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Boundary conditions and the generalized metric formulation of the double sigma model

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Abstract

Double sigma model with strong constraints is equivalent to the ordinary sigma model by imposing a self-duality relation. The gauge symmetries are the diffeomorphism and one-form gauge transformation with the strong constraints. We consider boundary conditions in the double sigma model from three ways. The first way is to modify the Dirichlet and Neumann boundary conditions with a fully $O(D, D)$ description from double gauge fields. We perform the one-loop β function for the constant background fields to find low-energy effective theory without using the strong constraints. The low-energy theory can also have $O(D, D)$ invariance as the double sigma model. The second way is to construct different boundary conditions from the projectors. The third way is to combine the antisymmetric background field with field strength to redefine an $O(D, D)$ generalized metric. We use this generalized metric to reconstruct a consistent double sigma model with the classical and quantum equivalence.

1. Introduction

Duality shows the nontrivial equivalence between two theories. It gives us a hope to unify all known theories. It is one of the important problems in the M-theory. For the ten dimensional theories, we have the T- and S-duality. The T-duality is an equivalence between different radii. We exchange the momentum and winding modes in closed string theory and the Dirichlet and

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Neumann boundary conditions in open string theory. The T-duality suffers from the T-fold problem in the closed string field theory [1,2]. Even for simple constant flux situation, we still find non-single valued fields because of non-isometry. The T-duality is not a well-defined transition function as gauge transformation or diffeomorphism. The S-duality is an equivalence between strong and weak coupling constants. Therefore, the S-duality is a non-perturbative duality so we cannot use perturbation with the coupling constant parameter. Invalidity of perturbation gives rise to a trouble. As a familiar example, it is the electric–magnetic duality of the abelian gauge theory. In the case of the non-abelian groups, the electric–magnetic duality is an open issue. In eleven dimensions, we combine the T- and S-duality to form the U-duality. The U-duality is expected to be a symmetry of the eleven dimensional supergravity.

The method of solving the T-fold problem is to extend from local to global geometry. So far many low-energy effective theories [3–10] are defined on local geometry. The brane theory with global geometry is the generalized Dirac–Born–Infeld (DBI) theory [11,12]. A non-commutative geometry of this theory at the semi-classical level (constant field strength) is governed by the generalized metric, which is an important element to combine tangent with cotangent bundle. This gives us a new perspective to construct the low-energy effective theory or extend understanding on the T-fold. The low-energy effective theory has a corresponding sigma model [11–14] from the new generalized metric. If we combine vector with one-form, a double geometry appears in their theories. This new geometry possibly be a good description to describe string theory [15,16]. They double coordinates (normal and dual coordinates) to embed the T-duality rule in the $O(D, D)$ structure for the closed string theory [17–28]. This extension gives the Courant bracket, which shows a way to solve the T-fold problem [29,30]. Its extension helps us to define exotic brane. The source of exotic brane is non-geometric flux (Q- and R-flux). One example is the 5_2^2-brane theory [31]. This brane theory comes from the Neveu–Schwarz five-brane (NS5-brane) by performing two times T-duality. The double geometry suffers from constraints. The relaxing constraints [32] is a hard problem due to the generalized Lie derivative is not a closed algebra without applying constraints in the double geometry. Recent reviews of double geometry are in [33–35]. For the same understanding of the U-duality as the T-duality, we need to extend this double geometry to the exceptional field theory or exceptional generalized geometry [36,37].

Double geometry of open string is proposed from [38]. They use the similar ways with the closed string theory and suggest that the projectors should satisfy the boundary conditions. The gauge transformation and properties [39,40] of a theory can be understood from the generalized geometry [41,42]. The extension of the gauge transformation from the generalized geometry to double geometry of the ten dimensional supergravity is governed by the F-bracket [43]. The strong constraints (removing the dependence of the dual coordinates) of the F-bracket has an exact one-form difference from the Courant bracket. A double sigma model with open string is found from this gauge transformation with classical equivalence and quantum equivalence at one-loop level [44]. Quantum fluctuation of string theory also gives a higher derivative gravity theory at low-energy level [45]. One-loop β function of a double sigma model for the closed string with the dilaton gives the consistent low-energy effective action [46,47]. The conditions of the quantum conformal and Lorentz invariance are also shown in [48]. The most interesting case of the one-loop quantum fluctuation is to simultaneously consider the fluctuation of the ordinary and dual coordinates, which gives us the correct equation of motion for the generalized metric [49]. This calculation exactly shows a low-energy effective action of the generalized metric formulation [27]. The covariant version of the double sigma model is constructed in [50].

Double geometry only shows the manifest formulation for the T-duality rule from the $O(D, D)$ description. We never discuss the manifest S-duality rule in this $O(D, D)$ formulation. It possibly be embedded in the $O(D, D)$ structure. The electric–magnetic duality in the electromagnetism is exchanging the electric and magnetic fields. It is equivalent to exchanging the field strength. The standard procedure of the electric–magnetic duality at quantum level is the auxiliary field method. But this method is not manifest because we do not put ordinary and dual gauge fields together. We double gauge fields to study a manifest electric–magnetic duality. We expect that exchanging gauge fields gives the manifest electric–magnetic duality rule. It possibly be a new way to embed the S-duality in a new perspective as the manifest T-duality rule. We naively double gauge fields in the double sigma model. Then the boundary term does not break $O(D, D)$ invariance. The one-loop β function gives a low-energy effective action consistent with the $O(D, D)$ description. We also calculate the non-commutative relation at the semi-classical level. The non-commutativity of the double gauge fields relies on the field strength. This situation is the consistent with the D-brane theory without doubling coordinates. In this formulation, we use a new degrees of freedom of the gauge field to enlarge symmetries and modify boundary conditions. For a consistent gauge symmetry, we need to use more constraints to kill this new degrees of freedom. The double gauge fields is an interesting way to enlarge symmetry structure although we need to put one more field which is not in the supergravity description. We propose projectors to consider more choices of boundary conditions on σ^1-direction. This method is a way to extend double gauge fields formulation with different boundary conditions. In this case, we can find a particular choice of the projects to obtain consistent DBI action for the one-loop β function. Finally, we combine the antisymmetric background field and field strength to obtain a different $O(D, D)$ generalized metric to construct a low-energy action from this generalized metric and scalar dilaton [44]. We also use this generalized metric to build a new double sigma model with the classical and quantum equivalence. The consistency check on the quantum equivalence from two ways. The first way is the one-loop β function for the constant background fields and the second way is to obtain the ordinary sigma model when integrating out the dual coordinates. This double sigma model shows a different perspective to observe the manifest semi-classical non-commutative geometry. It should have more different theoretical viewpoints than [43]. In this construction, we have a new degrees of freedom on the bulk. In the closed string theory, we do not have one-form gauge potential. In this double sigma model, we have one-form gauge potential without using the strong constraints on the bulk. By using the strong constraints, the degree of freedom for the one-form gauge potential should disappear. The interesting issue in this double sigma model is that we have one-form gauge symmetry without the strong constraints. We remind one truth that the Seiberg–Witten map comes from the one-form gauge transformation. This truth implies that the low-energy effective action of the double sigma model should have the Seiberg–Witten map and Moyal product. Because of the new degree of freedom (one-form gauge potential) on the bulk, we have possibility to find the Moyal product on the bulk.

The plan of this paper is to first review the double sigma model in Section 2. Then we double gauge fields, compute the one-loop β function, show the low energy effective action and non-commutative relation in Section 3. We also use projectors to realize boundary conditions on σ^1-direction in Section 4. Combining the antisymmetric background field and field strength to form a different generalized metric, constructing a double sigma model from this different generalized metric, and showing classical and quantum equivalence in Section 5. Finally, we discuss and conclude in Section 6.

2. Review of the double sigma model

We first review the double sigma model, then show classical equivalence for the double sigma model. At the end of the section, we write the gauge transformation.

