Fixed Point Theorems for Nonlinear Operators

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Some new fixed point theorems are presented for operators of accretive, nonlinear contractive, or nonexpansive type. These results are then used to establish a new existence principle for second order boundary value problems in Hilbert spaces.

1. INTRODUCTION

Some general fixed point theorems for nonlinear operators between complete metric spaces are presented in this paper. These results were motivated from the study of differential equations in abstract spaces.

The paper is divided into three sections. Section 2 presents notation, definitions, and some well known fixed point results. In Section 3 we begin by presenting some general continuation type theorems for maps between complete metric spaces. These results were motivated by the papers of Schöneberg [13, 14] and more recently by a paper of Granas [8]. Our continuation theorems are then used to establish Leray–Schauder type alternatives for accretive, nonlinear contractive, or nonexpansive maps. Properties of projection maps together with the theory described above allow us to establish new fixed point theorems for nonlinear contractive and nonexpansive maps. Finally Section 4 uses results from Section 3 to prove an existence principle for second order boundary value problems in Hilbert spaces. An interesting feature of this existence principle is that no compactness condition is assumed on the nonlinearity; this extends a result of Mawhin [11].
2. PRELIMINARIES

In this section we gather together some known fixed point results which will be used in Section 3. To begin with let \( X \) be a Hilbert space and \( \Omega \subseteq X \). A mapping \( T: \Omega \to X \) is said to be

(i) \textbf{strongly monotone} if there exists \( c > 0 \) with
\[
\Re\langle T(x) - T(y), x - y \rangle \geq c\|x - y\|^2 \quad \text{for all } x, y \in \Omega
\]

(ii) \textbf{monotone} if
\[
\Re\langle T(x) - T(y), x - y \rangle \geq 0 \quad \text{for all } x, y \in \Omega
\]

(iii) \textbf{demicontinuous} if for every sequence \( (x_n) \) in \( \Omega \), \( x_n \to x \in \Omega \) implies \( T(x_n) \to T(x) \); here \( \to \) denotes weak convergence.

\textbf{Theorem 2.1 [13]}. Let \( X \) be a Hilbert space, \( U \subseteq X \) be open, and let \( I - T: U \to X \) be demicontinuous and strongly monotone. Then \( T \) has a fixed point in \( U \) iff there is an \( x \in U \) such that \( \|x_0 - T(x_0)\| \leq \|x - T(x)\| \) for all \( x \in \partial U \).

Now let \( X \) be a real Banach space. Let \( \mathcal{M} \) denote the set of all continuous strictly increasing mappings \( \psi: [0, \infty) \to [0, \infty) \) such that \( \psi(0) = 0 \) and \( \psi(x) \to \infty \) as \( x \to \infty \). Also let \( X^* \) denote the dual of \( X \). Notice from the Hahn–Banach theorem that
\[
\{x^* \in X^*: x^*(x) = \|x\|^2, \|x^*\| = \|x\|\} \neq \emptyset
\]
for every \( x \in X \). The mapping \( F: X \to 2^{X^*} \) defined by
\[
F(x) = \{x^* \in X^*: x^*(x) = \|x\|^2 = \|x^*\|^2\}
\]
is called the \textbf{duality map} \([3, 6, 9]\) of \( X \). By means of \( F \), the semi-inner product \((.,.): X \times X \to \mathbb{R} \) is defined by
\[
(x, y)_+ = \sup\{y^*(x): y^* \in F(y)\}.
\]

Let \( \Omega \subseteq X \). A mapping \( T: \Omega \to X \) is said to be

(i) \textbf{\( \psi \)-accretive (\( \psi \in \mathcal{M} \))} if
\[
(T(x) - T(y), x - y)_+ \geq \psi(\|x - y\|)\|x - y\| \quad \text{for all } x, y \in \Omega
\]

(ii) \textbf{\( \psi \)-accretive} if \( T \) is \( \psi \)-accretive for some \( \psi \in \mathcal{M} \).

\textbf{Theorem 2.2 [14]}. Let \( X \) be a real Banach space, \( U \subseteq X \) be open, and let \( I - T: U \to X \) be continuous and strongly accretive. Then \( T \) has a fixed point
in \( U \) iff there is an \( x_0 \in U \) such that \( \|x_0 - T(x_0)\| \leq \|x - T(x)\| \) for all \( x \in \partial U \).

Now let \((X, d)\) be a complete metric space. Let \( \Omega \subseteq X \). A mapping \( T: \Omega \to X \) is called a \emph{nonlinear contraction} on \( \Omega \) if for all \( x, y \in \Omega \) we have

\[
d(T(x), T(y)) \leq \phi(d(x, y)),
\]

where \( \phi: [0, \infty) \to [0, \infty) \) is a continuous function satisfying \( \phi(z) < z \) for \( z > 0 \).

\textbf{Remark.} When \( \phi(z) = \alpha z, 0 \leq \alpha < 1 \) then \( T \) is a contraction. When \( \phi(z) = \alpha(z)z, \alpha(z) < 1 \) for \( z > 0 \) and \( \alpha \) is decreasing, then \( T \) is a Rakotch contraction.

\textbf{Theorem 2.3} [2]. Let \( \Omega \) be a closed subset of a complete metric space \( X \) and \( T: \Omega \to \Omega \) is a nonlinear contraction. Then \( T \) has a unique fixed point.

\textbf{Remark.} Of course in a Hilbert space \( X \) (and more generally in a Banach space) there are connections between contractive and monotone maps. For example, if \( T: X \to X \) (\( X \) is a Hilbert space) is nonexpansive then \( I - T: X \to X \) is monotone.

Finally to conclude this section we discuss the notation of measure of noncompactness \([1, 9]\). Let \( X \) be a Banach space and \( \Omega_X \) the bounded subsets of \( X \). The \textit{Kuratowski measure of noncompactness} is the map \( \alpha: \Omega_X \to [0, \infty) \) defined by

\[
\alpha(Y) = \inf \{ \varepsilon > 0 : Y \subseteq \bigcup_{i=1}^{n} Y_i \text{ and } \operatorname{diam}(Y_i) \leq \varepsilon \}; \quad \text{here } Y \in \Omega_X.
\]

Recall the following properties of \( \alpha \). Let \( Y_1, Y_2 \in \Omega_X \). Then

\begin{enumerate}
    \item \( \alpha(Y_1) = 0 \) iff \( Y_1 \) is compact;
    \item \( \alpha(Y_1) = \alpha(Y_1') \);
    \item if \( Y_1 \subseteq Y_2 \) then \( \alpha(Y_1) \leq \alpha(Y_2) \);
    \item \( \alpha(Y_1 \cup Y_2) = \max(\alpha(Y_1), \alpha(Y_2)) \);
    \item \( \alpha(\operatorname{co}(Y_1)) = \alpha(Y_1) \).
\end{enumerate}

\section{3. FIXED POINT THEORY}

Let \( U \) be an open subset of a set \( C \) in a complete metric space \((X, d)\). Here \( C \) is a convex set if \( X \) is a Banach space whereas \( C = X \) if \( X \) is not a Banach space. Throughout we let \( K_{xU}(\bar{U}, C) \) denote the family of mappings from \( \bar{U} \) to \( C \) which are fixed point free on \( \partial U \).
Remark. \( \overline{U} \) and \( \partial U \) denote the closure of \( U \) in \( C \) and the boundary of \( U \) in \( C \), respectively.

