0 \end{cases} \tag{5.5b}$$

Substitution of Eqs. (5.3), (5.4), and (5.5) into Eq. (5.2) leads to a final expression:

$$\sigma_j\overline{(\nabla\cdot\mathbf{u}\varphi)}_j^* = \overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j + \frac{1}{\rho}[\sum_{i=1,i\neq j}^{n} E_{j,i}\varphi_i - D_j\varphi_j], \tag{5.6}$$

where

$$D_j = \sum_{i=1,i\neq j}^{n} D_{j,i}$$

is the total detrainment from the jth subcomponent.

Note an important rule

$$E_{j,i} = D_{i,j},$$

because $\partial S_{j,i} = -\partial S_{i,j}$, and the air detrained out from an ith segment into a jth segment, at the same time, entrains into the jth segment from the ith segment.

5.4. Entrainment–detrainment hypothesis

These entrainment and detrainment terms may be considered merely short-handed expressions for the actual line integral defined by Eq. (5.5). In order to perform this line integral, we have to know the effective horizontal velocity, $\mathbf{u}^* - \dot{\mathbf{r}}_{b,j}$, crossing the segment boundary under a given position, $\partial S_{j,i}$, of the segment boundary.

The entrainment–detrainment hypothesis, on the other hand, emphatically insists that both the entrainment and the detriment rates are their own stand-alone physical parameters prescribing lateral exchanges between the different subgrid-scale components. If such a procedure ever works, a computational burden is substantially reduced by removing a need for performing the line integral in Eq. (5.5). As a result, it is no longer necessary to specify the boundary position, ∂S_j, for each segment, either.

As we are going to see immediately below, under the entrainment–detrainment hypothesis, a quantity called the mass flux

$$M_j = \rho\sigma_j w_j \tag{5.7}$$

(Eqs. AS.2, YEC.17), is going to play a key role. This is the main reason that this parameterization formulation is coined "mass flux". This quantity characterizes the convective vertical transport of physical variables.

By substituting the definition of the entrainment and the detrainment rates (5.5), the mass continuity (4.9) reduces to

$$\frac{\partial}{\partial t}\sigma_j + \frac{1}{\rho}(D_j - E_j) + \overline{\nabla}\cdot\sigma_j\mathbf{u}_j. + \frac{1}{\rho}\frac{\partial}{\partial z}M_j = 0, \tag{5.8}$$

where

$$E_j = \sum_{i=1,i\neq j}^{n} E_{j,i} \tag{5.9}$$

is the total entrainment into the jth segment. Note that the mass continuity (5.8) defines the time-evolution of the fractional area, σ_j, under a balance of horizontal divergence (2nd and 3rd terms) and the vertical divergence (4th term).

Under the NAM–SCA formulation, the horizontal divergence is diagnosed from the vertical divergence, with the vertical-velocity equation explicitly evaluated. The basic idea of the entrainment–detrainment formulation is to reverse this logic by prescribing the horizontal divergence term by the entrainment and the detrainment rates. As a result, the vertical divergence can be, essentially, evaluated diagnostically by mass continuity, thus the mass flux can be diagnosed without time-integrating the vertical-velocity equation. Note, however, a full advantage of this approach can be taken only after all the standard approximations are introduced so that Eq. (5.8) reduces to Eq. (7.12).

5.5. Prognostic equations

By substituting Eq. (5.6), the prognostic model Eq. (4.5) reduces to

$$\frac{\partial}{\partial t}\sigma_j\varphi_j + \frac{1}{\rho}(D_j\varphi_j - \sum_{i=1,i\neq j}^{n} E_{j,i}\varphi_i) + \frac{1}{\rho}\frac{\partial}{\partial z}M_j\varphi_j + \overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j = \sigma_j F_j. \tag{5.10}$$

For a later purpose, we also write down the same equation in the advective form, which is obtained with the help of the mass continuity (5.8):

$$\sigma_j\frac{\partial}{\partial t}\varphi_j + \sum_{i=1,i\neq j}^{n}\frac{E_{j,i}}{\rho}(\varphi_j - \varphi_i) + \sigma_j w_j\frac{\partial}{\partial z}\varphi_j + \sigma_j\mathbf{u}_j\cdot\overline{\nabla}\varphi_j = \sigma_j F_j. \tag{5.11}$$

5.6. Historical notes and perspectives

It would be fair to say that the concept of entrainment and detrainment is introduced into convection parameterization more as a historical consequence rather than for any objective reason (cf., Yano, 2014). A seminal work by Morton et al. (1956) is a starting point of history (see also Scorer, 1957; Turner, 1962 as well as Turner, 1969, 1986 as a review). They have performed a water tank experiment of convective plumes, and identified that an entrainment plumes hypothesis, assumption that the fractional entrainment rate, E/M, can be assumed a constant of a problem, only depending on a radius of a plume. In retrospect, the concept of entraining plume is introduced for interpreting observed atmospheric convection by Stommel (1947, 1951, see also Simpson 1983a, b, Lilly 1983, Morton 1997 as reviews).

This entrainment plume hypothesis is adopted by Simpson et al. (1965), Simpson and Wiggert (1969) for verification of their cloud seeding experiments, and they have obtained consistent results with observation with this model (see Simpson, 1983c as a review). In turn, Arakawa and Schubert (1974) adopt the entraining plume as a basic element of atmospheric convection.

Since then, extensive efforts (e.g., Raymond and Blyth, 1986; Blyth, 1993; Kain and Fritsch, 1990; Emanuel, 1991: see Raymond, 1993 as a review) are under way to make the concept of entrainment and detrainment more elaborated in the context of atmospheric moist convection. However, it is still far from reaching a consensus (cf., de Rooy et al., 2013).