2.1. Classical equivalence

We start from

$$S = -\int d^2\sigma \; \frac{1}{2}\partial^\alpha X^A \mathcal{H}_{AB}\partial_\alpha X^B, \tag{1}$$

where $\alpha = 0, 1$ (we use the Greek indices to indicate the worldsheet coordinates), $A = 0, 1, \cdots, 2D-1$ (we define the double target indices from A to K), and

$$X^A \equiv \begin{pmatrix} \tilde{X}_m \\ X^m \end{pmatrix},$$
$$\mathcal{H}^{-1} \equiv \mathcal{H}_{\bullet\bullet} = \left(\mathcal{H}^{AB}\right)^{-1} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}. \tag{2}$$

The index $m = 0, 1, \cdots, D-1$ (we define the non-double target indices from m to z). The ordinary coordinates are defined to be X^m and dual coordinates are defined to be $\tilde{X}_m$. Under the T-duality on all dimensions, ordinary and dual coordinates are exchanged. The metric field is g and antisymmetric background field is B. We also define

$$\mathcal{H} \equiv \mathcal{H}^{\bullet\bullet}. \tag{3}$$

The name for $\mathcal{H}$ is generalized metric. For double target indices, we use $\eta \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ to raise and lower indices for the $O(D, D)$ tensors. The index α is raised and lowered by the flat metric. The worldsheet metric is $(-, +)$ signature. If $h \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (a, b, c and d are $D \times D$ matrices) is an $O(D, D)$ tensor, it satisfies $h^T\eta h = \eta$, where T means the transpose of matrix. The equation of motion for X^A in the constant background is

$$\partial^\alpha(\mathcal{H}_{AB}\partial_\alpha X^B) = 0. \tag{4}$$

We need to eliminate the half degrees of freedom to show classical equivalence with the ordinary sigma model so we impose a self-duality relation

$$\partial_\alpha X^A = \epsilon_{\alpha\beta}\eta^{AB}\mathcal{H}_{BC}(\partial^\beta X^C), \tag{5}$$

where $\epsilon^{01} = -\epsilon^{10} = 1$. The matrix form is

$$\begin{aligned}\begin{pmatrix} \partial_\alpha \tilde{X} \\ \partial_\alpha X \end{pmatrix} &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}\begin{pmatrix} \epsilon_{\alpha\beta}\partial^\beta \tilde{X} \\ \epsilon_{\alpha\beta}\partial^\beta X \end{pmatrix} \\ &= \begin{pmatrix} Bg^{-1} & g - Bg^{-1}B \\ g^{-1} & -g^{-1}B \end{pmatrix}\begin{pmatrix} \epsilon_{\alpha\beta}\partial^\beta \tilde{X} \\ \epsilon_{\alpha\beta}\partial^\beta X \end{pmatrix} \\ &= \begin{pmatrix} Bg^{-1}(\epsilon_{\alpha\beta}\partial^\beta \tilde{X}) + (g - Bg^{-1}B)(\epsilon_{\alpha\beta}\partial^\beta X) \\ g^{-1}(\epsilon_{\alpha\beta}\partial^\beta \tilde{X}) - g^{-1}B(\epsilon_{\alpha\beta}\partial^\beta X) \end{pmatrix}.\end{aligned} \tag{6}$$

We use two equations to represent the matrix form

$$\partial_\alpha\tilde{X} = Bg^{-1}(\epsilon_{\alpha\beta}\partial^\beta\tilde{X}) + (g - Bg^{-1}B)(\epsilon_{\alpha\beta}\partial^\beta X),$$
$$\partial_\alpha X = g^{-1}(\epsilon_{\alpha\beta}\partial^\beta\tilde{X}) - g^{-1}B(\epsilon_{\alpha\beta}\partial^\beta X). \tag{7}$$

Hence, we can solve $\partial_\alpha\tilde{X}$ from (7).

$$\partial_\alpha\tilde{X} = \epsilon_{\alpha\beta}\, g\partial^\beta X + B\partial_\alpha X. \tag{8}$$

The equations of motion can be rewritten as

$$\partial^\alpha\begin{pmatrix} g^{-1}\partial_\alpha\tilde{X} - g^{-1}B\partial_\alpha X \\ Bg^{-1}\partial_\alpha\tilde{X} + (g - Bg^{-1}B)\partial_\alpha X\end{pmatrix} = 0. \tag{9}$$

We can obtain

$$\begin{aligned}
&\partial^\alpha\left(Bg^{-1}\partial_\alpha\tilde{X} + (g - Bg^{-1}B)\partial_\alpha X\right)_m \\
&\quad= \partial^\alpha\left(Bg^{-1}(\epsilon_{\alpha\beta}\, g\partial^\beta X + B\partial_\alpha X) + (g - Bg^{-1}B)\partial_\alpha X\right)_m \\
&\quad= \partial^\alpha\left(\epsilon_{\alpha\beta} B\partial^\beta X + Bg^{-1}B\partial_\alpha X + g\partial_\alpha X - Bg^{-1}B\partial_\alpha X\right)_m \\
&\quad= \partial^\alpha\left(\epsilon_{\alpha\beta} B\partial^\beta X + g\partial_\alpha X\right)_m
\end{aligned} \tag{10}$$

for the lower component of the equations of motion. This matches with the equation of motion for the ordinary sigma model by changing from B to $-B$. The ordinary sigma model is

$$\frac{1}{2}\int d^2\sigma\,\left(\partial_\alpha X^m g_{mn}\partial^\alpha X^n - \epsilon^{\alpha\beta}\partial_\alpha X^m B_{mn}\partial_\beta X^n\right). \tag{11}$$

This double sigma model (1) with the strong constraints describes the ordinary sigma model with constant background. A difficulty for extension from the constant background to non-constant background is the self-duality relation. The same self-duality relation cannot be used for the non-constant background. However, we use

$$S_{\text{bulk}} = \frac{1}{2}\int d^2\sigma\,\left(\partial_1 X^A\mathcal{H}_{AB}\partial_1 X^B - \partial_1 X^A\eta_{AB}\partial_0 X^B\right) \tag{12}$$

to discuss the non-constant background case. Using the strong constraints $\tilde{\partial}^m = 0$ ($\partial_m \equiv \frac{\partial}{\partial x^m}$, $\tilde{\partial}^m \equiv \frac{\partial}{\partial\tilde{x}_m}$ and $\partial_A \equiv \begin{pmatrix}\tilde{\partial}^m \\ \partial_m\end{pmatrix}$) and a self-duality relation

$$\mathcal{H}^m{}_B\partial_1 X^B - \eta^m{}_B\partial_0 X^B = 0 \tag{13}$$

to guarantee classical equivalence with the ordinary sigma model. If we consider the Neumann boundary condition on σ^1-direction, we should put

$$S_{\text{boundary}} = -\int d\sigma^0\; A_m\partial_0 X^m \tag{14}$$

to obtain the gauge invariance on the boundary. The one-loop β function of this double sigma model (12) for the constant background fields which should give the DBI model [44].

2.2. Gauge transformation

The gauge transformation is

$$\begin{aligned}\delta_\xi X^A &= \xi^C \partial_C X^A + (\partial^A \xi_C - \partial_C \xi^A) X^C,\\ \delta_\xi \mathcal{H}^{AB} &= \xi^C \partial_C \mathcal{H}^{AB} + (\partial^A \xi_C - \partial_C \xi^A)\mathcal{H}^{CB} + (\partial^B \xi_C - \partial_C \xi^B)\mathcal{H}^{AC},\\ \delta_\xi A_m &= \Lambda_m + \mathcal{L}_\epsilon A_m,\end{aligned} \tag{15}$$

where δ_ξ is the gauge transformation, $\xi^A \equiv \begin{pmatrix} \tilde{\xi}_m \\ \xi^m \end{pmatrix} \equiv \begin{pmatrix} \Lambda_m \\ \epsilon^m \end{pmatrix}$, and $\mathcal{L}_\epsilon$ is the Lie derivative along the vector field ϵ. We assume that the gauge parameters do not depend on the worldsheet coordinates. Then the double sigma model is gauge invariant and the gauge algebra is closed under the F-bracket with $\tilde{\partial}^m = 0$ [43].

$$[\delta_1, \delta_2] = -\delta_{[\xi_1,\xi_2]_F}, \tag{16}$$

where

$$\begin{aligned}[\xi_1, \xi_2]_F^A = &\Big(\xi_1^D \partial_D \xi_2^A - \xi_2^D \partial_D \xi_1^A\Big) - \frac{1}{2}\Big(\xi_1^D \partial^A \xi_{2D} - \xi_2^D \partial^A \xi_{1D}\Big)\\ &- \frac{1}{2}\partial^A \Big(\xi_{2D} Z^D{}_E \xi_1^E\Big),\end{aligned} \tag{17}$$

where

$$Z \equiv Z^A{}_B \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

The indices of Z are raised or lowered by η.