**Definition 3.1.** A mapping \( T \in K_{u}(\overline{U}, C) \) is said to be *essential* if \( T \) has a fixed point. Otherwise we say \( T \) is *inessential*.

**Definition 3.2.** Two maps \( T, S \in K_{u}(\overline{U}, C) \) are *homotopic* in \( K_{u}(\overline{U}, C) \), denoted \( T \sim S \), if there is a mapping \( H: [0, 1] \times \overline{U} \to C \) such that \( H(t, \cdot) = H_{t}(\cdot) \in K_{u}(\overline{U}, C) \) for each \( t \in [0, 1] \), \( H_{0} = S, H_{1} = T \), and \( H(t, u) \) is continuous in \( t \), uniformly for \( u \in \overline{U} \).

We now present two continuation type results.

**Theorem 3.1.** Assume \( T \sim S \in K_{u}(\overline{U}, C) \) and let \( H: [0, 1] \times \overline{U} \to C \) be a homotopy in \( K_{u}(\overline{U}, C) \) with \( H_{0} = S \) and \( H_{1} = T \). In addition assume the following conditions are satisfied:

\[
\begin{align*}
&\text{(3.1)} & &\text{for any sequence } \{x_{n}\}_{n=1}^{\infty} \subseteq U \text{ and any } \lambda \in [0, 1] \text{ with } d(x_{n}, H(\lambda, x_{n})) \to 0 \text{ as } n \to \infty, \text{ we have that } \{x_{n}\} \text{ is a Cauchy sequence} \\
&\text{(3.2)} & &H(t, \cdot) \text{ is continuous for each } t \in [0, 1] \\
&\text{(3.3)} & &\text{for any } x_{0} \in U \text{ there exists an } \varepsilon > 0 \text{ such that if } d(x_{0}, H(\lambda, x_{0})) < \varepsilon \\
& & &\text{for some } \lambda \in [0, 1], \text{ then } H(\lambda, \cdot) \text{ has a fixed point in } U.
\end{align*}
\]

If \( S \) is essential then so is \( T \).

**Proof.** Consider the set \( A = \{ \lambda \in [0, 1]: x = H_{\lambda}(x) \text{ for some } x \in U \} \).

Notice \( A \) is nonempty since \( 0 \in A \), i.e., \( H_{0} = S \) is essential. We will show that \( A \) is both open and closed in \([0, 1]\) and hence by connectedness \( A = [0, 1] \). As a result all the maps are essential so in particular \( H_{1} = T \) is essential.

We first show that \( A \) is closed in \([0, 1]\). To see this let \( \{\lambda_{n}\}_{n=1}^{\infty} \subseteq A \) with \( \lambda_{n} \to \lambda \in [0, 1] \) as \( n \to \infty \). We now show \( \lambda \in A \). Since \( \lambda_{n} \in A \) there exists \( x_{n} \in U \) with \( x_{n} = H_{\lambda_{n}}(x_{n}) \). Also \( d(x_{n}, H_{\lambda_{n}}(x_{n})) = d(H_{\lambda_{n}}(x_{n}), H_{\lambda_{n}}(x_{n})) \to 0 \) as \( n \to \infty \) since \( H_{\lambda}(u) \) is continuous in \( t \), uniformly for \( u \in \overline{U} \). Now assumption (3.1) implies \( \{x_{n}\} \) is a Cauchy sequence. By the completeness of \( X \) there exists \( x \in U \) with \( x_{n} \to x \). In addition since

\[
\begin{align*}
d(H(\lambda_{n}, x_{n}), H(\lambda, x)) \\
&\leq d(H(\lambda_{n}, x_{n}), H(\lambda, x_{n})) + d(H(\lambda, x_{n}), H(\lambda, x))
\end{align*}
\]

...
then (3.2) together with the fact that $H_i(u)$ is continuous in $t$, uniformly for $u \in \overline{U}$ implies

$$d(\lambda_n, H(\lambda, x)) = d(\lambda, H(\lambda, x)) \to 0.$$ 

Thus $x = H(\lambda, x)$ and since $H_i \in K_{\mathcal{L}}(\overline{U}, C)$ then $x \in U$. Consequently $\lambda \in A$ and so $A$ is closed in $[0, 1]$.

Next we show $A$ is open in $[0, 1]$. Let $\lambda_0 \in A$. Then there exists $z_0 \in U$ with $z_0 = H(\lambda_0, z_0)$. Let $\varepsilon > 0$ be chosen as in condition (3.3). Since $H_i(u)$ is continuous in $t$, uniformly for $u \in \overline{U}$ there exists a neighborhood $\Omega$ of $\lambda_0$ in $[0, 1]$ such that $d(z_0, H(\lambda, z_0)) < \varepsilon$ for all $\lambda \in \Omega$. Fix $\lambda \in \Omega$. Now condition (3.3) implies that there exists $x \in U$ with $H(\lambda, x) = x$. Thus $\lambda \in A$ so $\Omega \subseteq A$ and hence $A$ is open in $[0, 1]$.

Assumption (3.2) may be relaxed if we are in a Banach space setting.

**Theorem 3.2.** Let $X$ be a Banach space. Assume $T \sim S$ in $K_{\mathcal{L}}(\overline{U}, C)$ and let $H: [0, 1] \times \overline{U} \to C$ be a homotopy in $K_{\mathcal{L}}(\overline{U}, C)$ with $H_0 = S$ and $H_1 = T$. In addition assume (3.1), (3.3) (of course $d$ is the metric induced by the norm), and

$$H(t, \cdot) \text{ is demicontinuous for each } t \in [0, 1]$$

hold. If $S$ is essential then so is $T$.

