Three-dimensional equilibria of nonlinear pre-curved beams using an intrinsic formulation and shooting

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ABSTRACT

This article describes a shooting method for computing three-dimensional equilibria of pre-curved nonlinear beams with axial and shear flexibility using the intrinsic beam formulation. For distributed and concentrated follower loads acting on a cantilevered beam, the method amounts to a direct solution approach requiring only a single shot (zero iterations) to compute the equilibria. This is possible since the system equations are defined in a local coordinate system that rotates and translates with the beam, akin to the follower loads themselves. A general procedure employing nonconservative follower loads, which invokes the Picard–Lindelöf theorem on uniqueness and existence of solutions, is also introduced for finding all solutions for three-dimensional pre-curved beam problems with conservative loading. This is particularly useful in beam buckling problems where multiple stable and unstable solutions exist.

1. Introduction

Beams are common elements used in a variety of machine and structural elements. As a result, quantifying the nonlinear deformation of beams is of interest to several fields of engineering such as aerospace, mechanical, biomedical, and civil engineering. Several authors, including Timoshenko and Gere (1961), Barten (1944), Bisshopp and Drucker (1945), Frisch-Fay (1962), Argyris and Symeonidis (1981) and Pai and Lee (2002), have performed in-depth studies on the nonlinear deformation of beams. In particular, beam problems with nonconservative follower loads are a subset which have received renewed interest. The follower load problem was first introduced in the context of elastic stability by Nikolai (1928), Nikolai (1929) and was further studied by Ziegler (1953) and Bolotin (1963); however, this research was mainly viewed as a pure theoretical endeavor at the time (Herrmann, 1967). With advances in technology and material science, this category of beam problems was recognized to have practical engineering applications. The most prominent applications are in the medical and aerospace industries and include engine-supporting aircraft wings, robotic arms for spacecraft, and manipulators used in endoscopic surgery. A commonly observed follower load phenomena is the flutter instability experienced by aircraft wings due to aeroelastic effects. In robotic arms, follower moments exist at joints where servos control motion. With endoscopic manipulators, robotic catheters are in development which utilize shape memory alloys to control the motion of the beam-like catheter by applying distributed follower loads to the catheter body (Veeramani, 2009). Gaining greater insight into the nonlinear deformation of beams subjected to follower loads continues to aid in the design and control of these and other devices.

Solutions to large deformation beam problems have been investigated using several methods such as nonlinear finite element methods (Argyris and Symeonidis, 1981), iterative shooting methods (Wang and Kitipornchai, 1992; Navaee and Elling, 1992; Pai, 2011), the finite difference method (Al-Sadder and Al-Rawi, 2006; Gatti-Bono and Perkins, 2002), and less general analytical methods such as the elliptic integral formulation (Timoshenko and Gere, 1961). Numerical solutions are of particular interest since these solution methods are applicable to general problems. In this subset of solution methods, traditional finite element methods, finite difference methods and shooting methods require...
multiple iterations in order to solve the beam boundary value problem (BVP). The finite element method must update load and stiffness matrices as the geometry varies until the final load case is reached, which increases computational expense and simulation time (Hughes, 2000). Finite difference methods require convergence of a set of residuals with the use of a Newton–Raphson technique or may employ iterative shooting methods in order to solve a nonlinear BVP (Roberts and Shipman, 1972). Also, finite element and finite difference methods require a pre-processing step where the domain is discretized. For beams dependent on only a single spatial variable, iterative shooting methods are convenient since the equations fall neatly into the required equation format without requiring further modeling steps.

Iterative shooting methods are numerical techniques for solving BVP's posed with one independent variable. These methods solve systems of first-order ordinary differential equations (ODE's) over an interval \([x_0, x_1]\) where \(x_1\) is the independent variable. For the case of a beam with parameters and equilibrium equations dependent on \(x_1\), a set of fourth-order ODE's is reduced to a set of first-order ODE's and a set of boundary conditions is defined at both \(x_0\) and \(x_1\). With iterative shooting, this set of ODE's is treated as an initial value problem (IVP) in which multiple initial condition iterations at \(x_0\) are performed in order to correctly arrive at the final conditions at \(x_1\). In general, one cannot define a complete set of boundary conditions at \(x_0\) such that the requisite boundary conditions at \(x_1\) are satisfied. As a result, a minimization routine is typically invoked that varies the initial values at \(x_0\) until all requisite boundary conditions at \(x_1\) are satisfied (Roberts and Shipman, 1972). The varying of the initial values gives a different solution trajectory for each "shot," until the requisite boundary conditions are "hit" by the correct trajectory.

In contrast to traditional iterative shooting techniques for beam BVP's, Shvartsman introduced a method for solving planar cantilever beam problems subjected to follower loads by a direct shooting method (Shvartsman, 1999; Shvartsman, 2009; Shvartsman, 2007). This method uses variable substitution to arrive at a set of first-order ODE's and a set of boundary conditions is defined at both \(x_0\) and \(x_1\). The governing method is apparent when studying buckling beam problems: – all stable and unstable solutions can be found for conservatively-loaded BVP's as a result of the Picard–Lindelöf theorem guaranteeing the uniqueness of the solutions to IVP's with requisite smoothness. Due to this theorem, the follower load beam BVP defined by the intrinsic formulation gives unique solutions for unique sets of initial conditions. As such, conservatively-loaded beam BVP's are posed as follower load BVP's whose deformed configuration satisfy the conservative BVP's loading orientation. Many follower load orientations may exist that satisfy this criterion. By sampling the entire follower load orientation space, all solutions for conservatively-loaded problems are guaranteed to be found. Note that other authors (Navaee and Elling, 1993; Raboud et al., 2001; Batista and Kosel, 2005) have described alternate solution techniques for determining all solutions to initially-planar beam buckling problems, with (Batista and Kosel, 2005) describing an approach with the most similarity to ours in that it employs a root solver on a single algebraic equation, this algebraic equation being unrelated to any used herein.

