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# Planar Graphs with Circular Chromatic Numbers between 3 and 4

Xuding Zhu

*Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan 80424*
E-mail: zhu@math.nsysu.edu.tw

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This paper proves that for every rational number $r$ between 3 and 4, there exists

number $r$ between 2 and 4 is the circular chromatic number of a planar graph.



## 1. INTRODUCTION

The circular chromatic number (also known as the star chromatic number) $\chi_c(G)$ of a graph $G$ is a natural generalization of the chromatic number of a graph introduced by Vince in 1988 [14]. For a pair of positive integers $k$, $d$ with $k \geqslant d$, a $(k, d)$-colouring of a graph $G$ is a mapping $c$ of $V(G)$ to the set $\{0, 1, ..., k-1\}$ such that for any adjacent vertices $x$, $y$ of $G$, $d \leqslant |c(x) - c(y)| \leqslant k - d$. The *circular chromatic number* $\chi_c(G)$ of a graph $G$ is the infimum of the ratios $k/d$ for which there exists a $(k, d)$-colouring of $G$. It was proved in [1, 14] that for finite graphs the infimum in the definition is always attained and hence can be replaced by the minimum. This also implies that the circular chromatic number of a finite graph is always a rational number. Some other basic properties of the circular chromatic number of a graph can be found in [1, 14, 16]. The proof in [14] use the continuous methods, while a simple combinatorial treatment of the concept was given in [1]. An alternate definition of the circular chromatic number was given in [16].

It is easy to see that a $(k, 1)$-colouring of a graph $G$ is just an ordinary $k$-colouring of $G$. Therefore $\chi_c(G) \leqslant \chi(G)$ for any $G$. On the other hand, it is proved in [14] that $\chi_c(G) > \chi(G) - 1$. Thus if we know the circular chromatic number of a graph $G$ then $\chi(G)$ is just the ceiling of $\chi_c(G)$.



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However, two graphs of the same chromatic number may have different circular chromatic numbers. In this sense, $\chi_c(G)$ is a refinement of $\chi(G)$, and it contains more information about the structure of the graph than $\chi(G)$ does.

The concept of circular chromatic number has attracted considerable attention in the past ten years, see [21] for a survey. Questions concerning circular chromatic number of planar graphs were asked by Vince when he introduced the concept, and have been studied by many authors [4, 7, 10, 13, 15–18]. It is natural to ask for which rational number $r$ there is a planar graph $G$ with $\chi_C(G)=r$? By the Four Colour Theorem, for any planar graph $G$ we have $\chi_c(G)\leqslant\chi(G)\leqslant 4$. On the other hand, any non-trivial graph has circular chromatic number at least 2. Therefore $2\leqslant\chi_c(G)\leqslant 4$ for any non-trivial planar graph $G$. For a long time there was little progress on this problem except that $\chi_c(G)$ was determined for some special classes of planar graphs. In particular, it was unknown whether or not there are planar graphs whose circular chromatic numbers are less than 3, but arbitrarily close to 3, and planar graphs whose circular chromatic numbers are less than 4, but arbitrarily close to 4. This led to the suspicion that there might be some gaps between rational numbers which are the circular chromatic numbers of planar graphs. Recently, however, Moser [10] proved the somewhat surprising result that every rational number $r$ between 2 and 3 is the circular chromatic number of a planar graph. This result changed our expectation, and Moser [10] asked whether or not every rational number between 3 and 4 is the circular chromatic number of a planar graph. We shall answer this question in affirmative. We shall prove the following:

THEOREM 1.1. *For any rational number r between* 3 *and* 4, *there exists a planar graph G such that* $\chi_c(G)=r$.

Together with Moser's result, we arrive at the conclusion that every rational number between 2 and 4 is indeed the circular chromatic number of a planar graph. The basic idea of the construction in this paper is as in that of Moser's construction in [10], but the techniques are more complicated. The proof is similar to the proof of [17], but again more complicated.

## 2. AN OVERVIEW

This section gives an overview of the construction and the proof.