**Proof.** Let $A$ be as in Theorem 3.1. Now $A \neq \emptyset$ and $A$ (as in Theorem 3.1) is open in $[0, 1]$. We now show $A$ is closed in $[0, 1]$. Let $(\lambda_n)_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0, 1]$ as $n \to \infty$. Now (as in Theorem 3.1) there exists $x \in U$ with $x_n \to x$. Let $f \in X^*$ (the dual of $X$). Then

$$f(H(\lambda_n, x_n) - H(\lambda, x)) = f(H(\lambda_n, x_n) - H(\lambda, x_n)) + f(H(\lambda, x_n) - H(\lambda, x))$$

so (3.4) together with the fact that $H_i(u)$ is continuous in $t$, uniformly for $u \in \overline{U}$ implies

$$x_n = H(\lambda_n, x_n) \to H(\lambda, x);$$

here $\to$ denotes weak convergence. Thus $x = H(\lambda, x)$ and $x \in U$ since $H_\lambda \in K_{\mathcal{L}}(\overline{U}, C)$. Consequently $A$ is closed in $[0, 1]$.

We next use Theorems 3.1 and 3.2 to obtain results of Leray–Schauder type. Our results are given in the case when $X$ is a Banach space; however, it is worth remarking that there is an obvious formulation in the case when $X$ is a Fréchet space.
THEOREM 3.3. Let $U$ be an open subset of a convex set $C$ in a Banach space $(X, ||.||)$. Assume $0 \in U$, $T: \overline{U} \to C$ is a continuous mapping and $T(\overline{U})$ is bounded. In addition suppose

$$\begin{cases}
\text{for any sequence } \{x_n\}_{n=1}^{\infty} \subseteq U \text{ and any } \lambda \in [0, 1] \text{ with } x_n - \lambda T(x_n) \to 0 \\
\text{as } n \to \infty, \text{we have that } \{x_n\} \text{ is a Cauchy sequence}
\end{cases}$$

are satisfied. Then either

(A1) $T$ has a fixed point in $\overline{U}$; or

(A2) there exists $\lambda \in (0, 1)$ and $u \in \partial U$ such that $u = \lambda T(u)$.

Proof. Let $H: [0, 1] \times \overline{U} \to C$ be given by $H(\lambda, u) = \lambda T(u)$. Notice $H_0 = 0$ and $H_1 = T$. Also clearly $H$ satisfies conditions (3.1), (3.2), and (3.3) in Theorem 3.1. In addition $H(t, u)$ is continuous in $t$, uniformly for $u \in \overline{U}$ since $T(\overline{U})$ is bounded and

$$||H(\lambda_1, u) - H(\lambda_2, u)|| = |\lambda_1 - \lambda_2| ||T(u)||; \quad \lambda_1, \lambda_2 \in [0, 1] \text { and } u \in \overline{U}. $$

Now either $H$ is a homotopy in $K_{dcl}(\overline{U}, C)$ (i.e., $H_\lambda \in K_{dcl}(\overline{U}, C)$ for all $0 \leq \lambda \leq 1$) or it is not. If $H$ is a homotopy in $K_{dcl}(\overline{U}, C)$ then Theorem 3.1, since $H_0$ is essential, implies that $H_1$ is essential so property (A1) holds. On the other hand if $H$ is not a homotopy in $K_{dcl}(\overline{U}, C)$ then $\lambda T$ must have a fixed point on $\partial U$ for some $\lambda \in [0, 1]$. Notice $\lambda \neq 0$ since $0 \in U$. Thus either $\lambda = 1$ (in which case $T$ has a fixed point on $\partial U$ so (A1) holds) or $0 < \lambda < 1$ (so (A2) holds).

Essentially the same reasoning as in Theorem 3.3 (except we use Theorem 3.2) yields the following result.

THEOREM 3.4. Let $U$ be an open subset of a convex set $C$ in a Banach space $(X, ||.||)$. Assume $0 \in U$, $T: \overline{U} \to C$ is a demicontinuous mapping and $T(\overline{U})$ is bounded. In addition suppose (3.5) and (3.6) are satisfied.
Then either

(A1) $T$ has a fixed point in $\overline{U}$; or

(A2) there exists $\lambda \in (0, 1)$ and $u \in \partial U$ such that $u = \lambda T(u)$.

Theorems 3.3 and 3.4 will now be used to establish results for nonlinear operators of accretive, nonlinear contractive, or nonexpansive type.

**Theorem 3.5.** Let $U$ be an open subset in a real Banach space $(X, \|\|)$. Assume $0 \in U$, $T: \overline{U} \to X$ is a continuous mapping and $T(U)$ is bounded. In addition suppose $I - T: \overline{U} \to X$ is strongly accretive; here $I$ is the identity mapping. Then either

(A1) $T$ has a fixed point in $U$; or

(A2) there exists $\lambda \in (0, 1)$ and $u \in \partial U$ such that $u = \lambda T(u)$.

**Proof.** Take $H$ as in Theorem 3.3. Assume $\lambda T \in K_{ac}(\overline{U}, X)$ for each $\lambda \in (0, 1)$; otherwise property (A2) occurs if $\lambda \in (0, 1)$ and property (A1) occurs if $\lambda = 1$. Now since $I - T$ is strongly accretive then there exists $\psi \in \mathcal{M}$ with

$$((I - T)(x) - (I - T)(y), x - y)_+ \geq \psi(\|x - y\|)\|x - y\| \quad \text{for all } x, y \in \overline{U}.$$ 

The result follows immediately from Theorem 3.1 (with the ideas in Theorem 3.3) once we show (3.5) and (3.6) are satisfied (so (3.1), (3.2), and (3.3) hold). First we show $I - \lambda T$ is strongly accretive for any $\lambda \in [0, 1]$. To see this notice since $(z_1 + \alpha z_2, z_2)_+ = (z_1, z_2)_+ + \alpha\|z_2\|^2$ (here $z_1, z_2 \in \overline{U}$ and $\alpha$ is a scalar) we have for $x, y \in \overline{U}$ that

$$((I - \lambda T)(x) - (I - \lambda T)(y), x - y)_+$$

$$= (\lambda((I - T)(x) - (I - T)(y)) + (1 - \lambda)(x - y), x - y)_+$$

$$= \lambda((I - T)(x) - (I - T)(y), x - y)_+ + (1 - \lambda)\|x - y\|^2$$

$$\geq \psi_\lambda(\|x - y\|)\|x - y\|,$$

where $\psi_\lambda(t) = \lambda \psi(t) + (1 - \lambda)t$.

To see that (3.5) holds let $(x_n)_{n=1}^\infty$ be a sequence in $U$ and $\lambda \in [0, 1]$ with $x_n - \lambda T(x_n) \to 0$ as $n \to \infty$. Then for $n, m \in \{1, 2, \ldots\}$ we have

$$\psi_\lambda(\|x_n - x_m\|)\|x_n - x_m\|$$

$$\leq (x_n - \lambda T(x_n) - x_m + \lambda T(x_m), x_n - x_m)_+$$

$$\leq (\|x_n - \lambda T(x_n)\| + \|x_m - \lambda T(x_m)\|)\|x_n - x_m\|.$$
Consequently \( \psi_n(\|x_n - x_m\|) \to 0 \) as \( n, m \to \infty \). This together with the fact that \( \psi_n \) is strictly increasing (and \( \psi_n(0) = 0 \)) implies that \( (x_n) \) is a Cauchy sequence.