It is important to note that all these studies are performed under a framework of a steady plume (cf., Section 7.3) under the environment hypothesis. Indeed LES (large-eddy simulation) studies (e.g., Heus et al., 2008) point to a limit of the environment hypothesis and importance of taking into account an *immediate environment*, which is associated with downdrafts. Observations also report prominent downdrafts at cloud edges (e.g., Blyth et al., 2005). However, such a formulation is yet to be fully developed in operational contexts. The entrainment–detrainment hypothesis may not work well for transient plumes, as a preliminary investigation by Yano and Baizig (2012) suggests.

As already shown above, when full interactions between the subgrid-scale components are taken into account, the entrainment–detrainment formulations takes a matrix form. Though it would be straightforward to generalize the existing methodology (Siebesma and Cuijpers, 1995; Swann, 2001; Siebesma et al., 2003; de Rooy and Siebesma, 2010) for entrainment–detrainment estimates from CRM and LES under the matrix formulation, a general theoretical framework for developing such a formulation is missing.

6. Environment hypothesis: separation into convection and environment

6.1. Environment hypothesis (Weak Version)

Description of the subgrid-scale processes is quite involved, when full interactions between all possible subgrid-scale components are considered, as seen in the previous two sections. It is even not clear whether introduction of the entrainment–detrainment hypothesis helps to simply the description under full interactions (cf., Section 5.6).

The environment hypothesis a great deal helps to simplify the subgrid-scale descriptions. This hypothesis consists of the two major parts: (1) introduction of a special subgrid-scale component called "environment", and (2) assumption that all the other subgrid-scale components are solely surrounded by the environment. A discussion on merits and demerits for introducing the environment hypothesis in depth is found in Section 4 of Ooyama (1971). From now on, merely for economy of presentation, non-environmental subgrid-scale components are simply called "convection", although the generality of these subgrid-scale components is still maintained.

As a result, both entrainment and detrainment reduces to a vector from a matrix:

$$E_{j,i} = E_j \delta_{j,e} \tag{6.1a}$$

$$D_{j,i} = D_j \delta_{j,e} \tag{6.1b}$$

where $\delta_{j,e}$ is Kronecker's delta, with the subscript e stands for environment. As a result, for the non-environmental components, $j \neq e$, Eq. (5.6) reduces to

$$\rho\sigma_j \overline{(\nabla\cdot\mathbf{u}\varphi)}_j^* = \rho\overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j + D_j\varphi_j - E_j\varphi_e, \tag{6.2}$$

and the prognostic Eq. (5.10) to

$$\frac{\partial}{\partial t}\sigma_j\varphi_j + \frac{1}{\rho}(D_j\varphi_j - E_j\varphi_e) + \frac{1}{\rho}\frac{\partial}{\partial z}M_j\varphi_j + \overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j = \sigma_j F_j. \tag{6.3}$$

The corresponding advection form (5.11) becomes

$$\sigma_j\frac{\partial}{\partial t}\varphi_j + \frac{E_j}{\rho}(\varphi_j - \varphi_e) + \sigma_j w_j \frac{\partial}{\partial z}\varphi_j + \sigma_j\mathbf{u}_j\cdot\overline{\nabla}\varphi_j = \sigma_j F_j. \tag{6.4}$$

The mass continuity (5.8) remains the same.

Special attentions are required for the corresponding equations for the environmental component. This is best seen by examining the mass continuity for the environment:

$$\frac{\partial}{\partial t}\sigma_e + \overline{\nabla}\cdot\sigma_e\mathbf{u}_e + \frac{1}{\rho}\frac{\partial}{\partial z}M_e = \frac{1}{\rho}(D - E). \tag{6.5}$$

Here, the total entrainment E and detrainment D rates are introduced by

$$E = \sum_{j=1,i\neq e}^{n} E_j,$$
$$D = \sum_{j=1,i\neq e}^{n} D_j.$$

Note that the role of entrainment and detrainment for the environment reverses from that in Eq. (5.8) for an obvious reason: the detrained convective air enters the environment, and the environmental air entrains into a convective component.

The prognostic equations are given by

$$\frac{\partial}{\partial t}\sigma_e\varphi_e + \frac{1}{\rho}(E\varphi_e - \sum_{j=1,j\neq e}^{n} D_j\varphi_j) + \frac{1}{\rho}\frac{\partial}{\partial z}M_e\varphi_e + \overline{\nabla}\cdot\sigma_e\mathbf{u}_e\varphi_e = \sigma_e F_e \tag{6.6}$$

(cf., Eqs. AS.14, 15), and

$$\sigma_e \frac{\partial}{\partial t}\varphi_e + \frac{1}{\rho}(D\varphi_e - \sum_{j=1, j\neq e}^{n} D_j\varphi_j) + \sigma_e w_e \frac{\partial}{\partial z}\varphi_e + \sigma_e \mathbf{u}_e \cdot \overline{\nabla}\varphi_e = \sigma_e F_e \tag{6.7}$$

(Eqs. AS.16, 17), respectively, in flux and advective forms.

Here, the fractional area σ_e occupied by the environment is measured relative to the fraction σ_c of total convective area by the relation

$$\sigma_e = 1 - \sigma_c$$

with the latter defined by

$$\sigma_c = \sum_{j=1, j\neq e}^{n} \sigma_j.$$

Note that environmental mass flux is defined by

$$M_e = \rho\sigma_e w_e.$$

Finally, by taking a sum of Eq. (6.3) and (6.6) we obtain a prognostic equation for the grid-box mean:

$$\frac{\partial}{\partial t}\overline{\varphi} + \overline{\nabla}\cdot\overline{\mathbf{u}\varphi} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho\overline{w\varphi} = \left(\frac{\partial\varphi}{\partial t}\right)_e + \sum_{j=1, j\neq e}^{n}\left(\frac{\partial\varphi}{\partial t}\right)_j \tag{6.8}$$

(cf., Eqs. AS.33, 34 and Eqs. AS.35, 36), where the right-hand side terms are defined by

$$\left(\frac{\partial\varphi}{\partial t}\right)_j = \sigma_j F_j - \frac{1}{\rho}\frac{\partial}{\partial z}M_j\varphi_j' - \overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j', \tag{6.9a}$$

$$\left(\frac{\partial\varphi}{\partial t}\right)_e = \sigma_e F_e - \frac{1}{\rho}\frac{\partial}{\partial z}M_e\varphi_e' - \overline{\nabla}\cdot\sigma_e\mathbf{u}_e\varphi_e', \tag{6.9b}$$

where $\varphi_j' = \varphi_j - \overline{\varphi}$ is the deviation from the grid-box mean.