Since $K_3$ and $K_4$ are planar graphs with circular chromatic numbers 3 and 4 respectively, we only need to consider those rational numbers which are strictly between 3 and 4. Given a rational number $p/q$ such that $3 < p/q < 4$ and $(p, q) = 1$, let

$$\frac{3}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n} = \frac{p}{q}$$

be the Farey sequence of $p/q$. This means that for $0 \leqslant i \leqslant n-1$, any rational number $k/d$ strictly between $p_i/q_i$ and $p_{i+1}/q_{i+1}$ has numerator $k > p_{i+1}$ (see Section 3 for the precise definition).

In order to construct a planar graph $G$ with circular chromatic number equal to $p/q$, we recursively construct planar graphs $G_i$, for $i = 1, 2, ..., n$, such that $G_i$ has circular chromatic number $p_i/q_i$. The graph $G_i$ is constructed by "hooking" two parts $F_i$ and $H_i$ together. The parts $F_i$ and $H_i$ are not only the building blocks for $G_i$, they are also the building blocks for $F_{i+2}$ and $H_{i+1}$. This is how the structure of $G_i$ is weaved into that of $G_{i+1}$, allowing us to use induction in our proof. Indeed, the graph $G_{i+1}$ contains many (at least two) copies of $G_i$, and these copies of $G_i$ are intertwined, i.e. two copies of $G_i$ in $G_{i+1}$ may share many vertices. The graph $G_i$ will usually have exactly $p_i$ vertices. In the exceptional case, the graph $G_i$ has more than $p_i$ vertices, but not many more.

After the construction, we prove by induction on $i$ that each graph $G_i$ has circular chromatic number $p_i/q_i$. It is relatively easy to prove that $\chi_c(G_i) \leqslant p_i/q_i$. We simply give a $(p_i, q_i)$-colouring of $G_i$. It is more difficult to prove that $\chi_c(G_i)$ is not less than $p_i/q_i$. For the purpose of using induction, we shall prove a stronger result. Namely we shall prove that $G_i$ not only has circular chromatic number $p_i/q_i$, but also the $(p_i, q_i)$-colouring of $G_i$ is more or less unique. Then we show that the "unique" $(p_i, q_i)$-colouring of $G_i$ cannot be extended to a $(p_i, q_i)$-colouring of $G_{i+1}$. Therefore $\chi_c(G_{i+1}) > p_i/q_i$. It is well-known (cf. Corollary 5.1) that if a graph $G$ has circular chromatic number $k/d$, where $(k, d) = 1$, then $k \leqslant |V(G)|$. Because $G_{i+1}$ has $p_{i+1}$ vertices (in general), and any rational number $k/d$ strictly between $p_i/q_i$ and $p_{i+1}/q_{i+1}$ has $k > p_{i+1}$, we conclude that $\chi_c(G_{i+1}) \geqslant p_{i+1}/q_{i+1}$ and hence $\chi_c(G_{i+1}) = p_{i+1}/q_{i+1}$.

## 3. THE FAREY SEQUENCE

Given any rational number $p/q$ such that $3 < p/q < 4$ and $(p, q) = 1$, let $p'$, $q'$ be the unique positive integers such that $p' < p$, $q' < q$ and $pq' - qp' = 1$. Then it is straightforward to verify that $p'/q' < p/q$ and that $p'/q'$ is the largest fraction with the property that $p'/q' < p/q$ and $p' \leqslant p$. Similarly, let $p''$, $q''$ be positive integers such that $p'' < p'$, $q'' < q'$ and $p'q'' - p''q' = 1$. Then $p''/q''$ is the largest fraction with the property that $p''/q'' < p'/q'$ and that $p'' \leqslant p'$. Repeat this process of finding smaller and smaller fractions, we shall stop at the fraction 3/1 in a finite number of steps. Thus to each rational number $p/q$ between 3 and 4, there corresponds a unique sequence of fractions.

$$\frac{3}{1}=\frac{p_0}{q_0}<\frac{p_1}{q_1}<\frac{p_2}{q_2}< \cdots <\frac{p_n}{q_n}=\frac{p}{q}.$$

The sequence $(p_i/q_i\colon i=0, 1, ..., n)$ is called the *Farey sequence* of $p/q$ (This definition of Farey sequence was given in [10] and is slightly different from the Farey sequence found in number theory book such as in [11].)