It remains to show (3.6). Let

\[
\varepsilon = \inf\{\|x - \lambda T(x)\| : \lambda \in [0, 1] \text{ and } x \in \partial U\}.
\]

We will first show that \( \varepsilon > 0 \). If this is not true then there exists \( t_k \in [0, 1] \) and \( x_k \in \partial U \) with \( x_k - \lambda_k T(x_k) \to 0 \). Without loss of generality we may assume \( \lambda_k \to \lambda \in [0, 1] \) as \( k \to \infty \). Then \( x_k - \lambda T(x_k) \to 0 \) as \( k \to \infty \) since \( tT(u) \) is continuous in \( t \), uniformly for \( u \in \overline{U} \) (this follows since \( T(\overline{U}) \) is bounded). Hence (as above) for \( m, n \in \{1, 2, \ldots \} \) we have

\[
\psi_n(\|x_n - x_m\|)\|x_n - x_m\| \\
\leq (\|x_n - \lambda T(x_n)\| + \|x_m - \lambda T(x_m)\|)\|x_n - x_m\|,
\]

and so \( (x_n) \) is a Cauchy sequence in \( \partial U \). Consequently there exists \( x \in \partial U \) with \( x_n \to x \). Also since

\[
\lambda T(x_n) - \lambda T(x) = (\lambda_n - \lambda)T(x_n) + \lambda[T(x_n) - T(x)]
\]

we have that \( x = \lambda T(x) \). Now \( \lambda \notin (0, 1) \) since \( tT \in K_{id}^{+}(\overline{U}, X) \) for \( 0 < t \leq 1 \). Also \( \lambda \neq 0 \) since \( x \in \partial U \) and \( 0 \in U \). Thus we have a contradiction so \( \varepsilon > 0 \). Now take \( x_0 \in U \) and suppose \( \|x_0 - \lambda T(x_0)\| < \varepsilon \) for some \( \lambda \in [0, 1] \). Then

\[
\|x_0 - \lambda T(x_0)\| \leq \|x - \lambda T(x)\| \quad \text{for all } x \in \partial U.
\]

Theorem 2.2 implies that \( \lambda T(.) \) has a fixed point in \( U \) so (3.6) is satisfied. Theorem 3.1 implies that \( F \) is essential and we are finished. \( \blacksquare \)

Essentially the same reasoning as in Theorem 3.5 (except we use Theorem 3.2, with the ideas in Theorem 3.4, and Theorem 2.1) yields the following result (which was proved in [13]).

**Theorem 3.6.** Let \( U \) be an open subset in a Hilbert space \( X \). Assume \( 0 \in U, T: \overline{U} \to X \) is a demicontinuous mapping and \( T(\overline{U}) \) is bounded. In addition suppose \( I - T: \overline{U} \to X \) is strongly monotone. Then either

(A1) \( T \) has a fixed point in \( \overline{U} \); or

(A2) there exists \( \lambda \in (0, 1) \) and \( u \in \partial U \) such that \( u = \lambda T(u) \).

Theorem 3.6 immediately yields a result for monotone operators (see [13]).
THEOREM 3.7 [13]. Let \( X \) be a Hilbert space, \( \{x_n\}_{n=1}^{\infty} \) a bounded sequence in \( X \) with

\[
\Re \left( \frac{1}{n} x_n - \frac{1}{m} x_m, x_n - x_m \right) \leq 0 \quad \text{for all } n, m \in \{1, 2, \ldots \}.
\]

Then \( \{x_n\} \) converges strongly to some \( x \in X \).

THEOREM 3.8. Let \( U \) be a bounded open subset in a Hilbert space \( X \).
Assume \( 0 \in U, T : \overline{U} \to X \) is a demicontinuous mapping and \( T(U) \) is bounded.
In addition suppose \( I - T : \overline{U} \to X \) is monotone. Then either

(A1) \( T \) has a fixed point in \( \overline{U} \); or
(A2) there exists \( \lambda \in (0, 1) \) and \( u \in \partial U \) such that \( u = \lambda T(u) \).

Proof. Assume property (A2) does not occur. Consider for each \( n \in \{1, 2, \ldots \} \) the mapping

\[
S_n = \left(1 - \frac{1}{n}\right) I - (I - T) : \overline{U} \to X.
\]

Notice \( I - S_n \) is strongly monotone since \( I - T \) monotone implies

\[
\Re \langle (I - S_n)(x) - (I - S_n)(y), x - y \rangle \geq \frac{1}{n} \|x - y\|^2; \quad x, y \in \overline{U}.
\]

Apply Theorem 3.6 to \( S_n \) and we deduce that either \( S_n \) has a fixed point in \( \overline{U} \) or there exists \( \lambda \in (0, 1) \) and \( u \in \partial U \) with \( u = \lambda S_n(u) \).

Suppose there exists \( \lambda \in (0, 1) \) and \( u \in \partial U \) with \( u = \lambda S_n(u) \). Then

\[
u = \frac{n \lambda}{n + \lambda} T(u) = \eta T(u), \quad \text{where } 0 < \eta = \frac{n \lambda}{n + \lambda} < 1,
\]

a contradiction since property (A2) does not occur. Consequently for each \( n \in \{1, 2, \ldots \} \) we have that \( S_n \) has a fixed point \( u_n \in \overline{U} \).

Let \( n, m \in \{1, 2, \ldots \} \) and consider \( u_n = S_n(u_n) \) (i.e., \( 1/n)u_n = -(I - T)(u_n) \)) and \( u_m = S_m(u_m) \). Since \( I - T \) is monotone we have

\[
\Re \left( \frac{1}{n} u_n - \frac{1}{m} u_m, u_n - u_m \right) = -\Re \langle (I - T)(u_n) - (I - T)(u_m), u_n - u_m \rangle \leq 0.
\]
Theorem 3.7 implies that there exists \( u \in \overline{U} \) with \( u_n \to u \). Now since
\[
\left(1 + \frac{1}{n}\right)u_n = T(u_n) \to T(u)
\]
we have \( u = T(u) \), so (A1) is satisfied.