In summary then, this paper contributes the following to computing the nonlinear deformation of beams:

- a general procedure for computing three-dimensional equilibria of pre-curved cantilever beams subjected to follower loads, without the need for iteration;
- a new procedure for determining all equilibrium solutions for three-dimensional beam BVP's with conservative loading;
- a three-dimensional shooting method incorporating first-order shear modeling, which is particularly relevant to the study of thick beams.

2. Intrinsic beam formulation

This section presents the intrinsic formulation's governing equations utilized in the shooting method. In addition, a linearly elastic constitutive law is introduced, followed by a description for implementing shooting on the equilibrium equations.

2.1. Kinematics

Three configurations are used in developing the shooting technique, as shown in Fig. 1. The reference configuration, \(\Omega_{ref}\), represents a straight beam with zero curvature or strain. The initial configuration, \(\Omega_0\), represents the beam in an unstressed configuration exhibiting initial curvature and strain, \(k_0\) and \(\sigma_0\). The curvature vector contains three components describing twist and bending relative to the intrinsic basis of the beam, \(B_{01}, B_{02}, B_{03}\). The strain vector contains the axial and cross-sectional shear strains (analogous to the shear strain in Timoshenko beam theory). The deformed configuration, \(\Omega_d\), represents the beam in a stressed
state where internal moments and forces are present. The final curvature and strain vectors, $K_0$ and $\gamma_i$, denote the change in curvature and strain for $\Omega_f$ relative to $\Omega_{ref}$, while the net change in curvature and strain, $K$ and $\gamma$, denote the change from $\Omega_0$ to $\Omega_f$.

The intrinsic beam model uses three basis sets for each of the configurations. In the $\Omega_{ref}$ configuration, the Cartesian unit vectors $[I_1, I_2, I_3]$ are used as the basis, while the two sets of basis vectors, $[B_{01}, B_{02}, B_{03}]$ and $[B_1, B_2, B_3]$, are used for $\Omega_0$ and $\Omega_f$, respectively. The basis vectors in $\Omega_0$ and $\Omega_f$ follow the cross-section of the beam and are defined such that $B_0$ and $B_1$ initially correspond with $I_1$ in $\Omega_{ref}$, while $B_0$ and $B_2$ align with $I_2$ in $\Omega_{ref}$. The basis vectors $B_0$ and $B_1$ are defined by $B_0 \times B_3$ and $B_1 \times B_2$, completing the orthonormal set. Note that $B_0$ and $B_1$ are not necessarily tangent to the centerline $R$ in $\Omega_0$ and $\Omega_f$ due to the presence of strain.

The spatial rate of change of the centerline position and the basis vectors implicitly define the strains and curvatures of the beam in both $\Omega_0$ and $\Omega_f$ (Yu and Hodges, 2005) as shown in Eqs. (1) and (2):

$$R' = (1 + \gamma_{11})B_1 + 2\gamma_{12}B_2 + 2\gamma_{13}B_3,$$  \hspace{1cm} (1)

$$B' = K \times B.$$  \hspace{1cm} (2)

In these expressions, the distance along the center line is denoted by $x_I$ and the spatial derivative with respect to $x_I$ is denoted by a prime. Note that these equations hold in $\Omega_0$ and $\Omega_f$ and the subscripts have been removed for brevity.

2.2. Equilibrium equations

An intrinsic beam formulation, developed by Hodges (2003), defines the three-dimensional equations of motion governing the temporal and spatial changes of the beam’s velocity, angular velocity, curvature and strain

$$F' + K_0 \times F + f = P + \Omega \times P$$  \hspace{1cm} (3)

and

$$M' + K_0 \times M + (e_1 + \gamma_1) \times F + m = H + \Omega \times H + V \times P,$$  \hspace{1cm} (4)

where an over-dot represents a partial derivative with respect to time; $F$ and $M$ denote internal forces and moments; $f$ and $m$ denote external forces and moments per unit length; $P$ and $H$ denote the linear and angular momentum per unit length corresponding to linear velocity $V$ and angular velocity $\Omega$; and $e_1$ denotes a unit vector in the $B_1$ direction $\left[ 1 \ 0 \ 0 \right]^T$. The following two equations are necessary constraint equations to complete the set of four equations for the four field variables $(K, \gamma, V, \Omega)$:

$$\Omega' + K_0 \times \Omega = \dot{\Omega}$$  \hspace{1cm} (5)

and

$$V' + K_0 \times V + (e_1 + \gamma_1) \times \Omega = \dot{\gamma}.$$  \hspace{1cm} (6)