6.2. *Environment hypothesis (strong version)*

So far are the logical consequences by introducing the environmental hypothesis. However, Arakawa and Schubert (1974) have originally introduced a stronger hypothesis on the environment so that only environmental component crosses the gird box boundary, i.e.,

$$\overline{\nabla}\cdot\sigma_j\mathbf{u}_j\varphi_j = \delta_{je}\overline{\nabla}\cdot\mathbf{u}_e\varphi_e. \tag{6.10}$$

Here, on the left hand side, the index j includes the case with $j = e$. Note that this condition realizes in a formal manner only after introducing an asymptotic limit to $\sigma_j \to 0$ (with $j \neq e$) later. However, Arakawa and Schubert (1974) implicitly introduces the assumption (6.10) before they introduce this asymptotic limit.

Under the condition (6.10), the prognostic Eq. (6.3) reduces to

$$\frac{\partial}{\partial t}\sigma_j\varphi_j + \frac{1}{\rho}(D_j\varphi_j - E_j\varphi_e) + \frac{1}{\rho}\frac{\partial}{\partial z}M_j\varphi_j = \sigma_j F_j \tag{6.11}$$

(cf., Eqs. AS.44–46, 48–52). The corresponding advection form (6.4) becomes

$$\sigma_j\frac{\partial}{\partial t}\varphi_j + \frac{E_j}{\rho}(\varphi_j - \varphi_e) + \sigma_j w_j\frac{\partial}{\partial z}\varphi_j = \sigma_j F_j. \tag{6.12}$$

Furthermore, the mass continuity (4.6) reduces to

$$\rho\frac{\partial}{\partial t}\sigma_j+\frac{\partial}{\partial z}M_j=E_j-D_j \tag{6.13}$$

(cf., Eq. AS.10). Similar simplifications are also possible for Eqs. (6.5), (6.6) and (6.7).

The system under the environment hypothesis is arguably easier to handle than the case without, because each subgrid-scale (convective) component interacts only directly with the environment. Otherwise, generality of subgrid-scale description is still maintained.

7. Standard formulation: asymptotic limit of vanishing fractional area for convection

7.1. Scale separation principle: $\sigma_j \to 0$

The standard massflux formulation, as originally derived by Arakawa and Schubert (1974), is obtained finally by introducing an asymptotic limit that fractional area σ_j covered by each convective element is much smaller than σ_e for the environment (see a middle of their Section 2). Thus

$$\sigma_j \ll \sigma_e$$

for $j \neq e$ or equivalently

$$\sigma_j \ll 1, \quad \text{and} \quad \sigma_c \ll 1$$

with

$$\sigma_e \equiv 1-\sigma_c \simeq 1.$$

Under a standard notation for asymptotic expansion (e.g., Bender and Orszag, 1978; Olver, 1974), we may write

$$\sigma_j \to 0, \qquad \sigma_c \to 0, \quad \text{and} \quad \sigma_e \to 1. \tag{7.1}$$

Here, though the notion of asymptotic expansions is never explicitly evoked by Ooyama, Fraedrich, Arakawa and Schubert, this does not escape the fact that this limit is best understood in this manner (cf., Yano, 1999, 2003).

7.2. Consequences of the asymptotic limit $\sigma_j \to 0$

Under this asymptotic limit, we expect an asymptotic scaling that the magnitude of the convective variables is of the same order of magnitude as that of the environment, *i.e.*, $O(\varphi_j)\sim O(\varphi_e)$, except for the vertical velocity and forcing. The latter satisfy: $O(\sigma_j w_j)\sim O(w_e)$ and $O(\sigma_j F_j)\sim O(F_e)$.

A main consequence of the above scaling is that the grid-box means are approximated by the environmental values, i.e.,

$$\overline{\varphi} \simeq \sigma_e\varphi_e \simeq \varphi_e \tag{7.2}$$

by recalling the relation

$$\overline{\varphi}=\sigma_e\varphi_e+\sum_{j=1, j\neq e}^{n}\sigma_j\varphi_j$$

(cf., Eqs. AS.19, 20). However, keep in mind

$$\sigma_e w_e \simeq w_e \simeq \overline{w}-\sum_{j=1, j\neq e}^{n}\sigma_j w_j,$$

$$\sigma_e F_e \simeq F_e \simeq \overline{F}-\sum_{j=1, j\neq e}^{n}\sigma_j F_j,$$

and we *cannot* set $w_e \simeq \overline{w}$ nor $F_e \simeq \overline{F}$ even approximately.

The approximation (7.2) implies

$$\frac{\partial}{\partial t}\overline{\varphi} \simeq \frac{\partial}{\partial t}\sigma_e\varphi_e, \tag{7.3}$$

thus the tendency for the grid-box mean can be approximated by the environmental tendency. Also note that the tendency (6.9b) due the environmental component is approximated by

$$\left(\frac{\partial\varphi}{\partial t}\right)_e \simeq \sigma_e F_e \simeq F_e. \tag{7.4}$$

7.3. Steady-plume hypothesis

A simple corollary from the relation (7.3) is that the *total* convective tendency is negligible compared to that for the grid-box mean, i.e.,

$$\frac{\partial}{\partial t}\sum_{j=1,j\neq e}^{n}\sigma_j\varphi_j \simeq 0. \tag{7.5}$$

We may call this constraint the *subgrid-scale collective balance*.