**Remark.** In fact there is an analogue of Theorem 3.6 if \( X^* \) is a uniformly convex Banach space.

Next we obtain a Leray–Schauder type result for contractive type maps in a Banach space. One could use Theorem 3.5 to obtain an existence result. We will however give a direct proof using the Boyd–Wong theorem; this has the added advantage in that we need not assume \( \psi(z) = z - \phi(z) \to \infty \) as \( z \to \infty \) (here \( \phi \) is as described in Theorem 3.9) as we did in Theorem 3.5.

**Theorem 3.9.** Let \( U \) be an open subset of a convex set \( C \) in a Banach space \( (X, \|\cdot\|) \). Assume \( 0 \in U \) and \( T: \overline{U} \to C \) is a nonlinear contraction (i.e., there exists a continuous function \( \phi: [0, \infty) \to [0, \infty) \) satisfying \( \phi(z) < z \) for \( z > 0 \) such that \( \|T(x) - T(y)\| \leq \phi(\|x - y\|) \) for all \( x, y \in \overline{U} \)). In addition assume \( T(\overline{U}) \) is bounded and that there exists \( \theta_0 > 0 \) with \( \phi \) nondecreasing on \([0, \theta_0]\). Then either

1. \( T \) has a fixed point in \( \overline{U} \); or
2. there exists \( \lambda \in (0, 1) \) and \( u \in \partial U \) such that \( u = \lambda T(u) \).

**Remark.** For applications in integral and differential equations usually \( \overline{U} \) is bounded.

**Proof.** Let \( H \) be as in Theorem 3.3. The proof follows immediately from Theorem 3.3 once we show (3.5) and (3.6) are satisfied. To see that (3.5) is satisfied let \( (x_n)_{n=1}^\infty \) be a sequence in \( U \), \( \lambda \in [0, 1] \) with \( x_n - \lambda T(x_n) \to 0 \) as \( n \to \infty \). Now \( (x_n) \) is bounded since \( T(\overline{U}) \) is bounded and \( x_n - \lambda T(x_n) \to 0 \) as \( n \to \infty \). Suppose \( (x_n) \) is not Cauchy. Then we can find \( \delta > 0 \) and two sequences of integers \( (m(k)) \) and \( (n(k)) \), \( m(k) \geq n(k) \geq k \) with
\[
r_k = \|x_{n(k)} - x_{m(k)}\| \geq \delta, \quad k = 1, 2, \ldots \tag{3.7}
\]
Also
\[
r_k \leq \|x_{n(k)} - \lambda T(x_{m(k)})\| + \lambda\|T(x_{n(k)}) - T(x_{m(k)})\|
+ \lambda\|T(x_{m(k)}) - x_{m(k)}\|
\leq \|x_{n(k)} - \lambda T(x_{n(k)})\| + \phi(r_k) + \|T(x_{m(k)}) - x_{m(k)}\|
\]
and so

\[ 0 \leq \Phi(r_k) = r_k - \phi(r_k) \leq \|x_{n(k)} - \lambda T(x_{n(k)})\| + \|\lambda T(x_{m(k)}) - x_{m(k)}\|. \]

This together with the fact that \( x_n - \lambda T(x_n) \to 0 \) as \( n \to \infty \) yields

\[ \lim_{k \to \infty} \Phi(r_k) = 0. \quad (3.8) \]

Since \( \{x_n\} \) is bounded there exists a constant \( M_0 \) independent of \( k \) with \( \delta \leq r_k \leq M_0 \) for \( k = 1, 2, \ldots \). In this case

\[ \Phi(r_k) \geq \min_{x \in [\delta, M_0]} \Phi(x) = \Phi(x_0) > 0 \quad \text{for some} \ x_0 \in [\delta, M_0]. \]

This contradicts (3.8). Consequently \( \{x_n\} \) is Cauchy so (3.5) is true.

It remains to show (3.6). Now take \( x_0 \in U \). Choose \( \theta > 0, \varepsilon > 0 \) so that \( \varepsilon < \text{dist}(x_0, \partial U), \theta < \theta_0, \theta < \varepsilon \), and \( \phi(\theta) + \varepsilon = \theta \) (this is possible since \( \phi \) is continuous, \( \phi(0) = 0 \), and \( \phi(x) < x \) for \( x > 0 \)). Suppose \( \|x_0 - \lambda T(x_0)\| < \varepsilon \) for some \( \lambda \in [0, 1] \). Let \( B_\theta(x_0) = \{x: \|x - x_0\| \leq \theta\} \). For \( x \in B_\theta(x_0) \) we have since \( \phi \) is nondecreasing on \([0, \theta_0] \) that

\[ \|\lambda T(x) - x_0\| \leq \|\lambda T(x) - \lambda T(x_0)\| + \varepsilon \leq \phi(\|x - x_0\|) + \varepsilon \]

\[ \leq \phi(\theta) + \varepsilon = \theta \]

so \( \lambda T: B_\theta(x_0) \to B_\theta(x_0) \). Theorem 2.3 implies that \( \lambda T(.) \) has a fixed point in \( B_\theta(x_0) \subseteq U \) so (3.6) is satisfied.

Theorem 3.9 immediately yields a result for nonexpansive maps. First recall a well known result for nonexpansive maps [3, p. 103].

**Theorem 3.10.** Let \( X \) be a uniformly convex Banach space, \( \Omega \) a bounded closed convex subset of \( X \), and \( T: \Omega \to X \) a nonexpansive mapping. If \( \{u_n\} \) is a weakly convergent sequence in \( \Omega \) with weak limit \( u_0 \) and if \( (I - T)(u_n) \) converges strongly to an element \( w_0 \) in \( X \) then \( (I - T)(u_0) = w_0 \) (i.e., \( I - T \) is demiclosed in \( \Omega \)).

**Theorem 3.11.** Let \( U \) be a bounded, open, convex set in a uniformly convex Banach space \( (X, \|\|) \). Assume \( 0 \in U \) and \( T: \overline{U} \to X \) is a nonexpansive map. Then either

(A1) \( T \) has a fixed point in \( \overline{U} \); or

(A2) there exists \( \lambda \in (0, 1) \) and \( u \in \partial U \) such that \( u = \lambda T(u) \).

**Proof.** Assume property (A2) does not hold. Consider for each \( n \in \{2, 3, \ldots \} \) the mapping

\[ S_n = \left(1 - \frac{1}{n}\right)T: \overline{U} \to X. \]
Notice $S_n$ is a contraction with contraction constant $1 - 1/n$. Apply Theorem 3.9 (here $\phi(z) = (1 - 1/n)z$) to $S_n$ and we deduce that either $S_n$ has a fixed point in $U$ or there exists $\lambda \in (0,1)$ and $u \in \partial U$ with $u = \lambda S_n(u)$.