3. Double gauge fields

We double gauge fields on the boundary term in the double sigma model. Then we implement the self-duality relation and compute the one-loop β function to find the low-energy effective theory. We also discuss the semi-classical non-commutative geometry and the picture of the manifest electric–magnetic duality from the double gauge fields.

3.1. One-loop β function

When we double gauge fields, the boundary term becomes

$$-\int d\sigma^0 \; A_B \partial_0 X^B, \tag{19}$$

where

$$A_B \equiv \begin{pmatrix} \tilde{A}^m \\ A_m \end{pmatrix}. \tag{20}$$

The name for $\tilde{A}^m$ is the dual gauge field. The boundary conditions on σ^1-direction are

$$\mathcal{H}_{AB}\partial_1 X^B = F_{AB}\partial_0 X^B, \qquad \delta X^A \eta_{AB}\partial_0 X^B = 0, \tag{21}$$

where $F_{BC} \equiv \partial_B A_C - \partial_C A_B$, and the boundary condition on σ^0-direction is

$$\delta X^A = 0, \tag{22}$$

where δ is the variation. We set $B = 0$ and $g = I$ ($I \equiv$ identity matrix) to simplify the calculation without losing generality in the case of the constant background. We follow [51] to calculate the one-loop β function. The variation of the boundary term is

$$-\int d\sigma^0 \left(A_B \partial_0 X^B + \xi^B F_{BC} \partial_0 X^C + \frac{1}{2}\left(\xi^B \xi^C \partial_B F_{CD} \partial_0 X^D + \xi^B \partial_0 \xi^C F_{BC}\right)\right). \tag{23}$$

Therefore, the Green's function on the bulk is

$$\left(\mathcal{H}_{AB}\partial_1^2 - \eta_{AB}\partial_0\partial_1\right) G^B{}_C(\sigma, \sigma') = i I_{AC} \delta^2(\sigma - \sigma') \tag{24}$$

and the Green's function on the boundary is

$$\mathcal{H}_{AB}\partial_1 G^{BC} - F_{AB}\partial_0 G^{BC} = 0. \tag{25}$$

The counter term on the boundary is

$$-\frac{1}{2}\int d\sigma^0\, \Gamma_A \partial_0 X^A, \tag{26}$$

where

$$\Gamma_A = \lim_{\epsilon \to 0} G^{BC}(\epsilon \equiv \sigma^0 - \sigma^{0\prime})\partial_B F_{CA}. \tag{27}$$

The β function is defined by

$$\beta_A \equiv \epsilon \frac{\partial \Gamma_A}{\partial \epsilon}. \tag{28}$$

It is useful to change coordinates to solve the Green's function.

$$z = \sigma + \tau, \qquad \bar{z} = \sigma - \tau. \tag{29}$$

From the same procedure as [44], we can obtain

$$\sqrt{\det\left(\mathcal{H} + F\right)}, \tag{30}$$

from $\beta_A = 0$. This action has the $O(D, D)$ invariance as the double sigma model. For the non-constant background fields, we should obtain the same closed string theory from the bulk term and

$$e^{-d}\sqrt{\det\left(\mathcal{H} + F'\right)} \tag{31}$$

from the boundary term. We define

$$e^{-d} \equiv \left(-\det g\right)^{\frac{1}{4}} e^{-\phi}, \qquad F' \equiv \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B_{mn} - F_{mn} & -F_m{}^n \\ F^m{}_n & F^{mn} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \tag{32}$$

where d is called scalar dilaton and ϕ is called dilaton. When we exchange the ordinary gauge field and dual gauge field, perform the T-duality on the background fields and assume that the ordinary gauge and dual gauge fields only depend on the ordinary coordinates, we still obtain the consistent pull-back DBI action. The generalized metric implies the manifest T-duality and equivalence between the closed and open string parameters without the field strength. The leading order of the action on the flat background is the Yang–Mills term. If we assume that the dual gauge field is a constant field, we can obtain the ordinary Yang–Mills term. Alternatively, the ordinary gauge field is a constant field, we can obtain the dual Yang–Mills term. The situation exactly equals to the electric–magnetic duality. But we can find the electric–magnetic duality manifestly from the double gauge fields in all dimensions. The ordinary electric–magnetic duality only occurs in four dimensions. It implies that the exchanging gauge fields should have larger symmetry than the ordinary electric–magnetic duality. The primary reason should be that we lose the Poincaré lemma when exchanging ordinary and dual gauge fields. In the end of this section, we calculate the non-commutative relation at the semi-classical level. We first calculate

$$\begin{aligned}\langle X^A(z)X^B(z')\rangle = -\frac{1}{2\pi}\Bigg[&\mathcal{H}^{AB}\ln|z-z'| - \mathcal{H}^{AB}\ln|z+\bar{z}'| \\ &+\left(\frac{1}{\mathcal{H}^{-1}+\eta F\eta}\mathcal{H}^{-1}\frac{1}{\mathcal{H}^{-1}-\eta F\eta}\right)^{AB}\ln|z+\bar{z}'|^2 \\ &-\left(\frac{1}{\mathcal{H}^{-1}+\eta F\eta}\eta F\eta\frac{1}{\mathcal{H}^{-1}-\eta F\eta}\right)^{AB}\ln\frac{z+\bar{z}'}{\bar{z}+z'}\Bigg]\end{aligned} \tag{33}$$

on the boundary ($z=-\bar{z}$ and $z'=-\bar{z}'$). It can be solved from

$$\mathcal{H}_{AB}(\partial_z+\partial_{\bar{z}})X^B - F_{AB}(\partial_z-\partial_{\bar{z}})X^B = 0. \tag{34}$$

We restrict to the real z and z' and denote them to be τ and $-\tau'$. Hence, we obtain

$$\begin{aligned}< X^A X^B > = &-\frac{1}{2\pi}\left(\frac{1}{\mathcal{H}^{-1}+\eta F\eta}\mathcal{H}^{-1}\frac{1}{\mathcal{H}^{-1}-\eta F\eta}\right)^{AB}\ln|\tau-\tau'|^2 \\ &-\frac{i}{2}\left(\frac{1}{\mathcal{H}^{-1}+\eta F\eta}\eta F\eta\frac{1}{\mathcal{H}^{-1}-\eta F\eta}\right)^{AB}\epsilon(\tau-\tau'),\end{aligned} \tag{35}$$

where $\epsilon(\tau)=1$ when $\tau>0$ and $\epsilon(\tau)=-1$ when $\tau<0$. We interpret τ as time. The non-commutative relation at the semi-classical level is

$$\begin{aligned}[X^A(\tau), X^B(\tau)] &= T\Big(X^A(\tau)X^B(\tau^-) - X^A(\tau)X^B(\tau^+)\Big) \\ &= -i\left(\frac{1}{\mathcal{H}^{-1}+\eta F\eta}\eta F\eta\frac{1}{\mathcal{H}^{-1}-\eta F\eta}\right)^{AB}.\end{aligned} \tag{36}$$

It means that we can find the non-commutative geometry between the ordinary and dual coordinates. The non-commutativity is governed by the field strength. If we do not turn on the dual gauge field, we only have non-commutativity on the ordinary coordinates. For the constant field strength and the $O(D,D)$ boundary conditions, we can embed the semi-classical non-commutative geometry in the $O(D,D)$ structure [52].