For convenience, we let $p_{-1}=-1$ and $q_{-1}=0$. As $p_i q_{i-1}-p_{i-1}q_i=1$ and $p_{i-1}q_{i-2}-p_{i-2}q_{i-1}=1$ for all $1\leqslant i\leqslant n$, it follows that

$$p_{i-1}(q_i+q_{i-2})=q_{i-1}(p_i+p_{i-2}).$$

As $p_{i-1}$, $q_{i-1}$ are co-prime,

$$\alpha_i=\frac{p_i+p_{i-2}}{p_{i-1}}=\frac{q_i+q_{i-2}}{q_{i-1}}$$

is an integer which is greater than 1, and hence is at least 2. The sequence $(\alpha_1, \alpha_2, ..., \alpha_n)$ is called the *alpha sequence* of $p/q$ [10], and is obviously uniquely determined by $p/q$. The process of deducing the alpha sequence from the rational number $p/q$ can also be reversed. In other words, each sequence $(\alpha_1, \alpha_2, ..., \alpha_n)$ with $\alpha_i\geqslant 2$ determines a rational number $p/q$ between 3 and 4. Indeed, given the alpha sequence $(\alpha_1, \alpha_2, ..., \alpha_n)$, the fractions $p_i/q_i$ can be easily determined by solving the difference equations

$$p_i=\alpha_i p_{i-1}-p_{i-2}, \qquad q_i=\alpha_i q_{i-1}-q_{i-2}, \tag{$*$}$$

with the initial condition $(p_{-1}, q_{-1})=(-1, 0)$ and $(p_0, q_0)=(3, 1)$.

By repeatedly applying Eq. $(*)$, we may express $p_i$ (respectively $q_i$) in terms of $p_j$ and $p_{j-1}$ (respectively $q_j$ and $q_{j-1}$) for any $0\leqslant j\leqslant i-2$. Lemma 3.1 below gives the explicit expressions.

For $1\leqslant r\leqslant s\leqslant n$, let

$$\Lambda_{r,s}=\det\begin{pmatrix} \alpha_1 & 1 & 0 & \cdots & 0 & 0\\ 1 & \alpha_{r+1} & 1 & \cdots & 0 & 0\\ 0 & 1 & \alpha_{r+2} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \alpha_{s-1} & 1\\ 0 & 0 & 0 & \cdots & 1 & \alpha_s \end{pmatrix}.$$

LEMMA 3.1. *For* $0 \leqslant j \leqslant i-2$, *we have*

$$p_i = p_j A_{j+1,i} - p_{j-1} A_{j+2,i}, \qquad q_i = q_j A_{j+1,i} - q_{j-1} A_{j+2,i}. \qquad (**)$$

*Proof.* It suffices to prove the first equality. We shall prove it by induction on $i$. When $i = j+2$, by applying $(*)$ twice, we obtain $(**)$. Suppose that $i \geqslant j+3$ and that $(**)$ is true for any $i' < i$. Then by cofactor expansion,

$$\begin{aligned} p_j A_{j+1,i} - p_{j-1} A_{j+2,i} &= \alpha_i (p_j A_{j+1,i-1} - p_{j-1} A_{j+2,i-1}) \\ &\quad - (p_j A_{j+1,i-2} - p_{j-1} A_{j+2,i-2}) \\ &= \alpha_i p_{i-1} - p_{i-2} = p. \end{aligned}$$

The second equality uses the induction hypothesis. ∎

By letting $j = 0$ in $(**)$, and by using the initial condition, we have

$$p_i = 3A_{1,i} + A_{2,i}, \qquad q_i = A_{1,i}. \qquad (***)$$

LEMMA 3.2. *For* $0 \leqslant j \leqslant i-2$, $p_j q_i = p_i q_j - A_{j+2,i}$.