However, keep in mind that the approximation

$$\frac{\partial}{\partial t}\sigma_j\varphi_j \simeq 0 \tag{7.6}$$

for the individual convective component j does *not* follow immediately. The condition (7.6) is posed in Yanai et al. (1973) in deriving their Eqs. (27)–(30), which thus must be seen with caution. In the end, this has no unfavorable consequence in Yanai et al. (1973) because they mainly focus on the bulk formulation with a single convection type. Arakawa and Schubert (1974) introduce the same condition, however, in a more discreet manner.

In general, the left hand side of Eq. (7.6) can remain the order unity without contradicting with the remaining part of the formulation. It merely suggests the scaling, $\partial/\partial t{\sim}O(\sigma_j^{-1}) \to \infty$ under the asymptotic limit, $\sigma_j \to 0$, which states a simple fact that when the scale of convection is much smaller than that of the "large scale", convection evolves proportionally faster. A fast process for individual subgrid components is not quite of interests in parameterizations: it is better to avoid an explicit description of convective-scale transient processes such as life cycles for the sake of economy of computations. For this reason, traditionally, the condition Eq. (7.6) is introduced.

We call this assumption (7.6) the *steady-plume hypothesis*, because each convective element is traditionally modelled as a plume. By introducing this additional condition, we limit our attention to a subensemble of subgrid components designated by the indices j, and *do not consider* evolution of individual subgrid-scale components. In other words, when evolution of individual convection is to be considered explicitly (e.g., trigger, convective life cycles), the condition (7.6) must be removed. Unfortunately, the existing literature is not careful with this matter (cf., Yano, 2011; Yano et al., 2013).

7.4. Equations for the grid-box mean

The flux form of the equation for the grid-box mean is obtained by substituting Eq. (7.5) and (7.4) into (6.8):

$$\frac{\partial}{\partial t}\overline{\varphi} + \overline{\nabla}\cdot\overline{\mathbf{u}\varphi} + \frac{1}{\rho}\frac{\partial}{\partial z}\rho\overline{w\varphi} = -\frac{1}{\rho}\frac{\partial}{\partial z}\sum_{j=1,j\neq e}^{n}M_j\varphi_j' + \overline{F} \tag{7.7}$$

(Eqs. AS.74, 75, YEC.26), where $\overline{F} = \sum_{j=1,j\neq e}^{n}\sigma_j F_j + F_e$.

An alternative expression for the grid-box mean equation is more directly obtained from that for the environment given by Eq. (6.6). By applying the approximations (7.3), etc., we obtain

$$\frac{\partial}{\partial t}\overline{\varphi}+\overline{\nabla}\cdot\overline{\mathbf{u}\varphi}-\frac{1}{\rho}\left[\sum_{j=1,j\neq e}^{n}D_j\varphi_j-E\overline{\varphi}\right]+\frac{1}{\rho}\frac{\partial}{\partial z}\rho w_e\overline{\varphi}=F_e \tag{7.8}$$

(Eq. O71.26). The corresponding advective form is obtained by directly applying the approximations (7.1), (7.2), and (7.3) into Eq. (6.7):

$$\frac{\partial}{\partial t}\overline{\varphi}+\overline{\mathbf{u}}\cdot\overline{\nabla}\overline{\varphi}+w_e\frac{\partial}{\partial z}\overline{\varphi}=-\frac{1}{\rho}(D\overline{\varphi}-\sum_{j=1,j\neq e}^{n}D_j\varphi_j)+F_e \tag{7.9}$$

(Eqs. O71.28, AS.29, 30). The mass continuity for the environment reduces from Eq. (6.5) to

$$\overline{\nabla}\cdot\overline{\mathbf{u}}+\frac{1}{\rho}\frac{\partial}{\partial z}\rho w_e=D-E \tag{7.10}$$

(Eq. O71.27). Here, we set $\mathbf{u}_e\simeq\overline{\mathbf{u}}$.

An important implication from Eqs. (7.8) and (7.9) is that no information on the forcing F_c at convective scale is *directly* required in order to evaluate the tendencies of the grid-box mean itself. A main influence of convection to the environment is through detrainment of convective air as given in the third term on the left hand side.

7.5. Equations for the jth subgrid component

The prognostic equation for jth subgrid-scale component under the limit of $\sigma_j\to 0$ are the same as already given by Eqs. (6.11) and (6.12), but φ_e replacing with $\overline{\varphi}$. Introduction of the steady-plume hypothesis (7.6), however, further simplifies the matter, and the prognostic equation reduces to a diagnostic form:

$$\frac{\partial}{\partial z}M_j\varphi_j=E_j\overline{\varphi}-D_j\varphi_j+\rho\sigma_jF_j. \tag{7.11}$$

By taking the limit $\sigma_j\to 0$ in Eq. (6.13), the mass continuity for the jth convective element becomes:

$$\frac{\partial}{\partial z}M_j=E_j-D_j \tag{7.12}$$

(Eq. O71.20). Thus, the asymptotic limit to the vanishing fractional area for convection leads to a diagnostic description of mass continuity for the convective elements. Note especially that Eq. (4.10) reduces to

$$\frac{1}{S}\oint_{\partial S_j}\dot{\mathbf{r}}_b\cdot d\mathbf{r}=0.$$

Thus the fractional area does not change with time to leading order under this limit.

7.6. Conservative subgrid-scale processes

As a full formulation for the standard mass-flux parameterization takes the shape, it should also be becoming clear how the mass flux (Eq. (5.7)), M_j, plays the key role under this formulation. The point is best seen for *conservative* subgrid-scale processes.

When a variable, φ_j, is conserved under the subgrid-scale processes associated with a jth subcomponent, $F_j=0$, and Eq. (7.11) reduces to

$$\frac{\partial}{\partial z}M_j\varphi_j=E_j\overline{\varphi}-D_j\varphi_j. \tag{7.13}$$

The variable in concern may not be conservative in strict sense. However, so long as the time-scale for nonconservative processes is much longer than that for the subgrid-scale processes (e.g., convection), the variable may still be considered conservative for a parameteriztion purpose. For example, though moist entropy may be lost by radiative cooling, since it is a slow process, the moist entropy may be considered conserved as assumed by Arakawa and Schubert (1974).