The momenta and velocities of the beam are related using the mass per unit length $\mu$; cross-sectional mass moments and product of inertia $I_{1}, I_{2}$ and $I_{23}$; and the centroidal offsets from the center line $R_2$ and $R_3$ as given by Eq. (7):

$$F' + K_0 \times F + f = 0$$  \hspace{1cm} (8)

and

$$M' + K_0 \times M + (e_1 + \gamma_1) \times F + m = 0$$  \hspace{1cm} (9)

For determination of equilibria, these equations simplify greatly by removing time dependent terms and derivatives. As a result, the equilibrium equations are given by

$$F + K_0 \times F + f = 0$$  \hspace{1cm} (10)

and

$$M + K_0 \times M + (e_1 + \gamma_1) \times F + m = 0$$  \hspace{1cm} (11)

where $F$ and $M$ denote internal forces and moments; $f$ and $m$ denote external forces and moments per unit length; and $e_1$ denotes a unit vector in the $B_1$ direction $\left[ 1 \ 0 \ 0 \right]^T$. A constitutive law relating the internal forces and moments to curvature and strain completes the formulation. For all beams considered herein, the initial curvatures are small-enough to warrant a decoupled, linearized constitutive model (Leamy, 2012) of the form,

$$\{ F \} = [D] \{ \gamma \}$$  \hspace{1cm} (12)

where

$$[D] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2GJ/k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EI_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix}$$  \hspace{1cm} (13)

In Eq. (11), $E$ denotes the elastic modulus, $G$ denotes the shear modulus defined by $G = E/(2(1 + \nu))$, $\nu$ denotes Poisson’s ratio, $GJ$ denotes the torsional rigidity, $I_2$ and $I_3$ denote the cross-section’s area moments of inertia, and $k$ denotes a shear correction factor based on the cross-section’s shape. This constitutive law is valid for a beam with a symmetric cross-section composed of an isotropic material. For full constitutive modeling of more complex beam compositions, the reader is referred to Berdichevskii and Starolets’kii (1983), Kovvali and Hodges (2012). Note that in cases where strain is negligible, only the curvature-moment relations are required from Eq. (10). As a result, the constitutive law becomes

$$\{ M \} = [D] \{ \dot{K} \}$$  \hspace{1cm} (14)
The appropriate values for \( \mathbf{m} \) and \( \mathbf{f} \) in Eqs. (8) and (9) apply distributed follower loads to the beam. These distributed loads can be constant or defined as functions of \( x_i \). Furthermore, non-prismatic cantilever beams are easily modeled when the material properties and beam dimensions vary as a function of \( x_i \). The appropriate functions defining these beam parameters are implemented in the constitutive law given by Eq. (11). The equilibrium equations automatically enforce the fixed boundary condition at the end of the cantilever. After numerical integration of the \( \mathbf{K} \) and \( \mathbf{f} \) ODE set, the results are used to integrate the \( \mathbf{R} \) and \( \mathbf{B} \) equations from the fixed end of the beam yielding the deformed configuration in the global basis. The displacements of the beam are calculated with

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix} = \mathbf{R}^T - \mathbf{R}_0 (18)
\]

where \( u \) is the global X-displacement, \( v \) is the global Y-displacement and \( w \) is the global Z-displacement. The orientation of the follower loads in the global basis is given by

\[
\begin{bmatrix}
  L_x \\
  L_y \\
  L_z
\end{bmatrix} = \begin{bmatrix}
  \mathbf{B}_f & \mathbf{B}_i & \mathbf{B}_{fi}
\end{bmatrix} \begin{bmatrix}
  L_1 \\
  L_2 \\
  L_3
\end{bmatrix} (19)
\]

where \( \mathbf{L}_i \) denotes the global force or moment vector and \( \mathbf{L}_i \) denotes the applied follower force or moment vector. The flow chart in Fig. 2 outlines this procedure for modeling cantilever beams with follower loading.

This shooting method also generates solutions to general beam BVP’s with any combination of boundary conditions, conservative loads and nonconservative loads; however, iterative shooting must be utilized. Fig. 3 illustrates the procedure for solving general problems. First, the problem is converted into a series combination of cantilever beams. For general BVP’s, the reaction loads caused by imposed boundary conditions and loading are modeled as external loads on a cantilever beam. For example, Fig. 3 depicts a fixed–fixed beam transversely loaded with conservative force \( \mathbf{F}_0 \) located at \( x_1 = \zeta \). This BVP is modeled as a cantilever beam with a reaction force \( \mathbf{F}_1 \) and moment \( \mathbf{M}_0 \) on the free end in addition to \( \mathbf{F}_0 \) at \( x_1 = \zeta \). The next step involves decomposing the original BVP into the two cantilever beam problems denoted by Beam 1 and Beam 2. The initial values for Beam 1 are an initial guess for \( \mathbf{F}_2 \) and \( \mathbf{M}_0 \), while the initial values for Beam 2 include the arrived–at values from Beam 1, namely \( \mathbf{K}_2(\zeta) \) and \( \mathbf{f}_2(\zeta) \), along with an initial guess for the orientation of \( \mathbf{F}_0 \). An initial guess is required for \( \mathbf{F}_0 \) since conservative loads are modeled as follower loads that are rotated in \( \Omega_k \) until the follower load in \( \Omega_k \) is oriented in the direction of the desired conservative load. An optimization algorithm then determines the values for \( \mathbf{F}_2 \) and \( \mathbf{M}_0 \), and the correct orientation of \( \mathbf{F}_0 \) on Beam 2, which satisfy the original beam BVP. This optimization minimizes a cost function involving the deformed configuration’s adherence to boundary conditions and the \( \Omega_k \)-orientation of the applied external forces. The actual form of the cost function used in the optimization depends on the imposed boundary conditions and loads. Modification of this cost function allows modeling of any general beam BVP.