Suppose there exists $\lambda \in (0,1)$ and $u \in \partial U$ with $u = \lambda S_n(u)$. Then

$$u = \lambda \left(1 - \frac{1}{n}\right) T(u) = \eta T(u), \quad \text{where } 0 < \eta = \lambda \left(1 - \frac{1}{n}\right) < 1,$$

a contradiction since property (A2) does not occur. Consequently for each $n \in \{1,2,\ldots\}$ we have that $S_n$ has a fixed point $u_n \in \overline{U}$. A standard result in functional analysis (if $E$ is a reflexive Banach space then any norm bounded sequence in $E$ has a weakly convergent subsequence) implies (since $\overline{U}$ is closed, bounded, and convex (so weakly closed)) that there exists a subsequence $S$ of integers and a $u \in \overline{U}$ with

$$u_n \to u \quad \text{as } n \to \infty \text{ in } S.$$

In addition since $u_n = (1 - 1/n) T(u_n)$ we have

$$\|(I - T)(u_n)\| = \frac{1}{n} \|T(u_n)\| \leq \frac{1}{n} (\|T(u_n) - T(0)\| + \|T(0)\|)$$

$$\leq \frac{1}{n} (\|u_n\| + \|T(0)\|).$$

Thus $(I - T)(u_n)$ converges strongly to 0. Theorem 3.10 implies $u = T(u)$ and as a result (A1) is satisfied.

We now use Theorems 3.9 and 3.11 to obtain new fixed point theorems for nonlinear contractive and nonexpansive maps.

**Theorem 3.12.** Let $E$ be a Hilbert space and $Q$ a closed, bounded, convex subset of $E$ with $0 \in Q$. Also suppose $T: Q \to E$ is a nonlinear contraction (i.e., there exists a continuous function $\phi: [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $\|T(x) - T(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in Q$). In addition assume $\phi$ is nondecreasing with

$$\frac{\phi(x)}{\|x\|} \geq \frac{\phi(y)}{\|y\|} \quad \text{for all } x, y \in Q.$$

Then $T$ has a fixed point.

(3.9)
Remark. Condition (3.9) was introduced for compact maps by Furi and Pera [7].

Proof. Define \( r: E \to Q \) by \( r(x) = P_Q(x) \), i.e., \( r \) is the nearest point projection on \( Q \). It is well known that \( r \) is continuous and in fact nonexpansive. Consider the set

\[ B = \{ x \in E : x = Tr(x) \}. \]

We first show \( B \) is nonempty. To see this we examine the map \( rT: Q \to Q \). Now \( rT \) is a nonlinear contraction since for \( x, y \in Q \) we have since \( r \) is nonexpansive

\[ \| rT(x) - rT(y) \| \leq \| T(x) - T(y) \| \leq \phi(\| x - y \|). \]

Thus Theorem 2.3 implies that \( rT \) has a fixed point, i.e., there exists \( y \in Q \) with \( rT(y) = y \). Hence \( z = Tr(z) \) with \( z = T(y) \) so \( B \neq \emptyset \). In addition the continuity of \( Tr \) implies that \( B \) is closed. We next claim that \( B \) is compact.

First notice if \( V \) is a bounded set in \( E \) then

\[ \alpha(T(\Omega)) \leq \phi(\alpha(\Omega)); \quad (3.10) \]

here \( \alpha \) is Kuratowski’s measure of noncompactness. To see (3.10) let \( \varepsilon > 0 \) and suppose \( \Omega \subseteq \bigcup_{i=1}^{n} \Omega_i \) with \( \text{diam}(\Omega_i) < \alpha(\Omega) + \varepsilon \). Now \( T(\Omega) \subseteq \bigcup_{i=1}^{n} T(\Omega_i) = \bigcup_{i=1}^{n} Y_i \). If \( w_0, w_1 \in Y_i \) for some \( i \) then there exists \( x_0, x_1 \in \Omega \) with \( T(x_0) = w_0 \) and \( T(x_1) = w_1 \) and so since \( \phi \) is non-decreasing we have

\[ \| T(x_0) - T(x_1) \| \leq \phi(\| x_1 - x_2 \|) \leq \phi(\alpha(\Omega) + \varepsilon). \]

Thus \( \text{diam}(Y_i) \leq \phi(\alpha(\Omega) + \varepsilon) \) and so \( \alpha(T(\Omega)) \leq \phi(\alpha(\Omega) + \varepsilon) \). Since \( \varepsilon > 0 \) is arbitrary then (3.10) follows. Now since \( B \subseteq Tr(B) \) we have

\[ \alpha(B) \leq \alpha(Tr(B)) \leq \phi(\alpha(r(B))). \quad (3.11) \]

If \( \alpha(r(B)) \neq 0 \) then

\[ \alpha(B) \leq \phi(\alpha(r(B))) < \alpha(r(B)) \leq \alpha(B) \]

since \( \phi(z) < z \) when \( z > 0 \), and \( r \) being nonexpansive implies \( \alpha(r(B)) \leq \alpha(B) \). This is a contradiction so \( \alpha(r(B)) = 0 \) and so (3.11) implies \( \alpha(B) \leq \phi(\alpha(r(B))) = \phi(0) = 0 \), i.e., \( \alpha(B) = 0 \). Hence \( B \) is compact.

We now show \( B \cap Q \neq \emptyset \). To do this we argue by contradiction. Suppose \( B \cap Q = \emptyset \). Then since \( B \) is compact and \( Q \) is closed there
exists \( \delta > 0 \) with \( \text{dist}(B, Q) > \delta \). Choose \( N \in \{1, 2, \ldots \} \) such that \( 1 < \delta N \).

Define
\[
U_i = \left\{ x \in E : d(x, Q) < \frac{1}{i} \right\} \quad \text{for } i \in \{N, N+1, \ldots\};
\]

here \( d \) denotes the metric induced by the norm. Fix \( i \in \{N, N+1, \ldots\} \).

Since \( \text{dist}(B, Q) > \delta \), then \( B \cap U_i = \emptyset \). Notice \( U_i \) is open and bounded (since \( Q \) is bounded). Also \( Tr: U_i \to E \) is a nonlinear contraction since for \( x, y \in U_i \) we have
\[
\|Tr(x) - Tr(y)\| \leq \phi(\|r(x) - r(y)\|) \leq \phi(\|x - y\|)
\]

since \( \phi \) is nondecreasing and \( r \) is nonexpansive. Now Theorem 3.9 implies that there exists \( (y_i, \lambda_i) \in \partial U_i \times (0, 1) \) with \( y_i = \lambda_i Tr(y_i) \).