*Proof.* By applying Lemma 3.1, we have

$$\begin{aligned} p_i q_j - p_j q_i &= (p_j A_{j+1,i} - p_{j-1} A_{j+2,i})\, q_j - p_j (q_j A_{j+1,i} - q_{j-1} A_{j+2,i}) \\ &= A_{j+2,i} (p_j q_{j-1} - p_{j-1} q_j) \\ &= A_{j+2,i}. \end{aligned}$$ ∎

Noting that $\alpha_j \geqslant 2$, we leave the easy induction proof of the following lemma to the reader.

LEMMA 3.3. *For any* $2 < t < i$, $A_{t,i} < A_{t-1,i}$.

## 4. THE CONSTRUCTION

Let $r = p/q$ be any rational number strictly between 3 and 4, where $(p, q) = 1$, let $(\alpha_1, \alpha_2, ..., \alpha_n)$ be the alpha sequence of $p/q$, and let $(p_i/q_i$: $i = 1, 2, ..., n)$ be the Farey sequence of $p/q$. We shall use the alpha sequence to construct, by induction, a sequence of planar graphs $G_i$ such that $\chi_c(G_i) = p_i/q_i$. It turns out that the construction for $\alpha_1 = 2$ and for $\alpha_1 \geqslant 3$ are quite different. In this section, we shall construct the graphs $G_i$ for $\alpha_1 \geqslant 3$. The case $\alpha_1 = 2$ will be dealt with in Section 6.

Before constructing the graphs $G_i$ we need to construct ordered graphs $F_i$ and $H_i$. We construct these graphs, together with liner orderings of their vertices, recursively. If we set $f_i = |V(F_i)|$ and $h_i = |V(H_i)|$, we can put

$V(F_i)=\{x_{i,1}, x_{i,2}, ..., x_{i,f_i}\}$ and $V(H_i)=\{y_{i,1}, y_{i,2}, ..., y_{i,h_i}\}$ and consider the vertices of each in the order indicated.

First, let $F_1$ be the edgeless graph on $\{x_{1,1}\}$. Let $H_1$ be the square of the path $y_{1,1}, y_{1,2}, ..., y_{1,3\alpha_1}$, that is, let $y_{1,i}y_{1,j}$ be an edge of $H_1$ if and only if $1 \leqslant |i-j| \leqslant 2$. Similarly, let $F_2$ be the square of the path $x_{2,1}, x_{2,2}, ..., x_{2,3\alpha_1-3}$.

For $i \geqslant 2$, the graph $H_i$ is constructed from copies of $F_{i-1}$ and $H_{i-1}$ by adding some connecting edges; for $i \geqslant 3$, $F_i$ is constructed from copies of $F_{i-2}$ and $H_{i-2}$ by adding some connecting edges.

In order to describe the connecting edges, we define four types of hooks as follows:

Definition 4.1. Suppose $X$ and $Y$ are disjoint ordered graphs whose vertex orderings are $(x_1, x_2, ..., x_s)$ and $(y_1, y_2, ..., y_t)$, respectively. To say *hook $X$ to $Y$ with*

- *Type* 1 *hook* means to add the following set of edges connecting $X$ to $Y$:

$$x_1 y_1, \qquad x_1 y_2, \qquad x_s y_t, \qquad x_{s-4} y_t.$$

- *Type* 2 *hook* means to add the following set of edges connecting $X$ to $Y$:

$$x_1 y_1, \qquad x_2 y_1, \qquad x_s y_t, \qquad x_s y_{t-4}.$$

- *Type* 3 *hook* means to add the following set of edges connecting $X$ to $Y$:

$$x_1 y_1, \qquad x_1 y_2, \qquad x_s y_t, \qquad x_{s-1} y_t.$$

- *Type* 4 *hook* means to add the following set of edges connecting $X$ to $Y$:

$$x_1 y_1, \qquad x_2 y_1, \qquad x_s y_t, \qquad x_s y_{t-1}.$$

In the definition above, if $X$ is a singleton, then set $x_j = x_1$ for all $j$.

Figure 1 illustrates the four types of hooks defined above.