This iterative method for general beam BVP’s possesses the ability to determine all solutions to the beam problem of interest for a single load case. The Picard–Lindelöf theorem guarantees the existence and uniqueness of the solutions to an initial-value ODE set satisfying certain continuity prerequisites, whether it be linear or nonlinear (Edwards and Penney, 2004). For the equilibrium beam equations utilized in this work, the function \( f(x_i, y(x_i)) \) appearing in Eq. (14) is Lipschitz continuous in \( y \) and continuous in \( x_i \). Furthermore, follower loads can be specified based solely on the initial configuration. Therefore, a unique solution exists for any initial
condition set specified for the follower load beam BVP as posed by
the intrinsic equilibrium equations. As a result, varying the force
and moment initial conditions for a follower load beam BVP over
their entire range (e.g., by rotating the applied loads through all
possible directions) and compiling the solutions that satisfy the de-
sired conservatively-loaded beam BVP in its deformed configura-
tion yields all solutions for the conservative beam BVP. By way of
example, for the two-dimensional Euler buckling of a fixed-free
beam loaded by a conservative axial force, rotating a follower
load over the range \([0, 2\pi]\) in \(\Omega_b\) and determining all solutions that satis-
ify the correct deformed load orientation in \(\Omega_l\) gives all buckled
solutions to this BVP. A later section presents results and discus-
sion for the Euler buckling problem in detail.

After determining all BVP solutions, the stability of each solu-
tion can be assessed using analytical linearization techniques on
the dynamic equations (Cook, 1986) or a dynamic numerical sol-
er. For this work, a dynamic finite element code written specifi-
cally for the intrinsic beam equations of motion (Leamy and Lee,
2009) makes determining system stability trivial. First, the general
shooting method calculates the \(\Omega_l\) configuration for a specific load
case. Then, an automated process ports the curvatures and strains
from this configuration, along with the necessary boundary condi-
tions and loads, into the dynamic finite element code. Since the fi-
nite element program accepts curvatures and strains as the nodal
degrees of freedom, the deformed geometry from shooting
transfers to the finite element code with ease. Next, the system

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**Fig. 2.** The shooting method for cantilever beams subjected to follower loads has four main steps: define problem; initiate shooting from the free end to integrate \(\mathbf{K}\) and \(\mathbf{\tilde{y}}\); integrate the ODE set until the desired length of the beam is reached; using the \(\mathbf{K}\) and \(\mathbf{\tilde{y}}\) solutions, integrate \(\mathbf{r}\) and \(\mathbf{b}\) from the fixed end with \(\mathbf{r}(0) = \mathbf{0}\) and \(\mathbf{b}\) aligned with the global coordinate system.

**Fig. 3.** The shooting method requires beam BVP's to be split into a series combination of cantilever beams. The external loads and reaction loads are then varied until loading and boundary conditions are satisfied.
configuration is perturbed slightly and the dynamic finite element code simulates system response for an extended period of time. An exponential growth of the system response over time demonstrates system instability, while a stable system experiences oscillations proportional to the disturbance.

In situations where neglecting strain is warranted, only the curvature-moment relations from the constitutive law in Eqs. (12) and (13) need to be substituted into the equilibrium equations. The equations neglecting strain will be utilized in later sections where quantitative results are compared with those from the literature. When neglecting strain, the set of field variables now include \( F \) instead of \( \gamma \), which makes the application of point force loads at the end of the beam straightforward. These equations are presented in their entirety in Appendix A.

4. Results

The presented shooting method exhibits its strongest advantages when applied to cantilever beams subjected to nonconservative follower loads – these cases do not require iteration. The following test cases validate the method using both literature comparisons (Pai and Palazotto, 1996; Argyris and Symeonidis, 1981) and unique comparisons of three-dimensional systems with a commercial finite element code. Note that all of the presented results in Section 4 consider large deformation elastic problems only and plasticity is ignored.

4.1. Follower load validation

4.1.1. Straight and pre-curved cantilever beams subjected to point and distributed forces

The first validation results are for straight and pre-curved cantilever beams loaded by point follower forces, and a straight
cantilever beam loaded by a follower distributed force. For these validation cases, the results from the presented shooting method compare well to results obtained by Argyris and Symeonidis (1981), who in addition determined the critical flutter loads using a dynamic FE solution. Using these flutter loads as a guideline, we present only those equilibrium configurations resulting from loading below the critical values. Tables 1 and 2 present the material properties and geometry for each validation case.

