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ON THE UNIT INTERVAL NUMBER OF A GRAPH

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The unit interval number of a simple undirected graph G, denoted $i_u(G)$, is the least nonnegative integer r for which we can assign to each vertex in G a collection of at most r closed unit intervals on the real line such that two distinct vertices v and w of G are adjacent if and only if some interval for v intersects some interval for w. This concept generalizes the notion of a unit interval graph in the same way as the previously studied interval number generalizes the notion of an interval graph We present the following results on the unit interval number Let G be a graph on n vertices Then $i_u(G) \le \lceil \frac{1}{2}(n-1) \rceil$ For even n, the extremal graphs are $K_{1,n-1}$ and C_4 For odd $n \ge 3$, the extremal graphs are C_5, those graphs which contain induced copies of $K_{1,n-2}$ and (if $n=5$) those graphs with an induced C_4. These results suggest the question whether the unit interval number is unbounded for claw-free graphs, which we answer in the affirmative On the other hand, we find that $i_u(G) \le 3$ when G is the complement of a forest. In addition, we also present an upper bound on $i_u(G)$ in terms of the edge number of G, together with a characterization of the corresponding extremal graphs

1. Introduction

All graphs considered in the present paper are finite, simple, and undirected. For graph-theoretic terminology not defined in this paper, we refer to the book of Bondy and Murty [4]. The letters G and H always denote graphs. $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The *interval number* of G, denoted $i(G)$, is the least nonnegative integer r for which we can assign to each vertex of G a collection of at most r intervals on the real line such that two distinct vertices v and w of G are adjacent if and only if some interval for v intersects some interval for w. Graphs with interval number at most one are called *interval graphs*. If, in these definitions, we consider closed unit intervals rather than arbitrary intervals, then we obtain the concept of the *unit interval number* $i_u(G)$ of a graph G and the notion of a *unit interval graph*.

Interval graphs and unit interval graphs have extensively been studied; see e.g. the books of Fishburn [6], Golumbic [7] and Roberts [15] for a survey on the various results on interval graphs and unit interval graphs. (The latter are also known *as indifferences graphs* [6].) The interval number was introduced by Trotter and Harary [19] and, independently, by Griggs and West [9]. More recent results on the interval number are contained in [1,2,5,10-14,16-18,20]; for a survey see West

[21, pp. 322–325]. We also mention that interval graphs as well as unit interval graphs and the interval number have various applications, mainly in scheduling, allocation and storage problems; see e.g. [6, 7, 13–16].

In contrast to the interval number which was studied in several papers, almost no results concerning the unit interval number occur in the literature. As an exception, we mention a result of Maas [13, Corollary 2] which computes $i_u(G)$ for the case that G is triangle-free. Let us denote by $\Delta(G)$ the maximum degree of G. Then [13, Corollary 2] implies that, for connected triangle-free G, $i_u(G)=\frac{1}{2}\Delta(G)$ if $\Delta(G)$ is even and G has at least one vertex of degree less than $\Delta(G)$, and, otherwise, $i_u(G)=\lceil\frac{1}{2}(\Delta(G)+1)\rceil$. (This result is closely related to [9, Theorem 2 and Corollary].)