To construct $H_{i+1}$ for $i \geqslant 1$, we take $\alpha_{i+1}$ copies of $F_i$ and $\alpha_{i+1}-1$ copies of $H_i$, and hook them together in the order $F_i^1, H_i^1, F_i^2, H_i^2, ..., H_i^{\alpha_{i+1}-1}, F_i^{\alpha_{i+1}}$. Here $F_i^j$ denotes the $j$th copy of $F_i$, and $H_i^j$ denotes the $j$th copy of $H_i$. The type of hooks used are as follows:

- if $\alpha_1$ is odd and $i=1$, then for $j=1, 2, ..., \alpha_{i+1}-1$, $F_i^j$ is hooked to $H_i^j$ with a type 1 hook, and $F_i^{j+1}$ is hooked to $H_i^j$ with a type 2 hook;

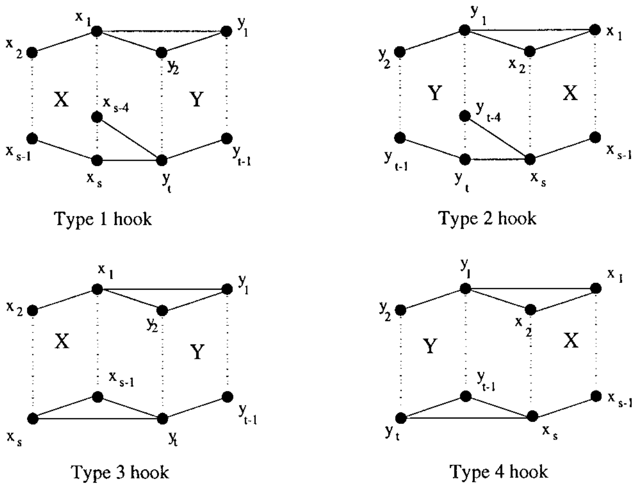


**FIG. 1.** The four types of hooks.

- if $\alpha_1$ is odd and $i \geqslant 2$, then for $j = 1, 2, ..., \alpha_{i+1} - 1$, $F_i^j$ is hooked to $H_i^j$ with a type 3 hook, and $F_i^{j+1}$ is hooked to $H_i^j$ with a type 4 hook;
- if $\alpha_1$ is even and $i = 1$, then for $j = 1, 2, ..., \alpha_{i+1} - 1$, $F_i^j$ is hooked to $H_i^j$ with a type 3 hook, and $F_i^{j+1}$ is hooked to $H_i^j$ with a type 4 hook;
- if $\alpha_1$ is even and $i$ is even, then for $j = 1, 2, ..., \alpha_{i+1} - 1$, $F_i^j$ is hooked to $H_i^j$ with a type 1 hook, and $F_i^{j+1}$ is hooked to $H_i^j$ with a type 4 hook;
- if $\alpha_1$ is even and $i \geqslant 3$ is odd, then for $j = 1, 2, ..., \alpha_{i+1} - 1$, $F_i^j$ is hooked to $H_i^j$ with a type 3 hook, and $F_i^{j+1}$ is hooked to $H_i^j$ with a type 2 hook.

The linear order of the vertices of $H_{i+1}$ is as follows: the vertices of $F_i^1$ in order, followed by the vertices of $H_i^1$ in the reverse order, followed by the vertices of $F_i^2$ in order, followed by the vertices of $H_i^2$ in the reverse order, etc.. To be precise, let the vertices of $F_i^j$ be $x_{i,1}^j, x_{i,2}^j, ..., x_{i,f_i}^j$ in that order, and let the vertices of $H_i^j$ be $y_{i,1}^j, y_{i,2}^j, ..., y_{i,h_i}^j$ in that order. Then the linear order of the vertices of $H_{i+1}$ is

$$x_{i,1}^1, x_{i,2}^1, ..., x_{i,f_i}^1, y_{i,h_i}^1, y_{i,h_i-1}^1, ..., y_{i,1}^1,$$
$$x_{i,1}^2, x_{i,2}^2, ..., x_{i,f_i}^2, y_{i,h_i}^2, ..., y_{i,1}^2, ..., x_{i,1}^{\alpha_{i+1}}, ..., x_{i,f_i}^{\alpha_{i+1}}.$$

The graph $F_{i+2}$ $(i \geqslant 1)$ is constructed in the same way as $H_{i+1}$, except that for $F_{i+2}$ we only take $\alpha_{i+1} - 1$ copies of $F_i$ and $\alpha_{i+1} - 2$ copies of $H_i$. The linear order of its vertex set is also defined in the same way.