![Intrinsic beam zero-iteration shooting calculated the deformed configuration of a cantilever beam loaded out of plane with the loads given in Table 3. The deformed configuration is in gray and the initial configuration is in black.](image)

**Fig. 5.**

Table 3
Validation beam dimensions and material properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (cm) (X-direction)</td>
<td>100.0</td>
</tr>
<tr>
<td>Width and height (cm)</td>
<td>2.00</td>
</tr>
<tr>
<td>Young’s modulus (N/cm²)</td>
<td>$2.10 \times 10^5$</td>
</tr>
<tr>
<td>Poison’s ratio</td>
<td>0.30</td>
</tr>
<tr>
<td>Initial $F_x$ (N)</td>
<td>$4 \times 10^5$</td>
</tr>
<tr>
<td>Initial $F_y$ (N)</td>
<td>$2 \times 10^5$</td>
</tr>
<tr>
<td>Initial $M_x$ (N/cm)</td>
<td>$1 \times 10^5$</td>
</tr>
</tbody>
</table>

**Fig. 6.** The $\mathbf{K}$, $\mathbf{\gamma}$ and displacement components from the three-dimensional cantilever beam load case described in Fig. 5 compared to results computed using Abaqus. Solid lines represent shooting method results, and circles represent Abaqus results.

Table 4
Helix properties given by Pai and Lee (2002).

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>2Rtan(\phi)</td>
</tr>
<tr>
<td>$n_1$</td>
<td>6</td>
</tr>
<tr>
<td>Helix radius (m)</td>
<td>0.02</td>
</tr>
<tr>
<td>Helix pitch angle $\phi$</td>
<td>10.00°</td>
</tr>
<tr>
<td>Cross section radius $r$ (m)</td>
<td>0.0010</td>
</tr>
<tr>
<td>Young’s modulus (N/m²)</td>
<td>$200.00 \times 10^9$</td>
</tr>
<tr>
<td>Poison’s ratio</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Table 5
Example helical beam dimensions and material properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (m)</td>
<td>0.77</td>
</tr>
<tr>
<td>$K_{01}$ (m⁻¹)</td>
<td>8.55</td>
</tr>
<tr>
<td>$K_{02}$ (m⁻¹)</td>
<td>0.00</td>
</tr>
<tr>
<td>$K_{03}$ (m⁻¹)</td>
<td>48.49</td>
</tr>
</tbody>
</table>

compare well to results obtained by Argyris and Symeonidis (1981), who in addition determined the critical flutter loads using a dynamic FE solution. Using these flutter loads as a guideline, we present only those equilibrium configurations resulting from loading below the critical values. Tables 1 and 2 present the material properties and geometry for each validation case.

Fig. 4(a) presents the deformed configurations of an initially straight beam subjected to a perpendicular, point, follower force at the free end for multiple load magnitudes. Fig. 4(c) depicts the...
deformed configurations of an initially straight beam subjected to a perpendicular, distributed, follower force across the length of the beam for various magnitudes. For both of these cases, the results obtained with zero-iteration shooting compare favorably to the results obtained from the finite element analysis (FEA) work done by Argyris and Symeonidis (1981).

The quantitative results used for comparison (see Figs. 4(b) and (d)) include the normalized displacements and the rotation $\phi_z$ about the $Z$ axis for the end of the beam. It is evident from this figure that results generated using the presented method compare well with the FEA results presented by Argyris, other than minor disagreement in Fig. 4(b) for $-u/L = 1$.

Figs. 4(e) and (f) validate the ability of the shooting method to model pre-curved beams without iteration. The case considered is a follower point force perpendicular to the pre-curved beam at the free end. As before, Fig. 4(e) depicts the final configurations of the beam for several different load magnitudes, and Fig. 4(f) compares the normalized beam tip displacements and rotations from zero-iteration shooting to numerical results from Argyris. Once again, the results show strong agreement with those presented in the literature.

### 4.1.2. Straight cantilever beam subjected to point forces and moments resulting in out of plane deformation

This follower load case illustrates the method’s ability to solve three-dimensional cantilever beam BVP’s in a single shot. A straight beam is subjected to a transverse follower load $F_Y$, an axial follower load $F_X$ and a torsional moment $M_X$ resulting in a non-planar deformed configuration. The beam properties and load conditions for this follower load BVP are described in Table 3. The resulting deformed configuration is shown in Fig. 5. Additionally, the zero-iteration shooting curvature, strain and displacement components are compared to the results obtained from the commercial finite element package Abaqus in Fig. 6. As shown in Fig. 6, the results from shooting compare well to the results from Abaqus. A small discrepancy does appear in the $\hat{K}_1$ result in Fig. 6(a), where shooting gives a constant $\hat{K}_1$ and the results from Abaqus vary on the order of $1 \times 10^{-4}$ along the arc length. This variance is due to a slight coupling between $\hat{\gamma}_{11}$ and $\hat{K}_1$ in Abaqus’s constitutive model that is not included in the present method – this is discussed at length in the next example.

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**Fig. 7.** Intrinsic beam zero-iteration shooting calculated the deformed configuration of a cantilever helical spring subjected to a 4.0N follower load in the $-X$ direction at the free end.

**Fig. 8.** The $\hat{K}$, $\hat{\gamma}$ and displacement components from the helical spring follower load case described in Fig. 7 compared to results computed using Abaqus. Solid lines represent shooting method results, dashed lines represent corrected shooting results, circles represent Abaqus results.
4.1.3. Helical cantilever beam subjected to large follower load

The final follower load case illustrates the method’s ability to solve three-dimensional, pre-curved, cantilever beam BVP’s without iteration. A helix is loaded on the free end with a follower force of magnitude of $-4.0$ N tangent to the centerline. Table 4 provides the helix properties for this validation case. Pai and Lee used these same values in Pai and Lee (2002), where they presented a shooting method that modeled the extension and compression of a spring.