4. Boundary conditions from projectors

We use projectors to study more boundary conditions. This way is also suitable for the double gauge fields. We implement the boundary conditions on the σ^1-direction from the projectors. The boundary conditions on the σ^1-direction are

$$\Pi_N \mathcal{H}^{-1} \partial_1 X = 0, \qquad \Pi_D \partial_0 X = 0. \tag{37}$$

The first one is the Neumann-like boundary condition and the other one is the Dirichlet-like boundary condition. Then we will use the projectors to project out the Dirichlet-like boundary condition on σ^1-direction. The projectors (Π_N and Π_D) should satisfy

$$\Pi_N^2 = \Pi_N, \qquad \Pi_N + \Pi_D^T = 1. \tag{38}$$

We can derive

$$\Pi_D^2 = \Pi_D \tag{39}$$

from (38). The equation of motion on the boundary is

$$\Pi_N \left(\mathcal{H}^{-1} \partial_1 X + \eta \partial_0 X \right) = 0. \tag{40}$$

Then we can obtain

$$\Pi_N \left(\mathcal{H}^{-1} \partial_1 X + \eta \Pi_N^T \partial_0 X \right) = 0. \tag{41}$$

If we want to get the Neumann-like boundary condition and remove the Dirichlet-like boundary condition, we need to assume

$$\Pi_N \eta \Pi_N^T = 0. \tag{42}$$

The Neumann-like boundary condition is equivalent to projecting out the dual coordinates. Alternatively, we use η to go to the dual frame. We equivalently exchange the ordinary and dual coordinates. It implies that we have the Dirichlet-like boundary condition with respect to the dual frame. Then we either project out the dual coordinates with respect to the ordinary frame or project out the ordinary coordinates with respect to the dual frame to obtain the Neumann-like boundary condition. If we use η to transform X, we obtain

$$X' = \eta X. \tag{43}$$

Therefore, we can deduce

$$X'^T \left(\Pi_N + \Pi_D^T \right) X' = X^T \eta \left(\Pi_N + \Pi_D^T \right) \eta X. \tag{44}$$

Assuming

$$\eta \Pi_N \eta = \Pi_D, \tag{45}$$

we can obtain

$$X^T \left(\Pi_N^T + \Pi_D \right) X. \tag{46}$$

We use

$$\Pi_N\eta = \eta\Pi_D, \qquad \Pi_N\eta\Pi_N^T = 0 \tag{47}$$

to show

$$\Pi_D\Pi_N^T = 0. \tag{48}$$

It is equivalent to showing

$$\Pi_D\eta\Pi_D^T = 0. \tag{49}$$

We assume

$$\Pi_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{50}$$

From $\Pi_D^T\eta\Pi_D = 0$, we obtain

$$\begin{aligned}
&\begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1-a^T & -c^T \\ -b^T & 1-d^T \end{pmatrix} \\
&\quad = \begin{pmatrix} -b & 1-a \\ 1-d & -c \end{pmatrix}\begin{pmatrix} 1-a^T & -c^T \\ -b^T & 1-d^T \end{pmatrix} \\
&\quad = \begin{pmatrix} -b(1-a^T)-(1-a)b^T & bc^T+(1-a)(1-d^T) \\ (1-d)(1-a^T)+cb^T & -(1-d)c^T-c(1-d^T) \end{pmatrix} = 0.
\end{aligned} \tag{51}$$

The conditions are

$$\begin{aligned}
&b(1-a^T) = -(1-a)b^T, \qquad bc^T = -(1-a)(1-d^T), \\
&(1-d)c^T = -c(1-d^T).
\end{aligned} \tag{52}$$

From $\Pi_N\eta\Pi_N^T = 0$, we obtain

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} &= \begin{pmatrix} b & a \\ d & c \end{pmatrix}\begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \\
&= \begin{pmatrix} ba^T+ab^T & bc^T+ad^T \\ da^T+cb^T & dc^T+cd^T \end{pmatrix} = 0.
\end{aligned} \tag{53}$$

Then we can get conditions

$$ba^T = -ab^T, \qquad bc^T = -ad^T, \qquad dc^T = -cd^T. \tag{54}$$

From $\Pi_N^2 = \Pi_N$, we can obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{55}$$

Therefore, we find

$$a^2+bc = a, \qquad ab+bd = b, \qquad ca+dc = c, \qquad cb+d^2 = d. \tag{56}$$

According to the above conditions, we obtain

$$\begin{aligned}
&b = -b^T, \qquad ba^T = ab, \qquad a+d^T = 1, \qquad bc = a(1-a), \\
&c^T = -c, \qquad a^Tc = ca.
\end{aligned} \tag{57}$$

The above construction is exactly consistent with [38]. The Green's function on the boundary is

$$\Pi_N^T\left(\mathcal{H}^{-1}\partial_1 G - \eta F\eta\partial_0 G\right) = 0. \tag{58}$$

If we use the same way as [51] to obtain the Green's function for all projectors, we should meet trouble. The problem is that the projectors are not invertible. However, we can solve them case by case. For example, we can choose

$$\Pi_N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{59}$$

to find the DBI theory as [44].

5. Generalized metric formulation

We combine the antisymmetric background field and field strength to form a different $O(D, D)$ generalized metric. We use this generalized metric and scalar dilaton to construct the low-energy effective action. From the same generalized metric, we reconstruct our double sigma model. We also check the one-loop β function in the case of the constant background field with the strong constraints. We consistently obtain the DBI action. Finally, we integrate out the dual coordinates of the double sigma model with the strong constraints, then we can obtain the ordinary sigma model. It shows quantum equivalence between the double and ordinary sigma model exactly.

5.1. Low-energy effective action

We consider low-energy effective theory for the closed and open string theory. The open string part is based on the diffeomorphism and one-form gauge transformation. The effective action is described by the DBI action. The closed string theory can be constructed from the $O(D, D)$ structure, $\mathbb{Z}_2$ symmetry, gauge symmetry with the strong constraints and two derivative terms. Here, we redefine our generalized metric by replacing B_{mn} by $B_{mn} - F_{mn}$. Then we can avoid using the field strength to write the low-energy effective action. The $\mathbb{Z}_2$ symmetry is

$$B_{mn} \to -B_{mn}, \qquad \tilde{\partial}^m \to -\tilde{\partial}^m. \tag{60}$$

We can rewrite

$$\tilde{\partial}^m \to -\tilde{\partial}^m \tag{61}$$

as

$$\partial_A \to Z\,\partial_A\,. \tag{62}$$

The transformation of the $\mathcal{H}^{AB}$ under the $\mathbb{Z}_2$ transformation. We have

$$\mathcal{H}^{AB} \to Z\mathcal{H}^{AB}Z\,, \qquad \mathcal{H}_{AB} \to Z\mathcal{H}_{AB}Z\,. \tag{63}$$

Then the action can be constructed from the gauge symmetry (with the strong constraints) by using all possible $O(D, D)$ elements (∂_A, $\mathcal{H}^{AB}$, $\mathcal{H}_{AB}$ and d) up to a boundary term and only considering two derivative terms. The action is

$$S_2 = \int dx\, d\tilde{x}\, e^{-2d}\Big(\frac{1}{8}\mathcal{H}^{AB}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \frac{1}{2}\mathcal{H}^{AB}\partial_B\mathcal{H}^{CD}\partial_D\mathcal{H}_{AC} - 2\partial_A d\partial_B\mathcal{H}^{AB} + 4\mathcal{H}^{AB}\,\partial_A d\partial_B d\Big). \tag{64}$$

The DBI action is

$$S_1 = \int dx\, d\tilde{x}\, e^{-d}\Big(-\det(\mathcal{H}_{mn})\Big)^{\frac{1}{4}}. \tag{65}$$

Because of the boundary conditions are not modified from the strong constraints except for the generalized metric does not depend on the dual coordinates in S_1, we only rewrite the DBI action in terms of the generalized metric and scalar dilation. We combine closed and open string to show the total action.

$$\begin{aligned} S_T &= S_1 + \alpha S_2 \\ &= \int dx\, d\tilde{x}\, \Bigg[e^{-d}\Big(-\det(\mathcal{H}_{mn})\Big)^{\frac{1}{4}} \\ &\quad + \alpha e^{-2d}\Bigg(\frac{1}{8}\mathcal{H}^{AB}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \frac{1}{2}\mathcal{H}^{AB}\partial_B\mathcal{H}^{CD}\partial_D\mathcal{H}_{AC} \\ &\quad - 2\partial_A d\partial_B\mathcal{H}^{AB} + 4\mathcal{H}^{AB}\,\partial_A d\partial_B d\Bigg)\Bigg], \end{aligned} \tag{66}$$

where α is an arbitrary constant. If we use the strong constraints, we obtain

$$\begin{aligned} &\int dx\, \sqrt{-\det g}\Bigg[e^{-\phi}\Big(-\det(g+B-F)\Big)^{\frac{1}{2}}\Big(-\det g\Big)^{-\frac{1}{2}} \\ &\quad + \alpha e^{-2\phi}\Big(R + 4(\partial\phi)^2 - \frac{1}{12}H^2\Big)\Bigg], \end{aligned} \tag{67}$$

where R is the Ricci scalar and $H = dB$ is the three form field strength. If we set $D = 10$, the action is the low-energy effective theory of the D9-brane from the one-loop β function [53].