Note that by assuming that $\alpha_1 \geqslant 3$, the graph $F_i$ is either a singleton or has at least 6 vertices, and that each of the graphs $H_i$ has at least 9 vertices.

Finally we construct graphs $G_i$ and $G'_i$ from a copy of $F_i$ and a copy of $H_i$ by hooking them using appropriate hooks so that $G_i$ is isomorphic to the subgraph of $H_{i+1}$ induced by the set $F_i^1 \cup H_i^1$, and $G'_i$ is isomorphic to the subgraph of $H_{i+1}$ induced by the set $H_i^1 \cup F_i^2$.

To be precise, we define $G_i$ and $G'_i$ as follows:

- if $\alpha_1$ is odd and $i=1$, then $G_1$ is obtained by hooking $F_1$ to $H_1$ with a type 1 hook; $G'_1$ is obtained by hooking $F_1$ to $H_1$ with a type 2 hook;
- if $\alpha_1$ is odd and $i \geqslant 2$, then $G_i$ is obtained by hooking $F_i$ to $H_i$ with a type 3 hook; $G'_i$ is obtained by hooking $F_i$ to $H_i$ with a type 4 hook;
- if $\alpha_1$ is even and $i=1$, then $G_1$ is obtained by hooking $F_1$ to $H_1$ with a type 3 hook; $G'_1$ is obtained by hooking $F_1$ to $H_1$ with a type 4 hook;
- if $\alpha_1$ is even and $i$ is even, then $G_i$ is obtained by hooking $F_i$ to $H_i$ with a type 1 hook; $G'_i$ is obtained by hooking $F_i$ to $H_i$ with a type 4 hook;
- if $\alpha_1$ is even and $i \geqslant 3$ is odd, then $G_i$ is obtained by hooking $F_i$ to $H_i$ with a type 3 hook; $G'_i$ is obtained by hooking $F_i$ to $H_i$ with a type 2 hook.

Figure 2 illustrates the construction of $F_i$, $H_i$ for the alpha sequence (3, 2, 3). The graphs $G_1$, $G'_1$, $G_2$, $G'_2$ are contained in $H_3$ as subgraphs. The graph $G_3$ can be easily obtained from $H_3$ and $F_3$.

Note that each of the graphs $F_i$ and $H_i$ is an ordered graph. We shall also regard the graph $G_i$ (as well as $G'_i$) as an ordered graph, the order of the vertices being inherited from the graph $H_{i+1}$, i.e., the vertices of $F_i$ in order, followed by the vertices of $H_i$ in the reverse order.

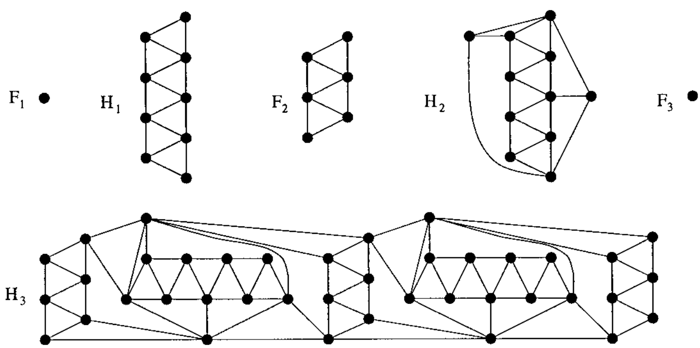


**FIG. 2.** $F_i$ and $H_i$ for the alpha sequences (3, 2, 3).

## 5. THE PROOF

In this section, we shall prove that for each $i \leqslant n$, the graphs $G_i$ and $G'_i$ are planar graphs, and that $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$.

THEOREM 5.1. *For each i, both graphs $G_i$, $G'_i$ are planar graphs.*

*Proof.* It suffices to show that for each $i$, the graph $H_i$ is planer (as $H_i$ contains $G_{i-1}$, $G'_{i-1}$ as subgraphs). We shall only consider the case that $\alpha_1$ is even. The case that $\alpha_1$ is odd is different, but can be proved similarly.