Under this set-up, a full advantage of entrainment–detrainmnet hypothesis introduced in Section 5 is fully appreciated. Once the entrainment–detrainment rate is specified, first of all, the mass flux, M_j, can be evaluated simply by vertically integrating Eq. (7.12) with given entrainment and detrainment rates. Note however, the bottom boundary condition must be specified, as further discussed in the next subsection.

Once the mass flux, M_j, is known, the vertical profile for the subgrid-scale state, φ_j, is also obtained by vertically integrating Eq. (7.13) from the bottom of convection. Here, again, the bottom boundary condition must be specified, but these values are commonly taken to be equal to the environmental values, with additional perturbations added in some convection schemes (e.g., Gregory and Rowntree, 1990; Kain and Fritsch, 1990, 1992).

7.7. Separation of the variables and closure

An important aspect of the evaluation procedure for the subgrid components outlined above is that all are *diagnostic* rather than prognostic. This is an important consequence of the steady plume hypothesis, which suggests that these subgrid-scale processes are in equilibrium against a given large-scale state, thus no fast evolution of the subgrid-scale must explicitly be evaluated *prognostically*. This hypothesis, furthermore, somehow separates a procedure of determining the vertical structure of the subgrid components and that of determining their total magnitudes.

In many existing entrainment–detrainment formulations, it is the fractional entrainment and detrainment rates, ϵ_j and δ_j, that are specified, rather than the full entrainment and detrainment rates themselves. These may be defined by

$$\epsilon_j = \frac{E_j}{M_j},$$

$$\delta_j = \frac{D_j}{M_j}.$$

As a result, Eq. (7.12) reduces to

$$\frac{1}{M_j}\frac{\partial}{\partial z}M_j = \epsilon_j - \delta_j. \tag{7.14}$$

When the mass flux equation is re-written in this manner, separation of the variables becomes possible, and we may write it as

$$M_j = \eta_j(z)M_{B,j}(t).$$

Here, $M_{B,j}(t)$ is the mass-flux value at the convection base (or base of any subgrid-scale component), and $\eta_j(z)$ is a normalized vertical profile determined by Eq. (7.14). Thus, the system is left with determining $M_{B,j}(t)$, which measures a total strength of convection. This *closure* problem is reviewed by Yano et al. (2013).

It is important to keep in mind that a separation of variables is possible only when the fractional entrainment and detrainment rates are considered independent parameters. When the entrainment and detrainment rates, E_j and D_j, are specified, instead, as parameters independent of M_j, separation of the variables is no longer possible.

7.8. Nonconservative subgrid-scale processes

When the subgrid-scale processes in concern are nonconservative, and $F_j \neq 0$, however, the procedure suddenly becomes involved. Here, the mass flux can still be computed in the same manner as before by Eq. (7.12). On the other hand, in order to evaluate the subgrid-scale component, φ_j, now, Eq. (7.11) must be integrated vertically in place of Eq. (7.13).

In doing so, we face the two difficulties. First, the source term, F_j, must be specified. Though this would involve some secondary difficulties (e.g., what kind of "physics" to adopt), in view of the whole formulation of the problem this is not fundamental.

The second is more fundamental: in order to account for a nonconservative tendency to a given subgrid-scale variable, a fractional area, σ_j, occupied by a given subgrid component must also be specified. However, any explicit equation for defining σ_j is not known (cf., Section 4.8). Worse than that, a need for defining a finite value for σ_j is even at odd with an assumed asymptotic limit to $\sigma_j \to 0$.

There are two major approaches for overcoming this difficulty: i) To introduce a formulation for the source in terms of $\sigma_j F_j$ instead of F_j so that a need for specifying σ_j can be avoided. This approach is taken by many earlier work including Arakawa and Schubert (1974) for describing precipitation. ii) To evaluate σ_j in certain manner. Under this second approach, a more common strategy is to try to evaluate the vertical velocity, w_j, instead of σ_j itself, because once the mass flux, M_j, is known, σ_j can be evaluated from M_j and w_j. Need for explicitly considering the convective vertical velocity for this purpose is emphasized by Donner (1993).

However, evaluation of the convective vertical velocity, w_j, poses its own problems. See in historical order: Levine (1959), Simpson et al. (1965), Simpson and Wiggert (1969), Donner (1993), Sud and Walker (1999), Bechtold et al. (2001), Siebesma et al. (2003), Bretherton et al. (2004), Zhang et al. (2005).

8. Discussions and conclusions

8.1. Summary

The main goal of this paper has been to show how the standard mass-flux convection parameterization, as originally proposed by Ooyama (1971), followed by Fraedrich (1973, 1974), and Arakawa and Schubert (1974), can be derived in a systematic manner starting from a full physical system (NAM) based on nonhydrostatic anelatic approximation. The derived formulation is applicable not only to convective processes but, in principle, also to any subgrid-scale processes due to the generality of the derivation (cf., Yano et al., 2005a; Yano, 2012a).

Key points to be re-iterated are:

i) Generality of mass-flux decomposition is exposed by identifying its basic geometrical constraint as a segmentally constant approximation (SCA). This re-interpretation enables us to incorporate various different dynamical and physical subgrid-scale components into the mass-flux formulation in a more consistent manner.
ii) The deductive derivation of the standard mass-flux formulation from NAM provides an interpretation of the standard formulation as an asymptotic limit. This interpretation enables us to find a way more easily relaxing the current parameterizations and adapting them for high-resolution regional modelling.
iii) A system obtained at each step of deduction constitutes its own stand-alone subgrid-scale model, thus it forms a hierarchy of subgrid-scale models as a whole. The hierarchy not only provides a range of choices for subgrid-scale schemes, but more importantly, it provides a mean for more systematically verifying mutual consistencies of the schemes with different degrees of complexities.