5.2. *Double sigma model*

We replace B_{mn} by $B_{mn} - F_{mn}$ to reconstruct our double sigma model. We discuss the classical equivalence and implement the self-duality relation at off-shell level. Finally, we calculate the one-loop β function to obtain the desirable DBI action for the constant background fields and integrate out the dual coordinates to get the ordinary sigma model.

5.2.1. *Action*

We replace B_{mn} by $B_{mn} - F_{mn}$ to rewrite our double sigma model without using the boundary term. But we still have the boundary conditions to obtain the effects of the open string. Although we replace B_{mn} by $B_{mn} - F_{mn}$, we still use B in the generalized metric for simplicity. The action is

$$\frac{1}{2}\int d^2\sigma\,\Big(\partial_1 X\mathcal{H}^{-1}\partial_1 X - \partial_1 X\eta\partial_0 X\Big). \tag{68}$$

The boundary conditions on σ^1-direction (the Neumann boundary condition) are

$$\mathcal{H}^m{}_A\partial_1 X^A - \eta^m{}_A\partial_0 X^A = 0, \qquad \mathcal{H}_{mA}\partial_1 X^A = 0, \qquad \eta_{mA}\partial_0 X^A = 0 \tag{69}$$

and the boundary condition on σ^0-direction (the Dirichlet boundary condition) is

$$\delta X^m = 0. \tag{70}$$

We remind that the boundary conditions are not modified from the strong constraints except for the generalized metric does not depend on the dual coordinates.

5.2.2. Classical equivalence

We use an on-shell self-duality relation and strong constraints to show the classical equivalence with the ordinary sigma model. It implies that we can find the same equations of motion as the ordinary sigma model. The equations of motion of (68) on the bulk are

$$\begin{aligned}
&\partial_1\bigg(\mathcal{H}_{mA}\partial_1 X^A - \eta_{mA}\partial_0 X^A\bigg) = \frac{1}{2}\partial_1 X^A\partial_m\mathcal{H}_{AB}\partial_1 X^B,\\
&\partial_1\bigg(\mathcal{H}^m{}_A\partial_1 X^A - \eta^m{}_A\partial_0 X^A\bigg) = \frac{1}{2}\partial_1 X^A\partial^m\mathcal{H}_{AB}\partial_1 X^B.
\end{aligned} \tag{71}$$

If we impose the strong constraints, we can obtain

$$\partial_1\bigg(\mathcal{H}^m{}_A\partial_1 X^A - \eta^m{}_A\partial_0 X^A\bigg) = 0. \tag{72}$$

A suitable self-duality relation is

$$\mathcal{H}^m{}_A\partial_1 X^A - \eta^m{}_A\partial_0 X^A = 0. \tag{73}$$

The self-duality relation is equivalent to

$$\partial_1\tilde{X}_m = g_{mn}\partial_0 X^n + B_{mn}\partial_1 X^n. \tag{74}$$

The other equation of motion is

$$\begin{aligned}
&\partial_1\bigg[\bigg(g - Bg^{-1}B\bigg)_{mn}\partial_1 X^n + \bigg(Bg^{-1}\bigg)_m{}^n\partial_1\tilde{X}_n - \partial_0\tilde{X}_m\bigg]\\
&\quad = \frac{1}{2}\partial_1 X^p\partial_m\bigg(g - Bg^{-1}B\bigg)_{pq}\partial_1 X^q + \partial_1 X^p\partial_m\bigg(Bg^{-1}\bigg)_p{}^q\partial_1\tilde{X}_q\\
&\qquad + \frac{1}{2}\partial_1\tilde{X}_p\partial_m g^{pq}\partial_1\tilde{X}_q.
\end{aligned} \tag{75}$$

We find the same equation of motion as the ordinary sigma model by using the self-duality relation to remove the dual coordinates.

$$\begin{aligned}
&\partial_1\bigg[\bigg(g - Bg^{-1}B\bigg)_{mn}\partial_1 X^n + \bigg(Bg^{-1}\bigg)_m{}^n\partial_1\tilde{X}_n - \partial_0\tilde{X}_m\bigg]\\
&\quad = \partial_1\bigg(g_{mn}\partial_1 X^n + B_{mn}\partial_0 X^n\bigg) - \partial_0\bigg(g_{mn}\partial_0 X^n + B_{mn}\partial_1 X^n\bigg).
\end{aligned} \tag{76}$$

$$\begin{aligned}
&\frac{1}{2}\partial_1 X^p\partial_m\bigg(g - Bg^{-1}B\bigg)_{pq}\partial_1 X^q + \partial_1 X^p\partial_m\bigg(Bg^{-1}\bigg)_p{}^q\partial_1\tilde{X}_q + \frac{1}{2}\partial_1\tilde{X}_p\partial_m g^{pq}\partial_1\tilde{X}_q\\
&\quad = -\frac{1}{2}\partial_0 X^p\partial_m g_{pq}\partial_0 X^q + \frac{1}{2}\partial_1 X^p\partial_m g_{pq}\partial_1 X^q + \partial_1 X^p\partial_m B_{pq}\partial_0 X^q.
\end{aligned} \tag{77}$$

Let us consider the effects of the one-form gauge potential on the bulk. Then the related terms of the equations of motion on the bulk are

$$\begin{aligned}
&\partial_1 B_{mn}\partial_0 X^n - \partial_0 B_{mn}\partial_1 X^n - \partial_1 X^p \partial_m B_{pq}\partial_0 X^q \\
&\quad = \partial_1 X^p \partial_p B_{mq}\partial_0 X^q - \partial_1 X^p \partial_q B_{mp}\partial_0 X^q - \partial_1 X^p \partial_m B_{pq}\partial_0 X^q \\
&\quad = \partial_1 X^p \partial_p B_{mq}\partial_0 X^q + \partial_1 X^p \partial_q B_{pm}\partial_0 X^q + \partial_1 X^p \partial_m B_{qp}\partial_0 X^q \\
&\quad = \partial_1 X^p H_{pmq}\partial_0 X^q.
\end{aligned} \tag{78}$$

It implies that the one-form gauge potential does not have degrees of freedom on the bulk at classical level. On the boundary, we have

$$g_{mn}\partial_1 X^n + B_{mn}\partial_0 X^n = 0. \tag{79}$$

This boundary condition is the ordinary Neumann boundary condition. We show that this double sigma model with the on-shell self-duality relation gives a consistent result with the ordinary sigma model.

5.2.3. Self-duality relation at off-shell level

We implement the self-duality relation at off-shell level in this section. The equations of motion on the bulk are

$$\begin{aligned}
&\partial_1\bigg(g^{-1}\partial_1\tilde{X} - g^{-1}B\partial_1 X - \partial_0 X\bigg)^m = 0, \\
&\partial_1\bigg(Bg^{-1}\partial_1\tilde{X} + \big(g - Bg^{-1}B\big)\partial_1 X - \partial_0\tilde{X}\bigg)_m \\
&\quad = \frac{1}{2}\partial_1 X\partial_m\bigg(g - Bg^{-1}B\bigg)\partial_1 X + \partial_1 X\partial_m\bigg(Bg^{-1}\bigg)\partial_1\tilde{X} + \frac{1}{2}\partial_1\tilde{X}\partial_m g^{-1}\partial_1\tilde{X}.
\end{aligned} \tag{80}$$

To obtain the self-duality relation and same equations of motion as the ordinary sigma model, we shift X^M ($X^M \to X^M + f^M(\sigma^0)$) and redefine g and B. Then we obtain

$$\begin{aligned}
&\partial_1\tilde{X}_m = B_{mn}\partial_1 X^n + g_{mn}\partial_0 X^n, \\
&\partial_1\bigg(g_{mn}\partial_1 X^n + B_{mn}\partial_0 X^n\bigg) - \partial_0\bigg(g_{mn}\partial_0 X^n + B_{mn}\partial_1 X^n\bigg) \\
&\quad = -\frac{1}{2}\partial_0 X^p\partial_m g_{pq}\partial_0 X^q + \frac{1}{2}\partial_1 X^p\partial_m g_{pq}\partial_1 X^q + \partial_1 X^p\partial_m B_{pq}\partial_0 X^q.
\end{aligned} \tag{81}$$

On the boundary, the equations of motion are

$$\begin{aligned}
\partial_1\tilde{X} - B\partial_1 X - g\partial_0 X &= 0, \\
Bg^{-1}\partial_1\tilde{X} + \big(g - Bg^{-1}B\big)\partial_1 X &= 0.
\end{aligned} \tag{82}$$

Therefore, we obtain

$$g\partial_1 X + B\partial_0 X = 0. \tag{83}$$

From the above discussion, the self-duality relation can be implemented at the off-shell level.