The following observations are straightforward from the construction:

- $H_1$ has an embedding in the plane such that the outer face contains the vertices $y_{1,1}, y_{1,2}, y_{1,h_1}, y_{1,h_1-1}$ in that cyclic order.
- $F_2$ has an embedding in the plane such that the outer face contains the vertices $x_{2,1}, x_{2,2}, x_{2,f_2}, x_{2,f_2-4}$ in that cyclic order.
- $H_2$ has an embedding in the plane such that the outer face contains the vertices $y_{2,1}, y_{2,2}, y_{2,h_2}, y_{2,h_2-1}$ in that cyclic order.
- $F_3$ has an embedding in the plane such that the outer face contains the vertices $x_{3,1}, x_{3,2}, x_{3,f_3}, x_{3,f_3-1}$ in that cyclic order. (Note that since $\alpha_1 \geqslant 3$, $F_3$ has more than $3\alpha_1$ vertices.)

We shall prove by induction that for each $i \geqslant 2$, if $i$ is even then the graph $H_i$ has an embedding on the plane such that the outer face contains

$$y_{i,1}, y_{i,2}, y_{i,h_i}, y_{i,h_i-1}$$

in that cyclic order; and that $F_i$ has an embedding on the plane such that the outer face contains

$$x_{i,1}, x_{i,2}, x_{i,f_i}, x_{i,f_i-4}$$

in that cyclic order. If $i$ is odd then the graph $H_i$ has an embedding on the plane such that the outer face contains

$$y_{i,1}, y_{i,2}, y_{i,h_i}, y_{i,h_i-4}$$

in that cyclic order; and that $F_i$ has an embedding on the plane such that the outer face contains

$$x_{i,1}, x_{i,2}, x_{i,f_i}, x_{i,f_i-1}$$

in that cyclic order.

This is straightforward, as the embedding of $H_i$ is simply obtained from the embeddings of those copies of $H_{i-1}$ and $F_{i-1}$ by adding the hooking edges. The induction hypothesis and the types of hooks we have chosen ensure that these added edges can be embedded on the plane. ∎

Now we proceed to prove that $\chi_c(G_i)=\chi_c(G'_i)=p_i/q_i$. We shall prove the conclusion for $G_i$ only since the arguments used are easily carried over to $G'_i$. We will simply assume the statements true for both $G_i$ and $G'_i$.

Lemma 5.1 and Corollary 5.1 below will be used extensively in our proof. Lemma 5.1 was proved in [6] and also implicitly used in [14, 16]. Given a $(k, d)$-colouring $c$ of a graph $G$, define a directed graph $D_c(G)$ on the vertex set of $G$ by putting a directed edge from $x$ to $y$ if and only if $(x, y)$ is an edge of $G$ and $c(x)-c(y)=d \pmod{k}$.

LEMMA 5.1. *For any $G$, $\chi_c(G)=k/d$ if and only if $G$ is $(k, d)$-colourable and for any $(k, d)$-colouring $c$ of $G$, the directed graph $D_c(G)$ contains a directed cycle.*

A simple calculation shows that the length of the directed cycle in $D_c(G)$ is a multiple of $k$, and hence is at least $k$ (under the assumption that $(k, d)=1$. Thus we have the following corollary [1, 6, 14, 16]:

COROLLARY 5.1. *For any graph $G$, if $\chi_c(G)=k/d$ where $(k, d)=1$, then $G$ has a cycle of length at least $k$. In particular $k \leqslant |V(G)|$.*

Thus if we know the number of vertices of a graph $G$, then by using Corollary 5.1, we can restrict the possible values of the circular chromatic number of $G$ to finitely many rational numbers. If, in addition, we know sharp upper and lower bounds for the circular chromatic number of $G$, then we might be able to determine the circular chromatic number of $G$ by using Corollary 5.1.