In Table 4, $r$ denotes the circular cross sectional radius of the beam, $n_c$ denotes the number of complete turns in the helix, $\psi$ denotes the pitch angle of the helix, and $R$ denotes the radius of the projection of the helix onto the $XY$ plane. The presented shooting method requires that these helix parameters be converted into initial curvatures. Pai and Lee (2002) defined the initial curvatures as functions of these parameters using

\begin{align*}
\beta_{\alpha} &= \cos(\psi)\sin(\psi)/R, \\
\beta_{\eta} &= \psi, \\
\beta_{\psi} &= \cos^2(\psi)/R,
\end{align*}

where for a helix $\psi = 0$. Table 5 gives the numerical values used for the initial curvature and length of the beam.

Fig. 7 displays the deformed and undeformed configurations of the helix computed using shooting with no iteration, while Fig. 8 presents quantitative results comparing the shooting method technique to finite element results obtained using the nonlinear analysis option available in Abaqus. Converged results were obtained using 500 B32 elements. The results show that the curvatures, strains and displacements calculated using the shooting method are in very good agreement with the finite element code; however, a small discrepancy exists between the two result sets for the $X$-displacement. This displacement component is an order of magnitude less than the other displacement components. As a result, the discrepancy has little effect on the overall displacement magnitude comparison between the two result sets. This $X$-displacement error results from a strain coupling present in Abaqus’s constitutive model that is not reproduced by the simple constitutive model used in this work – note that the constitutive model used by Abaqus is not readily available, and further, determining appropriate constitutive models via cross-sectional analysis is an ongoing research effort (Yu et al., 2002; Kumar Kovvali and Hodges, 2012). Close inspection of the Abaqus results show that three non-zero components of strain exist at the forced end of the helix, while the presented constitutive model (Eq. (17)) only imposes non-zero axial strain at this end. This suggests the use of a coupled constitutive model would improve results – such a model would simply

\begin{align*}
\frac{X}{F_X} = 32.5, \quad \frac{Y}{F_Y} = 27.5, \quad \frac{Z}{F_Z} = 25.05.
\end{align*}
replace off-diagonal terms in Eq. (11). To test this assertion, the shear strain values predicted by Abaqus at the forced end of the helix are used as the initial conditions for the shooting method, resulting in the shooting method producing notably better results – see light blue dashed vice dark blue solid lines in Fig. 8. This verifies that the constitutive model accounts for the small discrepancies, and not the presented shooting method itself.

4.2. Non-follower load validation

To demonstrate that the presented method does not have a disadvantage when compared to traditional shooting methods, shooting is used to solve standard conservatively-loaded beam BVPs. In addition, the ability of the method to obtain all solutions for a load case is demonstrated, and the stability of the post-buckled shapes is studied using an in-house finite element simulation tool (Leamy and Lee, 2009) based on the same intrinsic formulation.

4.2.1. Post-buckling deformation of an axially loaded cantilever beam

Studying the post-buckling behavior of a straight cantilever beam shows that the presented shooting method is able to solve beam problems with conventional non-follower loads. The results for this case are parameterized by beam dimensions and material properties. Using linear Euler–Bernoulli beam theory, the buckling loads for an axially-loaded (non-follower) fixed-free beam are well known:

$$F_m = (2m - 1)\frac{\pi^2EIzz}{4L^2}$$

where \(m = 1, 2, 3, \ldots\) denotes the buckling load number of the beam (Timoshenko and Gere, 1961). Since Euler–Bernoulli beam theory is used for these buckling cases, the intrinsic equations neglecting strain are used for sake of comparison.
Fig. 10 indicates that only one solution exists if \( b \neq \frac{\pi}{2} \). The solutions with \( a = \frac{\pi}{2} \) are desired; however, all of these solutions do not necessarily yield un

\[\text{Fig. 17. The force–displacement curves for the helical spring in (a) compression and (b) extension. The solid lines are results from shooting and the circles are results from Abaqus.}\]

Figs. 9 and 10 illustrate how buckled solutions are determined using the proposed shooting method. For planar buckling, two angles are defined: the angle of the follower load relative to the local \( \mathbf{B}_1 \) vector, \( \alpha \), and the angle of the follower load relative to the global \( X \)-direction in the deformed configuration, \( \beta \). The goal is to determine the post-buckled deformed configuration when \( \beta = \pi \).

Fig. 10 presents the resultant \( \beta \) as a function of \( \alpha \) for four different axial buckling loads. Also labeled are the necessary \( \alpha \)'s such that \( \beta = \pi \) in \( \Omega_2 \). Fig. 10 indicates that only one solution exists if \( \eta < 1 \), where \( \eta \) denotes a multiplication factor for the buckling load such that the applied axial load is \( F_X = -\eta F \). This solution is for \( \alpha = \pi \), resulting in axial compression of the beam. Furthermore, Fig. 10 depicts multiple solutions when \( \eta > 1 \) as a result of beam buckling. Note that all of the peaks in Fig. 10 represent a solution where the load in the deformed configuration is oriented as desired; however, all of these solutions do not necessarily yield unique configurations. The solutions with \( \alpha > \pi \) give the same deformed configuration shape as the solutions found with \( \alpha < \pi \), but have negative displacements. This is due to symmetry in the problem about the \( X \) axis. In fact, in the full three-dimensional problem, infinite solutions exist since the beam could buckle at any angle in the \( YZ \) plane. However, we are only concerned with the planar buckled configuration, so these solutions are suppressed.