5.2.4. One-loop β function for the constant background fields

We compute the one-loop β function for the constant background fields in this section. In the end of this section, we will obtain the consistent DBI action. We first expand X ($X \to \xi$) for the action of the double sigma model. Hence, we obtain

$$\begin{aligned}&\frac{1}{2}\partial_1\xi^m\big(g-Bg^{-1}B\big)_{mn}\partial_1\xi^n+\partial_1\xi^m\big(Bg^{-1}\big)_m{}^n\partial_1\tilde{\xi}_n+\frac{1}{2}\partial_1\tilde{\xi}_m\big(g^{-1}\big)^{mn}\partial_1\tilde{\xi}_n\\&-\frac{1}{2}\partial_1\xi^m\partial_0\tilde{\xi}_m-\frac{1}{2}\partial_1\tilde{\xi}_m\partial_0\xi^m+\partial_1\xi^m\xi^p\partial_p\big(g-Bg^{-1}B\big)_{mn}\partial_1X^n\\&+\partial_1\xi^m\xi^p\partial_p\big(Bg^{-1}\big)_m{}^n\partial_1\tilde{X}_n-\partial_1\tilde{\xi}_m\xi^p\partial_p\big(g^{-1}B\big)^m{}_n\partial_1X^n\\&+\frac{1}{4}\partial_1X^p\xi^m\xi^n\partial_m\partial_n\big(g-Bg^{-1}B\big)_{pq}\partial_1X^q+\frac{1}{2}\partial_1X^p\xi^m\xi^n\partial_m\partial_n\big(Bg^{-1}\big)_p{}^q\partial_1\tilde{X}_q,\end{aligned}\tag{84}$$

where X^m and $\tilde{X}_m$ satisfy the equations of motion. The linear order of ξ^m and $\tilde{\xi}_m$ disappear due to the equations of motion. We also use the strong constraints in our calculation. Because $\tilde{\xi}$ is at quadratic order, we can integrate out $\tilde{\xi}$. This integration is equivalent to the integration of

$$\int d^2\sigma\ \frac{1}{2}\partial_1\phi A\partial_1\phi+\phi\partial_1J,\tag{85}$$

where A ($A=A^T$) and J are not related to ϕ. Then we integrate out ϕ, we obtain

$$\exp\bigg(-\int d^2\sigma\ \frac{1}{2}J\partial_1\Big(\partial_1\big(A\partial_1\big)\Big)^{-1}\partial_1J\bigg)\sim\sqrt{\frac{1}{\det(A)}}\exp\bigg(-\int d^2\sigma\ \frac{1}{2}JA^{-1}J\bigg).\tag{86}$$

The result of the integration on the exponent is equivalent to using

$$A\partial_1\phi=J.\tag{87}$$

We also use

$$(\partial_1)^T=-\partial_1\tag{88}$$

and ∂_1^{-1} vanishes on the boundary. In our case, we use

$$\partial_1\xi^m\big(Bg^{-1}\big)_m{}^n+\partial_1\tilde{\xi}_m\big(g^{-1}\big)^{mn}-\partial_0\xi^n-\xi^p\partial_p\big(g^{-1}B\big)^n{}_m\partial_1X^m=0\tag{89}$$

to calculate the integration in the action. The determinant is not related to the fluctuation fields so we do not need to discuss this term on the calculation of the one-loop β function. When we integrate by parts during the Gaussian integration process, the boundary terms will vanish due to the boundary conditions. Then we separate two parts to discuss. The first part is not related to $\tilde{\xi}$. We calculate all terms related to $\tilde{\xi}$ in the second part. We start from the first part:

$$\begin{aligned}&\frac{1}{2}\partial_1\xi^m\bigg(g-Bg^{-1}B\bigg)_{mn}\partial_1\xi^n+\partial_1\xi^m\xi^p\partial_p\bigg(-Bg^{-1}B\bigg)_{mn}\partial_1X^n\\&\quad+\partial_1\xi^m\xi^p\partial_pB_{mn}\partial_0X^n+\partial_1\xi^m\xi^p\bigg(\partial_p\big(Bg^{-1}\big)B\bigg)_{mn}\partial_1X^n\\&\quad+\frac{1}{4}\partial_1X^p\xi^m\xi^n\partial_m\partial_n\bigg(-Bg^{-1}B\bigg)_{pq}\partial_1X^q+\frac{1}{2}\partial_1X^p\xi^m\xi^n\bigg(\partial_m\partial_n\big(Bg^{-1}\big)B\bigg)_{pq}\partial_1X^q\\&\quad+\frac{1}{2}\partial_1X^p\xi^m\xi^n\partial_m\partial_nB_{pq}\partial_0X^q\end{aligned}$$

$$= \frac{1}{2}\partial_1\xi^m\left(g - Bg^{-1}B\right)_{mn}\partial_1\xi^n - \partial_1\xi^m\xi^p\left(Bg^{-1}\partial_p B\right)_{mn}\partial_1 X^n + \partial_1\xi^m\xi^p\partial_p B_{mn}\partial_0 X^n$$
$$- \frac{1}{2}\partial_1 X^p\xi^m\xi^n\left(\partial_m Bg^{-1}\partial_n B\right)_{pq}\partial_1 X^q + \frac{1}{2}\partial_1 X^p\xi^m\xi^n\partial_m\partial_n B_{pq}\partial_0 X^q. \quad (90)$$

Then we discuss the second part:

$$\frac{1}{2}\partial_1\xi^m\left(Bg^{-1}\right)_m{}^n\partial_1\tilde{\xi}_n - \frac{1}{2}\partial_1\tilde{\xi}_m\partial_0\xi^m - \frac{1}{2}\partial_1\tilde{\xi}_m\xi^p\partial_p\left(g^{-1}B\right)^m{}_n\partial_1 X^n$$
$$= \frac{1}{2}\partial_1\xi^m\xi^p\left(Bg^{-1}\partial_p B\right)_{mq}\partial_1 X^q + \frac{1}{2}\partial_1\xi^m B_{mq}\partial_0\xi^q + \frac{1}{2}\partial_1\xi^m\left(Bg^{-1}B\right)_{mq}\partial_1\xi^q$$
$$- \frac{1}{2}\xi^p\partial_p B_{qm}\partial_1 X^m\partial_0\xi^q - \frac{1}{2}g_{qm}\partial_0\xi^m\partial_0\xi^q + \frac{1}{2}\partial_1\xi^m B_{mq}\partial_0\xi^q$$
$$- \frac{1}{2}\left(\xi^p\partial_p B_{qm}\partial_1 X^m\right)\xi^r\partial_r\left(g^{-1}B\right)^q{}_n\partial_1 X^n - \frac{1}{2}g_{qn}\partial_0\xi^n\xi^r\partial_r\left(g^{-1}B\right)^q{}_s\partial_1 X^s$$
$$+ \frac{1}{2}\partial_1\xi^m B_{mq}\xi^p\partial_p\left(g^{-1}B\right)^q{}_n\partial_1 X^n$$
$$= \partial_1\xi^m B_{mq}\partial_0\xi^q + \frac{1}{2}\partial_1\xi^m\left(Bg^{-1}B\right)_{mq}\partial_1\xi^q - \frac{1}{2}\partial_0\xi^n g_{qn}\partial_0\xi^q - \partial_0\xi^m\xi^p\partial_p B_{mq}\partial_1 X^q$$
$$+ \partial_1\xi^m\xi^p\left(Bg^{-1}\partial_p B\right)_{mq}\partial_1 X^q + \frac{1}{2}\partial_1 X^p\xi^m\xi^n\left(\partial_m Bg^{-1}\partial_n B\right)_{pq}\partial_1 X^q. \quad (91)$$