Let G be a graph on n vertices. The starting point for our investigations is a result of Griggs [8] which states that $i(G)\leq\lceil\frac{1}{4}(n+1)\rceil$; this bound is sharp for all n and, for $n\equiv 0 \pmod 4$, $K_{n/2,n/2}$ is known to be the unique extremal graph [1]. Theorem 2.1 of the present paper solves the corresponding extremal problem for the unit interval number: For each graph G on n vertices, $i_u(G)\leq\lceil\frac{1}{2}(n-1)\rceil$. For even n, the extremal graphs are $K_{1,n-1}$ and C_4; for odd $n\geq 3$, the extremal graphs are C_5, those graphs which contain induced copies of $K_{1,n-2}$ and (if $n=5$) those graphs with an induced C_4. (C_n denotes the cycle of length n and, as usual, $K_{n,m}$ denotes a complete bipartite graph.) As a consequence of Theorem 2.1 we shall also obtain a sharp upper bound on $i_u(G)$ in terms of the edge number of G together with a characterization of the corresponding extremal graphs. The result of Theorem 2.1 suggests the question whether the unit interval number is unbounded for *claw-free* graphs, i.e., for graphs which do not contain $K_{1,3}$ as an induced subgraph. We shall give an affirmative answer to this question by proving the following Theorem 3.1. For each positive integer t, there exists a bipartite graph G such that $i(\bar{G})\geq t$ for the complement $\bar{G}$ of G. On the other hand, there are certain subclasses of the class of claw-free graphs for which the unit interval number is bounded. It is easy to see that $i_u(G)\leq 2$ for line-graphs G; in addition, we shall show (Theorem 3.2) that $i_u(G)\leq 3$ when G is the complement of a forest. We conjecture that this is best possible for complements of forests.

We need some additional terminology. A *representation of* G is a mapping f that assigns to each $v\in V(G)$ a finite set of intervals on the real line such that for each pair of distinct $v,w\in V(G)$, v and w are adjacent if and only if some interval of $f(v)$ intersects some interval of $f(w)$. If $|f(v)|\leq k$ for each $v\in V(G)$, then f is called a *k-representation* of G. An interval $I\in f(v)$ will be called a *v-inverval* or an *interval for* v. A representation f of G is *proper* if no interval of f is properly contained in some other interval of f. It is not hard to see that G has a proper k-representation if and only if G has a k-representation in which all intervals are closed unit intervals. (This may be viewed as an immediate consequence of the well-known result that a graph is a proper interval graph if and only if it is a unit interval graph [7, p. 187].) For intervals $I_1=[a_1,b_1]$ and $I_2=[a_2,b_2]$, we write $I_1<I_2$ if $a_1<a_2$ and $b_1<b_2$. The *depth* of a representation f is the maximum number of intervals of f that have a common point. The symbol $\mathbb{N}$ denotes the positive integers.

2. Upper bounds and the corresponding extremal graphs

Theorem 2.1. *Let G be a graph on n vertices. Then* $i_u(G) \le \lceil \frac{1}{2}(n-1) \rceil$. *For even n, the extremal graphs are* $K_{1,n-1}$ *and* C_4. *For odd* $n \ge 3$, *the extremal graphs are* C_5, *those graphs which contain induced copies of* $K_{1,n-2}$ *and (if* $n=5$*) those graphs with an induced* C_4.

Proof. The proof of Theorem 2.1 splits into two parts of unequal difficulty. Below we shall only show $i_u(G) \le \lceil \frac{1}{2}(n-1) \rceil$, which (together with the proof that this bound is achieved by the graphs specified above) is the easy part. Owing to its length, the proof that there are no other extremal graphs is omitted; it is contained in the preprint version [3] of the present paper which interested readers can request from the author.

For the proof of $i_u(G) \le \lceil \frac{1}{2}(n-1) \rceil$ it clearly suffices to consider odd values of n and thus let $n=2k+1$. We proceed by induction on k. For $k=0$ our claim is trivial. Now let $k \ge 1$. If G is a complete graph, then $i_u(G)=1$ and thus it may be assumed that G is not complete. Pick distinct vertices x and y of G such that $(x,y) \notin E(G)$ and represent these vertices by a series of $2k$ disjoint intervals $I_i = [a_i, b_i]$ $(i=1,\dots,2k)$, where $a_i < b_i < a_{i+1} < b_{i+1}$ $(i=1,\dots,2k-1)$ and I_i is an interval for x (y) if i is odd (even). Let G' be the induced subgraph of G spanned by the vertices distinct from x and y. Let $I'_1, I'_2, \dots, I'_{2k-1}$ be disjoint closed intervals such that I'_i overlaps with both I_i and I_{i+1}. Since $|V(G')| = 2k-1$ we can represent each vertex of G' by exactly one of the intervals I'_i. Now, by the induction hypothesis, we can find a proper $(k-1)$-representation of G' in which all intervals are disjoint from the intervals placed so far. Thus if we appropriately shorten the intervals I'_i, we obtain a proper k-representation of G. This proves our claim, since (by a remark made in the introduction) each graph with a proper k-representation also has a k-representation with closed unit intervals.