8.2. Hierarchy of subgrid-scale models

From this very last perspective, the limit of the current extensive efforts for taking the CRM/LES outputs for verifications of parameterization may also be recognized: CRM/LES (NAM) and the standard mass flux formulation are at the two extreme ends of this whole hierarchy. A direct comparison of these two models, skipping various intermediate steps, would not be as useful as comparing a change of a behavior of a mass-flux scheme by removing only a single element of assumptions (i.e., asymptotic limit, entrainment–detrainment hypothesis). For example, though extensive evaluations

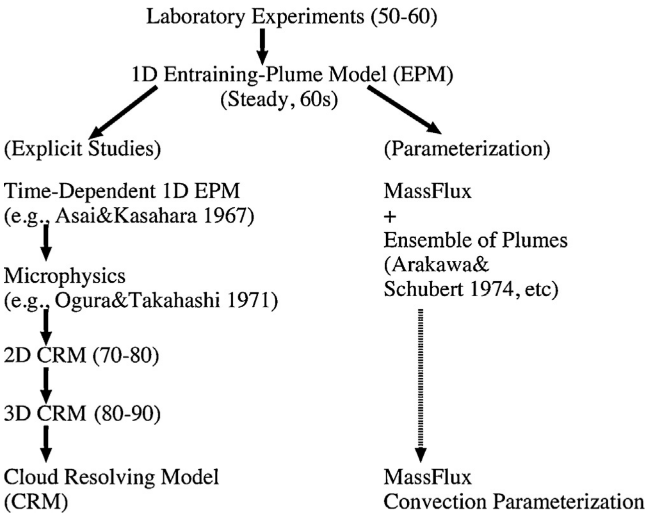

Fig. 4. Historical lines of atmospheric moist convection studies: historical evolution is shown by a flow chart with key publications and periods of key research activities indicated (see ref. Ogura and Takahashi (1971)).

of entrainment and detrainment rates from CRM/LES are available, their direct applicability in mass-flux parameterization is questionable due to different degrees of approximations adopted in both systems. Even harder is inferring a closure condition by CRM/LES analyses for the same very reason. A more systematic analysis of parameterized convective processes is indeed possible by running various intermediate models in stand-alone manner.

Fig. 4 schematically suggests how the steady-plume model originally introduced by Morton et al. (1956) and others in 50s and 60s (cf., Yano, 2014) gradually evolved into the two very distinctive lines of research: more explicit studies of convection itself (left side of the flow chart), which led to the current CRM/LES, and the parameterization studies (right side), which led to the current mass-flux parameterizations.

The present study shows how the full CRM/LESs and the operational mass-flux convection parameterizatinos, that have been developed along separate lines of research, can be linked together by a systematic stepwise deduction, with a stand-alone subgrid-scale representation scheme identified at each step of deduction. In verification efforts of parameterizations, in turn, such individual steps of the deduction should more explicitly be exploited rather than taking the models of the two extreme ends of the hierarchy.

Deduction is performed by introducing the following four elements:

i) Application of a segmentally-constant approximation (SCA) to a full CRM/LES (NAM) system as a basic geometrical constraint.
ii) Introduction of the entrainment–detrainment hypothesis in order to avoid a more direct but complicated estimation of horizontal winds.
iii) Introduction of "environment" in order to limit subgrid-scale interactions to those between each subgrid-scale component and the environment.
iv) Introduction of an asymptotic limit to vanishing fractional area for subgrid-scale processes (convection), leading to their steady description ("steady plume hypothesis").

Overall, each step introduces a certain simplification, but with a loss of generality retained in the original formulation. What is lost at each step may be important as much as what is gained by simplification.

A very first step for constructing a parameterization may more generally be called *caricatuarization*: i.e., a procedure of transforming, for example, a realistic convective structure simulated by a full model into a much simpler level, which may even be called *"caricature"*. Such a procedure may be, mathematically, that of posing a drastically simplified *geometrical* constraint to a full physical system. A basic geometrical constraint introduced for the mass-flux parameterization is named the segmentally constant approximation (SCA), and a resulting model is NAM–SCA.

Though these four steps are introduced in the above given order under the present analysis, this order is not unique. For example, the environment hypothesis can be introduced without adopting the entrainment–detrainment hypothesis. The above order allows us to introduce the entrainment–detrainment hypothesis in the most general manner. Alternatively, the environmental hypothesis can be introduced without the former, as an additional purely geometrical constraint. Horizontal winds can still be diagnosed explicitly without the entrainment–detrainment hypothesis under the environmental hypothesis. The asymptotic limit can even be introduced without entrainment–detrainment hypothesis.

8.3. Generalizations

Once the structure of the mass-flux parameterization formulation is well understood, it is more straightforward to generalize it in a more physically consistent manner. Importantly, the deduction of a full physical process into the standard mass-flux formulation is presented without explicitly specifying these subgrid-scale processes to be convective so long as the introduced hypotheses and an asymptotic limit are satisfied. This subsection reviews the deduction steps from this point of view.

8.3.1. SCA as a generalization of mass flux

Re-interpretation of "mass flux" as a geometrical constraint, SCA, makes it much easier to introduce any subgrid-scale processes into a mass-flux framework. As schematically suggested by Fig. 3, SCA is a purely geometrical constraint to a full physical system such as NAM. In principle, any full physical system can be run with any geometrical distributions of subgrid-scale components under SCA.

In an initial implementation of SCA into NAM, in order to retain its generality, Yano et al. (2010b) have introduced an adaptive procedure for re-defining distribution of constant segments with time by following the evolution of a system. Yano and Buniol (2010) show that such a procedure can describe a tropical squall-line system in highly efficient manner. Yano and Baizig (2012), in turn, show that NAM–SCA can describe evolution of a single convective plume by introducing only the two constant segments. A rather intriguing result by implementing a fully-adaptive NAM–SCA into operational single-column models is that NAM–SCA performs rather well under a limit to only two segments (Yano et al., 2012).