We combine two parts.

$$-\frac{1}{2}\partial_0\xi^m g_{mn}\partial_0\xi^n + \frac{1}{2}\partial_1\xi^m g_{mn}\partial_1\xi^n + \partial_1\xi^m B_{mn}\partial_0\xi^n$$
$$+ \partial_1\xi^m\xi^p\partial_p B_{mn}\partial_0 X^n - \partial_0\xi^m\xi^p\partial_p B_{mq}\partial_1 X^q + \frac{1}{2}\partial_1 X^p\xi^m\xi^n\partial_m\partial_n B_{pq}\partial_0 X^q. \quad (92)$$

The result is exactly consistent with the second order expansion of the ordinary sigma model. We redefine the one-form gauge field to absorb the constant antisymmetric background field into the one-form gauge field, then we obtain

$$-\frac{1}{2}\partial_0\xi^m g_{mn}\partial_0\xi^n + \frac{1}{2}\partial_1\xi^m g_{mn}\partial_1\xi^n - \partial_1\left(\frac{1}{2}\xi^n\xi^p\partial_n\partial_p A_m\partial_0 X^m + \xi^n\partial_n A_m\partial_0\xi^m\right)$$
$$+ \partial_0\left(\frac{1}{2}\xi^n\xi^p\partial_n\partial_p A_m\partial_1 X^m + \xi^n\partial_n A_m\partial_1\xi^m\right). \quad (93)$$

We impose the boundary conditions and integrate by parts on the boundary term, then we get

$$\int d^2\sigma\,\left(\frac{1}{2}\xi^m g_{mn}\partial_0^2\xi^n - \frac{1}{2}\xi^m g_{mn}\partial_1^2\xi^n\right)$$
$$+ \int d\sigma^0\,\left(\frac{1}{2}\xi^m g_{mn}\partial_1\xi^n + \frac{1}{2}\xi^m\partial_0\xi^n B_{mn} + \frac{1}{2}\xi^m\xi^n\partial_m B_{np}\partial_0 X^p\right). \quad (94)$$

The Green's function on the bulk is

$$g_{mn}\left(\partial_0^2 - \partial_1^2\right)G^{np} = 4g_{mn}\partial_z\partial_{\bar{z}}G^{np} = 2i\delta_m{}^p\delta^2(z - z'), \quad (95)$$

where

$$\delta^2(z-z') \equiv \frac{1}{2}\delta^2(\sigma-\sigma'). \tag{96}$$

The solution of the Green's function on the bulk is

$$G^{np} = -\frac{g^{np}}{4\pi}\ln(z-z') - \frac{g^{np}}{4\pi}\ln(\bar{z}-\bar{z}'). \tag{97}$$

The Green's function on the boundary is

$$g_{mn}\partial_1 G^{np} + B_{mn}\partial_0 G^{np} = \big(g_{mn}+B_{mn}\big)\partial_z G^{np} + \big(g_{mn}-B_{mn}\big)\partial_{\bar{z}} G^{np} = 0. \tag{98}$$

The solution is

$$\begin{aligned} G^{np} &= \mathcal{H}^{np}\ln|\,z-z'\,| + \frac{1}{2}(g+B)^{nq}(g-B)_{qw}\mathcal{H}^{wp}\ln|\,z+\bar{z}'\,| \\ &\quad + \frac{1}{2}(g-B)^{nq}(g+B)_{qw}\mathcal{H}^{wp}\ln(\bar{z}+z')\,|_{z=-\bar{z},\ z'=-\bar{z}'}\,. \end{aligned} \tag{99}$$

The counter term is

$$\frac{1}{2}\int d\sigma^0\ \Gamma_m\partial_0 X^m, \tag{100}$$

where

$$\Gamma_m \equiv \lim_{\epsilon\to 0} G^{np}(\epsilon \equiv \sigma^0-\sigma^{0\prime})\partial_n B_{pm}. \tag{101}$$

Therefore, the β function is

$$\begin{aligned} \beta_m &= -\Big(\mathcal{H}^{np} + \frac{1}{2}(g+B)^{nq}(g-B)_{qw}\mathcal{H}^{wp} + \frac{1}{2}(g-B)^{nq}(g+B)_{qw}\mathcal{H}^{wp}\Big)\partial_n B_{pm} \\ &= -2\Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)^{np}\partial_n B_{pm}. \end{aligned} \tag{102}$$

Multiplying $\Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)$ at both sides, we obtain

$$\begin{aligned} \Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)^{mn}\beta_n &= 2\bigg\{\partial^p\Big[\Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)^{mn}B_{np}\Big] \\ &\quad - \Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)^{mx}\mathcal{H}_x{}^w\partial_t B_{wq}\Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)^{qp}\mathcal{H}_p{}^t\bigg\}. \end{aligned} \tag{103}$$

The equation of motion of the DBI model is equivalent to

$$\sqrt{\det(g+B)}\Big(\big(\mathcal{H}_{yz}\big)^{-1}\Big)^{mn}\beta_n = 0. \tag{104}$$

Although we do not show the non-constant background case, it should be consistent with the ordinary sigma model. We can follow [49] to obtain the massless closed string theory from the bulk.

5.3. Quantum equivalence with the strong constraints

We show that this double sigma model with the strong constraints can be quantum equivalence with the ordinary sigma model. We integrate out the dual coordinates, then we obtain the same ordinary sigma model. When we do Gaussian integration, the result of the integration on the exponent is equivalent to using

$$\partial_1\tilde{X}_p = g_{pn}\partial_0 X^n + B_{pn}\partial_1 X^n. \tag{105}$$

Then we integrate out the dual coordinates:

$$\begin{aligned}
&\frac{1}{2}\partial_1 X^m\Big(g - Bg^{-1}B\Big)_{mn}\partial_1 X^n + \partial_1 X^m\Big(Bg^{-1}\Big)_m{}^n\partial_1\tilde{X}_n + \frac{1}{2}\partial_1\tilde{X}_m\Big(g^{-1}\Big)^{mn}\partial_1\tilde{X}^n \\
&\quad - \partial_1\tilde{X}_m\partial_0 X^m \\
&= \frac{1}{2}\partial_1 X^m\Big(g - Bg^{-1}B\Big)_{mn}\partial_1 X^n + \frac{1}{2}\partial_1 X^m\Big(Bg^{-1}\Big)_m{}^n\partial_1\tilde{X}_n - \frac{1}{2}\partial_1\tilde{X}_m\partial_0 X^m \\
&= \frac{1}{2}\partial_1 X^m\Big(g - Bg^{-1}B\Big)_{mn}\partial_1 X^n + \frac{1}{2}\partial_1 X^m B_{mn}\partial_0 X^n + \frac{1}{2}\partial_1 X^m\Big(Bg^{-1}B\Big)_{mn}\partial_1 X^n \\
&\quad - \frac{1}{2}\partial_0 X^m g_{mn}\partial_0 X^n + \frac{1}{2}\partial_1 X^m B_{mn}\partial_0 X^n \\
&= -\frac{1}{2}\partial_0 X^m g_{mn}\partial_0 X^n + \frac{1}{2}\partial_1 X^m g_{mn}\partial_1 X^n + \partial_1 X^m B_{mn}\partial_0 X^n.
\end{aligned} \tag{106}$$

Then we discuss about the measure. When we perform the Gaussian integration, we have a non-trivial determinant term. The measure of the double sigma model

$$\int DX^A \tag{107}$$

becomes

$$\int DX^m\sqrt{\det g} \equiv \int D'X^m \tag{108}$$

when we integrate out the dual coordinates. We obtain the diffeomorphism invariant measure ($D'X^m$) with shift symmetry. This measure exactly satisfies suitable conditions in the ordinary sigma model. This measure term has some interesting observations. The double sigma model with the global $O(D,D)$ invariant measure becomes the diffeomorphism invariant measure after we impose the strong constraints and integrate out the dual coordinates. This results may imply that restoration of diffeomorphism on the double space should change the measure. Therefore, the double sigma model with diffeomorphism symmetry may come from adding one more metric field. The consistency condition is that we choose a gauge fixing to go back to the double sigma model with an $O(D,D)$ symmetry. Although we need to have one more metric field from this thought, we can have the gauge symmetry. It possibly shed the light on defining a local symmetry on the double space.