The proof that the graphs specified in Theorem 2.1 are extremal graphs is left to the reader. By a refinement of the above inductive argument one can show that there are no other extremal graphs; for the details the reader is referred to [3]. Here we just mention that, perhaps not surprisingly, the case when n is even is a special case which is much easier to settle than the general case. □

As a consequence of Theorem 2.1, one gets the following corollary. We remark that finding a sharp upper bound for $i(G)$ in terms of the number of edges of G is an open problem; see [9].

Corollary 2.2. *Let G be a graph with l edges and suppose that G has no isolated vertices. Then* $i_u(G) \le \lceil \frac{1}{2} l \rceil$. *For l odd the unique extremal graph is* $K_{1,l}$. *For l even the extremal graphs are* C_4 *and those graphs that contain an induced copy of* $K_{1,l-1}$.

Proof. Let $n = |V(G)|$. One easily finds that it suffices to prove the theorem for connected graphs G; we may assume that $n-1\leq l$. Hence by Theorem 2.1, $\iota_u(G)\leq\lceil\frac{1}{2}(n-1)\rceil\leq\lceil\frac{1}{2}l\rceil$. Let l be odd. Then, clearly, $\iota_u(K_{1,l})=\lceil\frac{1}{2}l\rceil$. Conversely, if $\iota_u(G)=\lceil\frac{1}{2}l\rceil$, then $\iota_u(G)=\lceil\frac{1}{2}(n-1)\rceil$ and, consequently, since l is odd, $n-1=l$. Hence n is even and we conclude from Theorem 2.1 that $G\cong K_{1,l}$ or $G\cong C_4$; however, the latter would contradict $n-1=l$ and thus we are done. Now, let l be even. If a copy of $K_{1,l-1}$ is an induced subgraph of G or $G\cong C_4$, then clearly $i_u(G)=\lceil\frac{1}{2}l\rceil$. Conversely, if $\iota_u(G)=\lceil\frac{1}{2}(n-1)\rceil=\lceil\frac{1}{2}l\rceil$, then $n-1=l$ or $n=l$. If $n=l$, then one concludes from Theorem 2.1 that $G\cong C_4$ and if $n-1=l$, then Theorem 2.1 implies that a copy of $K_{1,l-1}$ is an induced subgraph of G. □

We close this section with the remark that $\lceil\frac{1}{2}(\Delta(G)+1)\rceil$ is a sharp upper bound for $\iota_u(G)$; this follows immediately from the proof and the corollary of [9, Theorem 2].

3. Claw-free graphs

Since in Theorem 2.1 all extremal graphs with at least six vertices contain $K_{1,3}$ as an induced subgraph it is natural to ask whether there are claw-free graphs of arbitrarily high (unit) inverval number. For related questions concerning bipartite $K_{m,m}$-free graphs, see also the paper of Erdos and West [5]. We shall consider certain graphs $G_{t,n}$ which are complements of bipartite graphs and which are defined below. Similar graphs were used by Trotter to show that the interval number of triangulated graphs is unbounded; Trotter's proof is unpublished but can be found in [17].