In order to investigate the post-buckling behavior of a cantilever beam, the deflection of the beam and the deformed post-buckled configurations are plotted as a function of \( \eta \). Fig. 11 depicts a bifurcation diagram of the deformed beam’s normalized end deflections, \( u/L \) and \( v/L \), and compares the shooting results with those obtained by Pai and Palazotto (1996). Note that as the load increases past \( F_1 \), or past the critical buckling load, more than one possible equilibrium solution satisfies the BVP. For the first buckling load, this includes the unstable straight configuration and a stable buckled shape. Each new solution branch in the bifurcation diagram corresponds to a higher buckling mode. Additionally, Fig. 12 illustrates that for each buckling load, \( m + 1 \) solutions exist, which consist of \( m \) buckled mode shapes and a straight configuration. For example, when \( \eta = 22.5 \) three solutions exist, one of which is straight and the other two are post-buckled.

Figs. 11(b) and 13 depict these two post-buckled configurations, which correspond to the initial follower load angles \( \alpha = 5.57 \) and \( \alpha = 6.28 \) radians, respectively. It is of practical interest which, if any, of these solutions are stable.

As described previously, the stability of the second buckling load mode shapes is tested with the use of a special-purpose explicit finite element code (Leamy and Lee, 2009) developed from the same intrinsic equations. Two equilibrium positions and loads are ported to the finite element code and allowed to simulate for an extended period of time after the application of a small perturbation. Fig. 14 presents the displacement of the beam end as a function of time for the mode shape shown in Fig. 11(b). These results illustrate instability as is evident by the exponential growth in the end displacement of the beam.

In contrast to this unstable solution, the buckled mode shape presented in Fig. 13 is a stable solution as verified by Fig. 15. Fig. 15 displays the displacement of the end of this case away from the buckled solution as a function of time. Unlike the unstable case discussed earlier, the displacement oscillates around the stable equilibrium value at a magnitude on the order of the applied perturbation. This oscillation is constant for a long period of time, indicating dynamic stability around the equilibrium point.

### 4.2.2. Deformation of a helical beam subjected to compression and extension

The shooting method presented is also capable of modeling more complex loading configurations with three-dimensional geometry. In the validation case presented next, a prescribed \( Z \)-displacement applied to the free end of a fixed-free helical spring allows for the creation of compression and extension force–displacement curves. Tables 4 and 5 define the helix properties and dimensions, which are the same values used for the follower loaded helix.

The presented method’s results are compared to results obtained from an Abaqus model using nonlinear analysis and 500 B32 beam elements. The force–displacement curves are created with the implementation of a multivariable minimization algorithm that uses shooting to find the correct end load magnitude and direction that results in the desired displacement. For this helix case with a prescribed \( Z \)-displacement, only the three force components at the helix free end govern the possible system response. Once the minimization algorithm determines the correct values for these forces, they are projected onto the global basis to give the global reaction forces and the force in the \( Z \) direction necessary to compress or extend the spring to the desired displacement. Displacements up to \( 5R \) and \( -5R \) are imposed on the spring. Fig. 16 illustrates the resulting deformed configurations along with the undeformed configurations for the largest imposed displacements of \( -5R \) for compression, and \( 5R \) for extension. Fig. 17 compares quantitative results for the force–displacement curves obtained from shooting to similar results obtained using Abaqus. The \( K, \gamma \) and displacement components from these two helical spring load cases are compared with the same results from Abaqus in Figs. 18 and 19. Strong agreement can be noted in all comparisons.
5. Conclusions

This article has developed and validated a shooting method for computing solutions to nonlinear, intrinsic beam equations governing three-dimensional equilibria. Test cases show that the presented method avoids iteration for pre-curved cantilever beams subjected to distributed and/or point follower loads. In addition, the article has presented a general approach for finding all solutions to conservatively-loaded beam problems. For beam buckling, solution stability has been assessed using a dynamic finite element code based on the same intrinsic equations. Due to the method avoiding iteration in follower load problems, it may be attractive for use in model-based control where the solution of a system’s response to follower loads is needed in a computationally-efficient manner.

Appendix A. Explicit definition of derivatives for intrinsic formulation shooting equations with strain

The following equations were derived using the process explained in Section 3 and the constitutive law defined by Eq. (10).