Currently, an intermediate version of NAM–SCA is under development in order to accomplish a more realistic description of convective systems with only few segments. Fig. 5 shows a snap shot from such a preliminary result with only four fixed constant segments in horizontal direction, but distributed in highly inhomogeneous manner. More specifically, a 512-km horizontal periodic domain is taken as the case in Yano and Baizig (2012). In a middle of this domain, a segment with a size of 8 km is placed, which is immediately adjacent to the two segments with a size of 16 km.

The study case taken is the same as that of Yano and Baizig (2012), based on an idealized forcing for a tropical situation under a realistic easterly wind over the Tropical Atlantic constructed from a GATE (Global Atmospheric Research Program's (GARP) Atlantic Tropical Experiment) period observation by Jung and Arakawa (2005). The system is initialized by a vertical profile as introduced by Jung and Arakawa (2005) with a random temperature perturbation in the first model layer. The convective circulation is sustained by adding to the lowest model layer a constant sensible heating of 10^{-4} K/s at the center segment and cooling of $-1/4 \times 10^{-4}$ K/s at the two side segments. Though very crude, a realistic-looking tropical squall-line like structure is maintained only with the four constant segments with a updraft-downdraft pair spontaneously generated to the expected propagation direction of the squall line with a cold pool underneath the downdraft (cf., Yano, 2012b).

This preliminary result further demonstrates that NAM–SCA can describe the subgrid-scale processes efficiently. Under this framework, a key question of the subgrid-scale description reduces to that of defining a distribution of constant segments over a large-scale grid box. Here, we do not have to (and cannot) specify a role or function of a given segment (e.g., updraft or downdraft), but the system

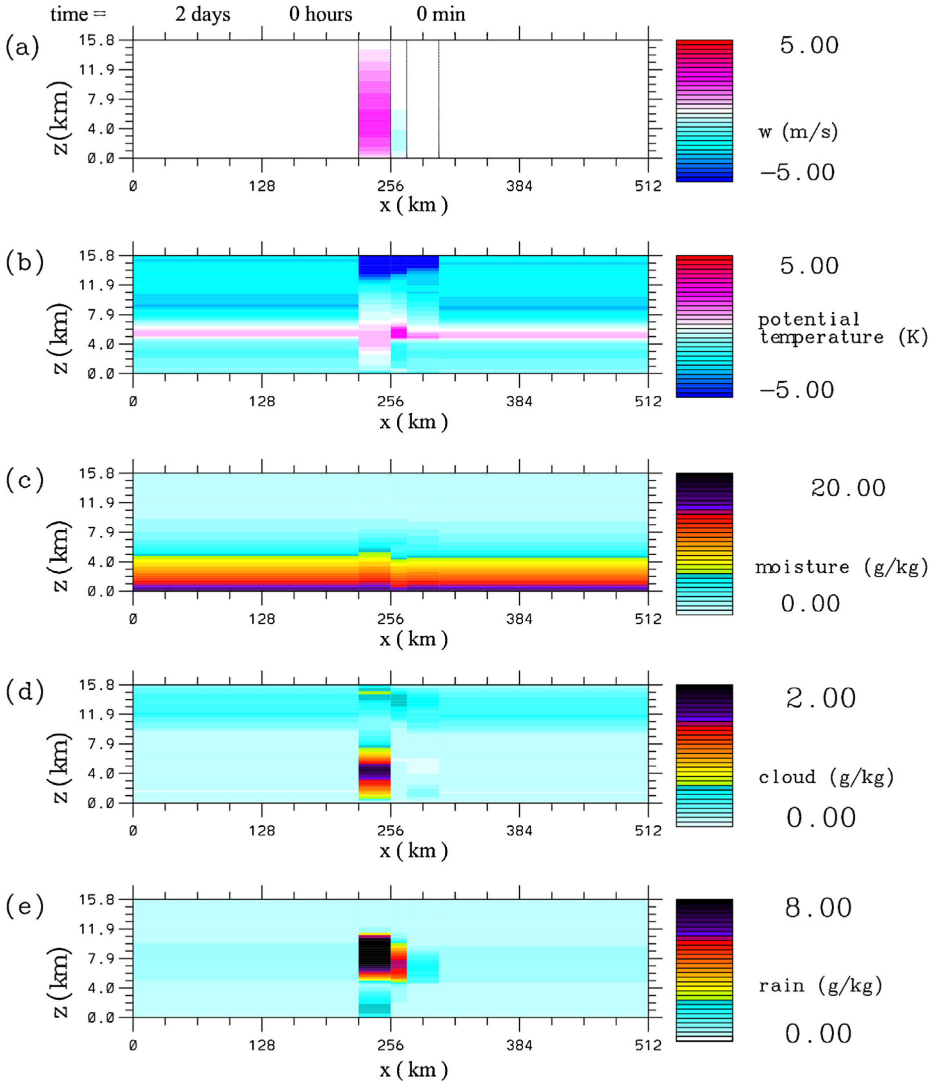

Fig. 5. A snap shot of an evolving NAM–SCA system consisting only of the four SCA segments on the 2nd day of a simulation under an idealized tropical situation. Shown from the top to bottom are: (a) vertical velocity (m/s), (b) potential temperature anomaly (deviation from a horizontal average, K), (c) water–vapor mixing ratio (g/kg), (d) cloud water (g/kg), (e) precipitating water (g/kg).

decides by itself with its evolution. In the above particular example, the middle smallest segment ends up as a downdraft in spite of the fact that heating is added at the surface for nudging it towards an updraft.

8.3.2. From prognostic to diagnostic: closure problem

The next question is a reduction of the fully-prognostic NAM–SCA into a standard diagnostic formulation, which is more appealing for operational implementations with potential numerical efficiencies. At the most formal level, a diagnostic version of NAM–SCA would be obtained by simply applying asymptotic limit of vanishing “convective” area relative to the environment.

However, practicality of a diagnostic approach must be judged carefully: a diagnostic problem is mathematically often more difficult to solve than a prognostic problem. Typically a certain iteration must be performed, and an effort for iterations could be more expensive than an original time-integration problem. Especially, so long as the formulation remains prognostic, no need for a closure, as in the standard formulation, is required. Clearly, the closure is far from being a trivial problem (cf., Yano et al., 2013).