Let $a_1,a_2,\dots$ be an infinite sequence of distinct vertices and let $A_n=\{a_1,\dots,a_n\}$, $n=1,2,\dots$. For all $n,t\in\mathbb{N}$, $n\geq t$, let $B_{t,n}$ be a set of $\binom{n}{t}$ vertices such that $B_{t,n}\cap A_n=\emptyset$. Consider a fixed one-to-one correspondence between members of $B_{t,n}$ and the t-element subsets of A_n. Let $G_{t,n}$ be the graph with vertex set $A_n\cup B_{t,n}$ and the following edges. Draw edges between any two vertices of A_n and between any two vertices of $B_{t,n}$; further draw an edge between a_i and $b\in B_{t,n}$ if and only if a_i is contained in the subset of A_n which corresponds to b. With these notations we shall show the following. (We assume in the proof of Theorem 3.1 that all intervals under consideration are closed and bounded; this can be done w.l.o.g. since it can easily be shown that each graph which has a k-representation also has a k-representation with closed and bounded intervals.)

Theorem 3.1. *For each $t\in\mathbb{N}$, there exists a positive integer $n(t)$ such that $\iota(G_{t,n})\geq t$ for all $n\geq n(t)$.*

Proof. Assume that, for a fixed $t\in\mathbb{N}$, the assertion does not hold. It follows that $\iota(G_{t,n})\leq t-1$ for all $n\geq t$. For all $n\geq t$, let us consider a fixed $(t-1)$-representation f_n of $G_{t,n}$. For all $n\geq t$ and $j\in\{1,\dots,n\}$, let $\mathcal{I}_n^j$ be the set of a_j-intervals of f_n which intersect at least one b-interval of f_n, $b \in B_{t,n}$. Let further $\mathscr{I}_n$ be the union of the sets $\mathscr{I}_n^1, \ldots, \mathscr{I}_n^n$.

(1) Let $n \geq t$ and let r be a point on the real line. Then r is contained in at most $2t$ intervals $I \in I_n$.

Proof of (1). Assume that r is contained in $2t+1$ intervals $I_i = [x_i, y_i] \in \mathscr{I}_n$, $i = 1, \ldots, 2t+1$. Then, clearly, there exists a certain $k \in \{1, \ldots, 2t+1\}$ such that $x_i \leq x_k$ for at least t subscripts $i \in \{1, \ldots, 2t+1\}$ and such that $y_k \leq y_j$ for at least t subscripts $j \in \{1, \ldots, 2t+1\}$. By the definition of $\mathscr{I}_n$, there is a b-interval I for some $b \in B_{t,n}$ such that $I \cap I_k \neq \emptyset$. It follows that I intersects at least $t+1$ of the intervals $I_1, \ldots, I_{2t+1}$, which is a contradiction to the fact that b has only t neighbors in A_n. (We assume w.l.o.g. that intervals representing the same vertex are disjoint.) Hence (1).

For all $n \geq t$ define the graph H_n as follows. Let $V(H_n) = A_n$ and let distinct $a_i, a_j \in A_n$ be joined by an edge of H_n if and only if some interval of $\mathscr{I}_n^i$ intersects some interval of $\mathscr{I}_n^j$. Then there exists a positive integer p (which depends only on t) such that

(2) for all $n \geq t$, H_n contains no clique with more than p vertices.

This immediately follows from (1) and the fact that the depth of a m-representation of a complete graph K_r exceeds $r/2m$, see Scheinerman [17, 18] or Maas [14].

From (2) and the theorem of Ramsey, one finds that, for each $k \in \mathbb{N}$, there exists a positive integer $r(k)$ such that $H_{r(k)}$ has a stable set S_k of k vertices. If k is sufficiently large then there exists a subset $T \subseteq S_k$, $|T| = t$, such that for each pair $a_i, a_j \in T$ $(i \neq j)$ the following holds. If $I \in \mathscr{I}_{r(k)}^i$ and $J \in \mathscr{I}_{r(k)}^j$ such that $I < J$, then for some $a_l \in S_k \setminus T$ there exists an a_l-interval $L \in \mathscr{I}_{r(k)}^l$ such that $I < L < J$. (This again can be obtained by using Ramsey's theorem. Draw an edge between two vertices $a_i, a_j \in S_k$ $(i \neq j)$ if there are intervals $I \in \mathscr{I}_{r(k)}^i$, $J \in \mathscr{I}_{r(k)}^j$, $I < J$, for which there exists no a_l-interval $L \in \mathscr{I}_{r(k)}^l$ such that $I < L < J$ and $a_l \in S_k \setminus \{a_i, a_j\}$. Then, clearly, the maximum degree of the resulting graph G_k is at most $2(t-1)$. Thus, if k is large, then G_k contains a stable set of t vertices. From this the assertion readily follows.)