We integrate out the dual coordinates to obtain the quantum equivalence with the ordinary sigma model. Alternatively, we can integrate out the ordinary coordinates with $\partial_M = 0$. Then we obtain the dual sigma model (replacing the ordinary coordinates by the dual coordinates and using the non-commutative variables in the ordinary sigma model). We find the same situation in the generalized metric formulation at low-energy level. This result shows that if we use the strong constraints, the role of the dual coordinates is like an auxiliary field. The double sigma model only gives us new understanding about the duality. Without considering the duality, double geometry with the strong constraints only contains the same information as the ordinary sigma model. However, double geometry lets us to redefine the T-duality rule by enlarging from the $O(d, d)$ to the $O(D, D)$ structure. It possibly gives us some connection about the non-commutative geometry and the manifest T-duality. In this double sigma model, we redefine the generalized metric. From this redefinition, we have a one-form gauge field on the bulk. This term does not have contributions with the strong constraints on the bulk. The interesting issue is that the one-form gauge symmetry still remains without using the strong constraints. The Seiberg–Witten map of the open string theory is based on the one-form gauge symmetry. From this point of view, we should have the Moyal product on the bulk in the case of the constant background. Although this one-form gauge potential does not have contributions in the double sigma model with the strong constraints. This one-form gauge potential enhance the symmetry structure to define the non-commutative geometry of closed string theory. We leave this interesting subject in the future.

6. Discussion and conclusion

We discuss boundary conditions and formulate the new double sigma model by combining the field strength and antisymmetric background field. This paper should be the first one to discuss the double geometry with boundary conditions. The first method is the double gauge fields. We double gauge fields on the boundary. Then this theory is fully $O(D, D)$ invariant. The $O(D, D)$ invariance was broken down due to the boundary conditions without doubling coordinates. After that, we compute the one-loop β function to obtain the DBI-like action. The difference between the DBI-like and DBI theories is that the gauge fields are doubled in the DBI-like case, but not in the DBI case. If we want to obtain the DBI action, we can let the dual gauge field be a constant. The generalized metric also appears in the action. The generalized metric govern the manifest T-duality rule and equivalence between the closed and open string parameters. The double gauge fields possibly be helpful in the manifest electric–magnetic duality. On the flat background, the electric–magnetic duality is equivalent to exchanging electric and magnetic fields. This situation is the same as the double gauge fields. At the end of the double gauge fields, we show the non-commutative relation at the semi-classical level. The non-commutativity exists in double coordinates and relies on the field strength. We also use the projectors to realize different boundary conditions on σ^1-direction. In this part, we only show conditions for the projectors. Because the projectors are not invertible, it causes a problem in considering generic cases for the one-loop β function. However, we can choose a particular projector to go back to the DBI action. The calculation is the same as [44]. Then we extend our understanding for the ordinary boundary conditions. We combine the field strength and antisymmetric background field to reconstruct a double sigma model. We show the classical equivalence and implement the self-duality relation at the off-shell level. At the end of the double sigma model, we check the one-loop β function for the constant background fields and integrate out the dual coordinates to obtain the ordinary sigma model.

Double gauge fields should be an idea framework to consider the manifest electric–magnetic duality. Although it is still far from solving this problem, we already show how to realize it in the case of the flat background. The electric–magnetic duality of the non-abelian group is still not understood at this stage. We want to use the picture of the doubled gauge fields to probe the electric–magnetic duality of the non-abelian group. It should teach us more about the multiple M5-brane theory.

We construct projectors to realize boundary conditions on σ^1-direction. We find the consistent DBI action in a particular projector. It should be interesting to study the one-loop β function for the general projectors. We believe that different choices of boundary conditions should give different low-energy effective theories. These theories should be beyond the ordinary string theory. This method is also valid for the double gauge fields. However, we leave unfinished parts in the future.

We construct a double sigma model from combination of the antisymmetric background field and field strength. The main difference are the boundary term and self-duality relation. This double sigma model do not have boundary term, but it has the consistent boundary conditions and equations of motion with the self-duality relation and strong constraints. This consistency comes from the modification of the self-duality relation. The field strength goes into the self-duality relation. It explains why we do not have the boundary term. The old double sigma model does not need the self-duality relation on the boundary, but this new double sigma model needs. We can say that the old double sigma model is a simplified version of this new double sigma model. This double sigma model naively shows that the bulk term has the effects of the one-form gauge field. But we can show that the effects of the one-form gauge field only appear in the boundary term at classical and quantum level with the strong constraints. The calculation of the one-loop β function should be harder than the ordinary string theory. Furthermore, we show that this double sigma model is calculable and we also get the consistent answers. The elements of the double sigma model are the full $O(D, D)$ elements, but it does not have the full $O(D, D)$ invariance because the boundary conditions break the $O(D, D)$ invariance. If we want to have the $O(D, D)$ boundary conditions, we need to do similar construction with the double gauge fields. The gauge symmetry of the double sigma model is also interesting. Due to the combination of the antisymmetric background and field strength, we have the one-form gauge symmetry without using the strong constraints. It implies that we should have the Moyal product on the bulk. The non-commutative description of the open string theory is an economic way to obtain background effects to all orders. If we can find the Seiberg–Witten map explicitly, we should find the background effects to all orders on the bulk. This double sigma model should shed the light on obtaining the α' correction from the non-commutative geometry. Therefore, we can mention that double field theory should be beyond rewriting theories. Double field theory gives us a hope to develop new methods on calculations.

We show quantum equivalence by integrating out the dual coordinates. The calculation shows that this double sigma model with the strong constraints should be exactly equivalent to the ordinary sigma model beyond the one-loop level. The dual sigma model can be obtained by integrating out the ordinary coordinates with $\partial_M = 0$. We use this way to observe the manifest invariance by exchanging the ordinary and dual coordinates as the generalized metric formulation at low-energy level. We also obtain a correct diffeomorphism invariant measure. This measure term possibly reflect that a diffeomorphism measure on the double space may be constructed by adding one more metric field. Then we point out some future directions. It should be interesting to study the double sigma model without the strong constraints for quantization and one-loop β function. Quantization should show the non-commutative relation between the ordinary and dual coordinates. It should be interesting to compare open string in the constant background with the closed string in the generic background. The most interesting direction of the one-loop β function should be the low-energy effective action of the double geometry. Then we can find what kind of low-energy theory arisen from the fluctuation of the ordinary and dual coordinates. For the closed string, it is already done in [49]. The unsolved problem is the boundary part. From the generalized metric formulation of the closed string theory without the strong constraints, we should expect that the field strength should have effects on the bulk. Even if we consider constant background fields, it is still non-trivial because we need to consider bulk and boundary terms simultaneously. We do not have any evidences to show that this low energy effective action can be found when considering the fluctuation of the ordinary and dual coordinates simultaneously. But we remind that the boundary conditions are not modified from the strong constraints. It may imply that the DBI term does not have modification when considering the fluctuation of the dual coordinates. However, it should be interesting to give a new perspective for the generalized metric formulation [27]. Finally, we also comment that the generalized metric, which is the combination of the field strength and antisymmetric background field. The generalized metric govern the semi-classical non-commutative geometry. Then we naively argue that the non-commutative geometry of closed and open theory *cannot* be decoupled in the double geometry. We should consider them simultaneously. This structure is not known before because we do not have the non-commutative structure on the closed string theory without doubling coordinates. However, this double sigma model should be more clearer on this point. It might be a clue that the T-duality should be more suitable on the non-commutative space. If we expect that the duality is a way to unify our theories, we should define string theory on the non-commutative space.

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