\[ \ddot{\gamma}_{12} = \frac{2G\left(K_0 + \tilde{K}_1\right)\dddot{\gamma}_{13} - Ek\left(K_0 + \tilde{K}_3\right)\dddot{\gamma}_{11}}{2G} \]  
(A.5)

\[ \ddot{\gamma}_{13} = \frac{Ek\left(K_0 + \tilde{K}_2\right)\dddot{\gamma}_{11} - 2G\left(K_0 + \tilde{K}_3\right)\dddot{\gamma}_{12}}{2G} \]  
(A.6)

\[ B'_{12} = \left(K_0 + \tilde{K}_2\right)B_2B_1 - \left(K_0 + \tilde{K}_2\right)B_2B_1 + \left(K_0 + \tilde{K}_3\right)B_3B_1 - \left(K_0 + \tilde{K}_3\right)B_3B_1 \]  
(A.7)

\[ B'_{13} = \left(K_0 + \tilde{K}_1\right)B_3B_1 - \left(K_0 + \tilde{K}_1\right)B_3B_1 + \left(K_0 + \tilde{K}_3\right)B_3B_1 - \left(K_0 + \tilde{K}_3\right)B_3B_1 \]  
(A.8)

\[ B'_{23} = \left(K_0 + \tilde{K}_1\right)B_1B_2 - \left(K_0 + \tilde{K}_1\right)B_1B_2 + \left(K_0 + \tilde{K}_3\right)B_3B_2 - \left(K_0 + \tilde{K}_3\right)B_3B_2 \]  
(A.9)

\[ B'_{22} = \left(K_0 + \tilde{K}_1\right)B_1B_2 - \left(K_0 + \tilde{K}_1\right)B_1B_2 + \left(K_0 + \tilde{K}_3\right)B_3B_2 - \left(K_0 + \tilde{K}_3\right)B_3B_2 \]  
(A.10)

\[ B'_{32} = \left(K_0 + \tilde{K}_1\right)B_3B_2 - \left(K_0 + \tilde{K}_1\right)B_3B_2 + \left(K_0 + \tilde{K}_3\right)B_3B_2 - \left(K_0 + \tilde{K}_3\right)B_3B_2 \]  
(A.11)

\[ B'_{33} = \left(K_0 + \tilde{K}_1\right)B_3B_2 - \left(K_0 + \tilde{K}_1\right)B_3B_2 + \left(K_0 + \tilde{K}_3\right)B_3B_2 - \left(K_0 + \tilde{K}_3\right)B_3B_2 \]  
(A.12)
Fig. 19. The $K$, $\gamma$ and displacement components from the helical spring extension load case described in Fig. 16(b) compared to results computed using Abaqus. Solid lines represent shooting method results and circles represent Abaqus results.

$$K_s' = (K_0 + \Delta K_1)B_1B_3 - (K_0 + \Delta K_1)B_1B_3$$
$$+ (K_0 + \Delta K_2)B_2B_3 - (K_0 + \Delta K_1)B_2B_3$$
$$B_y' = (K_0 + \Delta K_1)B_1B_3 - (K_0 + \Delta K_1)B_1B_3$$
$$+ (K_0 + \Delta K_2)B_2B_3 - (K_0 + \Delta K_2)B_2B_3$$
$$B_z' = (K_0 + \Delta K_1)B_1B_3 - (K_0 + \Delta K_1)B_1B_3$$
$$+ (K_0 + \Delta K_2)B_2B_3 - (K_0 + \Delta K_2)B_2B_3$$

$$K_s' = (1 + \gamma_{11})B_{zz} + 2\gamma_{12}B_{Bz} + 2\gamma_{13}B_{Bz}$$
$$K_y' = (1 + \gamma_{11})B_{yy} + 2\gamma_{12}B_{Bz} + 2\gamma_{13}B_{Bz}$$
$$K_z' = (1 + \gamma_{11})B_{zz} + 2\gamma_{12}B_{Bz} + 2\gamma_{13}B_{Bz}$$

Appendix B. Explicit definition of derivatives for intrinsic formulation shooting equations without strain

The following equations were derived using the process explained in Section 3 and the constitutive law defined by Eq. (12).

$$\dot{K}_1 = (GJ)^{-1}[I_2(K_0 + \Delta K_1)\dot{K}_2 - I_2(K_0 + \Delta K_1)\dot{K}_2]$$
$$\dot{K}_2 = E^{-1}I_2[\dot{B}_1(K_0 + \Delta K_1)\dot{K}_2 - GJ(K_0 + \Delta K_1)\dot{K}_1 - F_1F_1 + (1 + F_1)F_1]$$
\[ B_{2z} = (K_0 + \tilde{K}_1)B_1B_3 - (K_0 + \tilde{K}_1)B_1B_2 \\
+ (K_0 + \tilde{K}_1)B_3B_2 - (K_0 + \tilde{K}_1)B_3B_1 \] (B.12)

\[ B_{3z} = (K_0 + \tilde{K}_1)B_1B_3 - (K_0 + \tilde{K}_1)B_1B_2 \\
+ (K_0 + \tilde{K}_1)B_3B_2 - (K_0 + \tilde{K}_1)B_3B_1 \] (B.13)

\[ B_{3y} = (K_0 + \tilde{K}_1)B_1B_3 - (K_0 + \tilde{K}_1)B_1B_2 \\
+ (K_0 + \tilde{K}_1)B_3B_2 - (K_0 + \tilde{K}_1)B_3B_1 \] (B.14)

\[ B_{2y} = (K_0 + \tilde{K}_1)B_1B_3 - (K_0 + \tilde{K}_1)B_1B_2 \\
+ (K_0 + \tilde{K}_1)B_3B_2 - (K_0 + \tilde{K}_1)B_3B_1 \] (B.15)

\[ R'_{x} = B_{1x} \] (B.16)

\[ R'_{y} = B_{1y} \] (B.17)

\[ R'_{z} = B_{1z} \] (B.18)

References