For T with these properties let b be the vertex of $B_{t,r(k)}$ that corresponds to T. It follows that $t-1$ intervals for b are not enough to meet intervals for all vertices of T. (Note that no interval for b is allowed to intersect an interval L that "separates" I from J, where L, I and J are as above.) This contradicts the assumption that $f_{r(k)}$ is a $(t-1)$-representation of $G_{t,r(k)}$. □

Thus in general the interval number is not bounded for complements of bipartite graphs, however, we have the following result on complements of forests.

Theorem 3.2. *Let G be a graph such that $\bar{G}$ is a forest. Then* $i_u(G) \leq 3$.

Proof. We start with labelling the vertices of G in a one-to-one manner by positive integers $1, 2, \ldots, |V(G)|$ as follows. First pick an arbitrary vertex in each component of $\bar{G}$ as a root and label these roots $1,2,\dots,k$ where k is the number of components of $\bar{G}$. For $x\in V(G)$, let $h(x)$ denote the length of the unique path in $\bar{G}$ which connects x with one of the roots. For each $x\in V(G)$ which is not one of the roots, let us denote by x' the predecessor of x, i.e. x' is the unique vertex for which $h(x')=h(x)-1$ and $(x,x')\in E(\bar{G})$. Further for all $x\in V(G)$, let $S(x)=\{y\in V(G)\colon x=y'\}$. We now label the unlabeled vertices of $V(G)$ by $k+1,\dots,|V(G)|$ such that (i) if $h(x)<h(y)$, then the label of x is smaller than the label of y, (ii) if $h(x)=h(y)$ and if the label of x' is smaller than the label of y', then the label of x is smaller than the label of y. Clearly, such a labeling is possible. For the rest of our discussion, let us identify the vertices of G with their corresponding labels.

We now represent G. First, for all $j\in V(G)$ such that $h(j)$ is even, place a unit interval I_j for j such that these intervals have a point on the real line in common and such that $I_i<I_j$ whenever $i<j$. Thereafter, for each $j\in V(G)$ such that $h(j)$ is odd, place two unit intervals I_j^1 and I_j^2 for j such that the following hold. (i) $I_j^1<I_i<I_j^2$ for all i with $h(i)$ even and (ii) I_j^1 and I_j^2 intersect as many of the intervals I_i as possible, i.e., I_j^1 intersects exactly the intervals I_i with $i<j'$ and I_j^2 intersects exactly the intervals I_i for which $i>t$ for all $t\in S(j)\cup\{j'\}$. Note also that, clearly, it can be achieved that all intervals I_j^1 have a point on the real line in common.

By this construction certain (but not all) edges of G are represented. In order to represent the remaining edges, we place unit intervals J_i which are disjoint from the intervals placed so far. We proceed inductively as follows. Start with placing a unit interval J_1 for 1 and assume inductively that we have already placed unit intervals $J_1<J_2<\cdots<J_{t-1}$ for $2\le t\le |V(G)|$, where J_i is an interval for i $(1\le i\le t-1)$. Then place a unit interval J_t for t such that $J_{t-1}<J_t$ and such that J_t intersects exactly the intervals J_i for which $t^*<i\le t-1$, where $t^*=0$ if t is a root and $t^*=t'$ otherwise.

This completes our construction. One readily finds that this defines a representation of G as desired. (We leave the proof to the reader.) $\square$

I do not know whether the bound of Theorem 3.2 is best possible but conjecture that it is. In this context it would also be interesting to know if there exists a forest F such that $i(\bar{F})=3$.

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