**Remark.** Note \( A1 \) in Theorem 3.9 does not occur since \( B \cap U_i = \emptyset \).

Consequently for each \( j \in \{N, N+1, \ldots\} \) there exists \( (y_j, \lambda_j) \in \partial U_j \times (0, 1) \) with \( y_j = \lambda_j Tr(y_j) \). Notice in particular since \( y_j \in \partial U_j \) that
\[
\lambda_j Tr(y_j) \notin Q \quad \text{for } j \in \{N, N+1, \ldots\}. \quad (3.12)
\]

Next look at the set
\[
C = \{ x \in E : x = \lambda Tr(x) \text{ for some } \lambda \in [0, 1] \}.
\]

Clearly \( C \) is closed. Also \( C \) is compact (so sequentially compact). To see this notice
\[
C \subseteq \overline{\text{co}}(Tr(C) \cup \{0\}).
\]

If \( \alpha(r(C)) \neq 0 \) then
\[
\alpha(C) = \alpha(\overline{\text{co}}(Tr(C) \cup \{0\})) = \alpha(Tr(C)) \leq \phi(\alpha(r(C)))
\]
\[
< \alpha(r(C)) \leq \alpha(C).
\]

This is a contradiction so \( \alpha(r(C)) = 0 \) and so
\[
\alpha(C) \leq \phi(\alpha(r(C))) = \phi(0) = 0.
\]

Thus \( C \) is compact. This together with \( d(y_j, Q) = 1/j, |\lambda_j| \leq 1 \) (for \( j \in \{N, N+1, \ldots\} \)) implies that we may assume without loss of generality that \( \lambda_j \to \lambda^* \in [0, 1] \) and \( y_j \to y^* \in \partial Q \); also \( y_j = \lambda_j Tr(y_j) \to \lambda^* Tr(y^*) \) so \( y^* = \lambda^* Tr(y^*) \). If \( \lambda^* = 1 \) then \( y^* = Tr(y^*) \) which contradicts \( B \cap Q = \emptyset \). Hence we may assume \( 0 \leq \lambda^* < 1 \). But in this case, (3.9) with \( x_j = \)
$r(y_j) \in \partial Q$ and $x = y^* = r(y^*)$ implies $\lambda_j Tr(y_j) \in Q$ for $j$ sufficiently large. This contradicts (3.12). Thus $B \cap Q \neq \emptyset$ so there exists $x \in Q$ with $x = Tr(x)$, i.e., $x = T(x)$. 

**Remark.** If $0 \in \text{int}(Q)$ and $E$ is a Banach space then the result of Theorem 3.12 is again true. In this case the proof involves showing that condition (A2) in Theorem 3.9 does not hold.

**Theorem 3.13.** Let $E$ be a Hilbert space and $Q$ a closed, bounded, convex subset of $E$ with $0 \in Q$. Also suppose $T: Q \rightarrow E$ is a nonexpansive mapping and that (3.9) holds. Then $T$ has a fixed point.

**Proof.** For each $n \in \{2, 3, \ldots\}$ consider the mapping

$$S_n = \left(1 - \frac{1}{n}\right)T: Q \rightarrow E.$$ 

Now $S_n$ is a contraction. Let $((x_j, \lambda_j))_{j=1}^\infty$ be a sequence in $\partial Q \times [0, 1]$ converging to $(x, \lambda)$ with $x = \lambda S_n(x)$ and $0 \leq \lambda < 1$. Then

$$\lambda_j S_n(x_j) = \lambda_j \left(1 - \frac{1}{n}\right)T(x_j) = \mu_j T(x_j) \in Q \quad \text{for } j \text{ sufficiently large}$$

since $T$ satisfies (3.9) (note $\mu_j = \lambda_j(1 - 1/n)$ is a sequence in $[0, 1]$ with $\mu_j \rightarrow \lambda(1 - 1/n) = \mu$, $0 \leq \mu < 1$, and $x = \lambda S_n(x) = \lambda(1 - 1/n)T(x) = \mu T(x)$). Apply Theorem 3.12 to $S_n$ to deduce that $S_n$ has a fixed point $u_n \in Q$. Now (as in Theorem 3.11) there exists a subsequence $S$ of integers and a $u \in Q$ with $u_n \rightarrow x$ as $n \rightarrow \infty$ in $S$. In addition (as in Theorem 3.11) we have that $(I - T)(u_n)$ converges strongly to 0. Theorem 3.10 now implies $u = T(u)$. 

**4. APPLICATION**

The fixed point theory in Section 3 can be used to establish existence principles for the second order boundary value problems in abstract spaces. To illustrate the ideas involved we will use Theorem 3.8 to examine

$$\begin{cases}
y'' + f(t, y, y') = 0, \quad 0 \leq t \leq 1 \\
y(0) = y(1) = 0,
\end{cases}$$

where $f: [0, 1] \times H \times H \rightarrow H$ is continuous; here $H = (H, |.|)$ is a real Hilbert space. By a solution to (4.1) we mean a function $y \in C^2([0, 1], H)$ with $y$ satisfying the differential equation on $[0, 1]$ and the stated boundary conditions.
Consider the problem

\[
\begin{cases}
w'(t) + f(t, \int_0^t w(x) \, dx, w(t)) = 0, & 0 \leq t \leq 1 \\
\int_0^1 w(x) \, dx = 0.
\end{cases}
\] (4.2)

By a solution to (4.2) we mean a function \( w \in C^1([0, 1], H) \) with \( w' = -f(t, \int_0^t w(x) \, dx, w) \) on \([0, 1]\) and \( \int_0^1 w(x) \, dx = 0 \). Notice \( y \) is a solution of (4.1) iff \( w = y' \) is a solution of (4.2). As a result for the remainder of this section we will restrict ourselves to the problem (4.2). For notational purposes let

\[
L^2_0([0, 1], H) = \left\{ u \in L^2([0, 1], H) : \int_0^1 u(x) \, dx = 0 \right\}.
\]

Notice \( L^2_0([0, 1], H) \) is a closed subspace of \( L^2([0, 1], H) \) and consequently \( L^2_0([0, 1], H) \) is a Hilbert space.

The following well known result will be needed to prove our existence principle.