8.3.3. Entrainment–detrainment hypothesis

A further introduction of the hypotheses for the standard formulation may simplify a general parameterization formulation. However, the validity of each hypothesis must carefully be judged for every generalization. Among those, the entrainment–detrainment hypothesis is specifically introduced with the convective plume dynamics in mind, as already emphasized in Section 5.6. This is certainly a core of the original mass-flux parameterization. However, the applicability of this hypothesis into the other types of subgrid-scale processes is not immediately obvious, although the detrainment could be reinterpreted as a linear damping process with a characteristic time scale for stratiform clouds, for example (cf., Yano, 2012a).

8.3.4. Geometrical Information

A clear advantage of introducing entrainment and detrainment is that, on the other hand, we can evade a need for knowing the horizontal wind velocity explicitly in order to calculate the lateral exchange of physical quantities between a plume and the environment. As a result, furthermore, there is no longer a need for specifying positions of individual convective plumes, but only their fractional areas.

Specification of the geometrical positions of the subgrid-scale elements is a new feature found under NAM–SCA. The entrainment–detrainment hypothesis is a key ingredient that makes it possible to make the geometrical information on subgrid-scale variables implicit. Importance of subgrid-scale geometrical information under generalization is hardly overemphasized. We are so used to an idea of implicit geometrical treatment, it appears, that the issue is often even not noticed. Here, a subtle difference between the so-called top-hat distribution and the mass-flux concept must carefully be made. Unlike the former, the latter contains a geometrical connotation.

The strategy without geometrical information no longer works quite, either, when we generalize the problem to nonconservative processes: in order to account for a contribution of a subgrid-scale source term in a grid-box average, the convective vertical velocity must be known, as already discussed in Section 7.8. In order to perform the convective vertical-velocity evaluation consistently, we have to recover the geometrical information on the distribution of convective elements within a grid box explicitly, especially for diagnosing the aerodynamic pressure consistently, as already discussed in Section 4.7.

8.3.5. Interactions between the Subgrid-Scale Components

The notion of "environment" is so hard wired into the standard mass-flux formulation that in the literature, it is even hardly noticed as a basic constraint. This notion may most intuitively be understood by schematics presented by Fig. 1 of Arakawa and Schubert (1974), in which all the convective plumes are embedded into a larger portion of the grid-box domain called "environment". As a result, the convective plumes no longer interact with each other directly, but only indirectly through their interactions with the environment. Note that under the environment hypothesis, the interactions between the subgrid-scale components are very limited, and the convective updraft air is transported to the downdraft element only at the originating vertical level of the latter (e.g., Betts and Silva Dias, 1979; Fritsch and Chappell, 1980; Tiedtke, 1989; Grell et al., 1991; Emanuel, 1991; Zhang and McFarlane, 1995; Ducrocq and Bougeault, 1995; Bechtold et al., 2001), for example.

8.3.6. Plume hypothesis and convective life-cycle

Another core of the original mass-flux convection parameterization is to consider the atmospheric convective system as an ensemble of entraining plumes. See Section 5.6 for further discussions.

Relaxation of the plume hypothesis could be a prerequisite for a generalization of the mass-flux formulation into the other subgrid-scale processes.

Here, what appears not be widely recognized in the literature is the fact that the steady-plume hypothesis (7.6) is not an indispensable part of the standard mass-flux parameterization formulation. In order to maintain the consistency of the subrid-scale processes in the asymptotic limit, the subgrid-scale collective balance (7.5) is an only condition that must be satisfied. So long as the latter is satisfied, each subgrid-scale element can evolve in transient manner by following their own life cycles.

Absence of an explicit consideration of a convective life-cycle is an often-cited problem with the standard mass-flux formulation. Notably, Mapes' (1997) activation-control principle provokes for a need for such considerations. However, little effort is made to introduce such explicit description of subgrid-scale evolution (cf., Yano, 2011; Yano et al., 2013).

8.4. High-resolution limit

With increasing horizontal resolutions of the operational models, we face with needs for relaxing the assumptions behind the standard mass-flux parameterizations. The absolute minimum requirement under this limit is to remove the approximations as a direct consequence of the asymptotic limit to the vanishing fractional area for convection. It leads to a fully prognostic formulation presented in Section 6. Such a drastic measure itself is yet to be fully adopted in operational models.

Beyond this point, though there is no definite answer to the question of which hypotheses to be retained and which to be removed, some general remarks can be made. In principle, as outlined in Section 6, and also suggested by Yano (2012a) for the primitive equation system, it would be possible to construct a self-contained description of subgrid-scale processes without removing any further hypotheses. However, my own personal opinion, such a formulation would be less physically justifiable and also mathematically less tractable.

Yano (2012a) has already removed the environment hypothesis in his formulation. It would be hard to justify maintaining this hypothesis when every subgrid-scale element begins to take a finite fraction, and every subgrid-scale component begins to interact with each other as already discussed in Section 5. As an important consequence, we no longer have a unique equation for the domain mean any more, that is traditionally approximated by an environmental state. Instead, we have to integrate each subgrid-scale component in time, and the domain mean state is obtained only by taking a weighted average of all those subcomponents.

Whether to retain the entrainment–detrainment hypothesis would be a more delicate issue. As already discussed in Section 5, at a very formal level, generalization of entrainment–detrainment into a matrix formulation under an absence of environment is straightforward. However, reliability of this formulation under a fully-prognostic framework is still to be demonstrated. A preliminary investigation suggests that even a simple implementation with a constant fractional entrainment into a transient single plume leads to a explosively growing tendency (Yano and Baizig, 2012).

In my own personal opinion, the most robust choice would be to stay with a full NAM–SCA outlined in Section 4, with a preliminary prototype shown in Fig. 5. However, extensive investigations are still required before a final word can be heard. The present work provides a road map for making these investigations systematic.

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