**Theorem 4.1** [10, 12]. (i) (Wirtinger) Let \( u : [0, 1] \to H \) have a continuous derivative and satisfy \( u(0) = u(1) = 0 \). Then

\[
\pi^2 \int_0^1 |u(t)|^2 \, dt \leq \int_0^1 |u'(t)|^2 \, dt.
\]

(ii) (Opial) Let \( u : [0, 1] \to H \) have a continuous derivative and satisfy \( u(0) = u(1) = 0 \). Then

\[
4 \int_0^1 |u(t)||u'(t)| \, dt \leq \int_0^1 |u'(t)|^2 \, dt.
\]

**Theorem 4.2.** Assume the following conditions are satisfied:

\[
\begin{cases}
f : [0, 1] \times H \times H \to H \text{ is continuous and for each } r > 0 \text{ there exists} \\
h_r \in L^1[0, 1] \text{ such that } |f(t, u, v)| \leq h_r(t) \text{ for all } t \in [0, 1], \\
|u| \leq r, \text{ and } v \in \mathbb{R}
\end{cases}
\] (4.3)
and

\[ \begin{align*}
\begin{cases}
\text{there exists a subset } \Omega_0 \subseteq \mathbb{R}^2 \text{ and } a_0 \geq 0, b_0 \geq 0 \\
\quad \text{with } a_0 + b_0(\pi^2/4) \leq \pi^2 \text{ such that}
\quad \langle f(x, u_0, v_0) - f(x, u_1, v_1), u_0 - u_1 \rangle \\
\quad \quad \leq a_0|u_0 - u_1|^2 + b_0|u_0 - u_1||v_0 - v_1|
\end{cases}
\end{align*} \tag{4.4} \]

for all \( t \in [0, 1] \) and \((u_0, v_0), (u_1, v_1) \in \Omega_0 \).

In addition suppose there exists a constant \( M_0 \), independent of \( \lambda \), with

\[ ||w||_{L^2} \neq M_0 \]

for any solution \( w \) to

\[
\begin{align*}
\begin{cases}
w'(t) + \lambda f(t, \int_0^t w(x) \, dx, w(t)) = 0, & 0 \leq t \leq 1 \\
\int_0^1 w(x) \, dx = 0
\end{cases}
\end{align*} \tag{4.5}_\lambda
\]

for each \( \lambda \in (0, 1) \). Let

\[ U = \{ u \in L^2_0([0, 1], H) : ||u||_{L^2} < M_0 \} \]

and assume the following condition holds:

\[ u \in \overline{U} \text{ implies } \left( \int_0^1 u(x) \, dx, u(t) \right) \in \Omega_0 \quad \text{for a.e. } t \in [0, 1]. \tag{4.6} \]

Then (4.2) has a solution.

**Proof.** Define \( T : \overline{U} \to X = L^2_0([0, 1], H) \) by

\[ Tw(x) = \int_0^1 \int_0^t f \left( s, \int_0^s w(x) \, dx, w(s) \right) \, ds \, dt \]

\[ - \int_0^t f \left( s, \int_0^s w(x) \, dx, w(s) \right) \, ds. \tag{4.7} \]

Notice \( T \) is well defined because of (4.3). Also \( T : \overline{U} \to X \) is continuous. This follows from (4.3), the Lebesgue dominated convergence theorem (version where convergence almost everywhere is replaced by convergence in measure) and a result of Nemytskii. We claim that \( T \) has a fixed point \( w \in \overline{U} \). If the claim is true then \( w \) is a solution of (4.2) and we are finished.

To show \( T \) has a fixed point in \( \overline{U} \) we will apply Theorem 3.8. Firstly (4.3) together with (4.7) implies that \( T : \overline{U} \to X \) is continuous and that \( T(\overline{U}) \) is bounded. Next we show that \( I - T : \overline{U} \to X \) is monotone. Let \( u, v \in \overline{U} \).
Then
\[
\langle Tu - Tv, u - v \rangle \\
= - \int_0^1 \left[ \int_0^s \left[ f(s, \int_0^x u(s)) \right] ds, u(t) - v(t) \right] dt \\
- \left[ f(s, \int_0^x v(s)) \right] ds, u(t) - v(t) \right] dt \quad (4.8)
\]
since \( \int_0^1 [u(t) - v(t)] dt = 0 \) implies
\[
\int_0^1 \int_0^1 \int_0^s \left[ f(s, \int_0^x u(z) dz, u(s)) \right] \\
- f(s, \int_0^x v(z) dz, v(s)) \right] ds dx, u(t) - v(t) \right] dt = 0.
\]
Interchange the order of integration in (4.8) to obtain
\[
\langle Tu - Tv, u - v \rangle \\
= - \int_0^1 \left[ f(s, \int_0^x u(s)) \\
- f(s, \int_0^x v(s)) \right] \int_0^1 [u(t) - v(t)] dt \right] ds \\
= \left[ f(s, \int_0^x u(s)) - f(s, \int_0^x v(s)) \right] \int_0^s u(t) dt - \int_0^s v(t) dt \right] ds
\]
since \( \int_0^1 [u(t) - v(t)] dt = 0 \). This together with condition (4.3) and Lemma 4.1 will give
\[
\langle Tu - Tv, u - v \rangle \leq a_0 \int_0^1 \left| \int_0^s u(t) dt - \int_0^s v(t) dt \right|^2 ds \\
+ b_0 \int_0^1 \left| \int_0^s u(t) dt - \int_0^s v(t) dt \right| |u(s) - v(s)| ds \\
\leq \frac{a_0}{\pi^2} \|u - v\|_{L^2}^2 + \frac{b_0}{4} \|u - v\|_{L^2}^2.
\]
Consequently for \( u, v \in \overline{U} \) we have

\[
\langle (I - T)u - (I - T)v, u - v \rangle \geq \left( 1 - \frac{a_0}{\pi^2} - \frac{b_0}{4} \right) \|u - v\|_2^2 \geq 0 \quad (4.9)
\]

so \( I - T: \overline{U} \to X \) is monotone. Hence all the conditions in Theorem 3.8 are satisfied. Finally notice that condition (A2) cannot occur since if there exists a \( \lambda_0 \in (0, 1) \) and \( u \in \partial U \) with \( u = \lambda_0 T(u) \) then \( \|u\|_2^2 = M_0 \) and \( u \) is a solution of \((4.5)_0\), which is a contradiction. Thus \( T \) has a fixed point in \( \overline{U} \).

**Remarks.**

(i) There is an obvious analogue of Theorem 4.2 in the case when \( f: [0, 1] \times H \times H \to H \) is a \( L^1 \)-Carathéodory function.

(ii) Existence of a solution to (4.2) (if \( f \) satisfies (4.3) and (4.4)) reduces to obtaining a constant \( M_0 \), independent of \( \lambda \), as described in Theorem 4.2 (notice if \( a_0 + b_0(\pi^2/4) < \pi^2 \) then (4.9) with \( u = w \) and \( v = 0 \) implies that there exists a constant \( M_1 \), independent of \( \lambda \), with \( \|w\|_2 \leq M_1 \) for any solution \( w \) to (4.5)).

(iii) The results in Mawhin’s well known paper [11] follow from the above theorem. The idea is to construct a modified problem (whose solution is also a solution of Mawhin’s problem) so that Theorem 4.2 can be applied (in [11] we have \( \Omega_0 = \mathbb{R}^2 \)).

**REFERENCES**