LEMMA 5.2. *The graph $G_i$ has $p_i$ vertices.*

*Proof.* From the construction of $G_i$, we know that $G_i$ has $g_i=f_i+h_i$ vertices. From the construction of $F_i$, $H_i$, we know that

$$f_1=1, \qquad f_2=3\alpha_1-3, \qquad h_1=3\alpha_1,$$

and for $i \geqslant 2$,

$$h_i=\alpha_i f_{i-1}+(\alpha_i-1)\, h_{i-1},$$

for $i \geqslant 3$,

$$f_i=(\alpha_{i-1}-1)\, f_{i-2}+(\alpha_{i-1}-2)\, h_{i-2}.$$

Simple algebraic calculation shows that

$$h_i = \alpha_i g_{i-1} - h_{i-1}, \qquad f_i = (\alpha_{i-1} - 1)\, g_{i-2} - h_{i-2} = h_{i-1} - g_{i-2}.$$

Hence

$$g_i = \alpha_i g_{i-1} - g_{i-2}.$$

It is straightforward to verify that $g_1 = p_1$, $g_2 = p_2$. Thus $g_i$, $p_i$ satisfy the same difference equation and the same initial condition. Hence $|G_i| = g_i = p_i$. ▮

DEFINITION 5.1. Suppose $X$ is an ordered graph with vertices ordered as $(x_1, x_2, ..., x_\beta)$. For an edge $e = x_k x_s$ of $X$, we call $\ell(e) = |k - s|$ the order length of $e$.

LEMMA 5.3. *Let* $L_i = \{1, 2, 5\} \cup \{p_t - 1,\ p_t - 2,\ p_t - 5 :\ 1 \leqslant t \leqslant i - 1\}$. *Then for any* $i \geqslant 1$ *and for any edge* $e$ *of* $H_i$ *or* $F_i$, *we have* $\ell(e) \in L_i$.

*Proof.* We shall prove it by induction. When $i = 1$, this is true from the definition. Suppose $i \geqslant 2$ and the lemma is true for all $1 \leqslant j < i$. We shall prove that it is true for $i$. Note that when we construct the graph $H_i$ from copies of $F_{i-1}$ and $H_{i-1}$, the edges of $H_i$ are either those carried over from $F_{i-1}$ and $H_{i-1}$, or are the edges of the hooks. For those edges of the copies of $F_{i-1}$ and $H_{i-1}$, their order lengths remain unchanged in $H_i$. For the edges of the hooks, it is straightforward to verify that their order lengths belong to the set $\{1, 2, 5, p_{i-1} - 1, p_{i-1} - 2, p_{i-1} - 5\}$. The proof for the edges of $F_i$ is similar. ▮

COROLLARY 5.2. *For any* $i$ *and for any edge* $e$ *of* $G_i$ *or* $G'_i$, *we have* $\ell(e) \in L_{i+1}$.

THEOREM 5.2. *For each* $i$, $\chi_c(G_i) \leqslant p_i / q_i$.

*Proof.* Let $(c_1, c_2, ..., c_{p_i})$ be the vertices of $G_i$, in that order. Let $\Delta(c_k) = k q_i \pmod{p_i}$. We claim that $\Delta$ is a $(p_i, q_i)$-colouring of $G_i$, i.e., for any edge $e = c_k c_s$, we have $q_i \leqslant |\Delta(c_k) - \Delta(c_s)| \leqslant p_i - q_i$.

Note that $|\Delta(c_k) - \Delta(c_s)| = \ell(e)\, q_i \pmod{p_i}$ or $p_i - \ell(e)\, q_i \pmod{p_i}$. Therefore, it suffices to prove that for any edge $e$ of $G_i$, we have

$$q_i \leqslant \ell(e)\, q_i \qquad (\mathrm{mod}\ p_i) \leqslant p_i - q_i.$$

By Corollary 5.2, we know that $\ell(e) \in L_{i+1}$. If $\ell(e) = 1, 2$, then obviously we have $q_i \leqslant \ell(e)\, q_i \leqslant p_i - q_i$ (recall that $p_i / q_i > 3$). If $\ell(e) = 5$, then $\ell(e)\, q_i \pmod{p_i} = 5q_i - p_i$. Since $3 < p_i / q_i < 4$, we have $q_i < 5q_i - p_i < 2q_i < p_i - q_i$.

If $\ell(e) = p_i - 1$, $p_i - 2$ or $p_i - 5$, then it is also straightforward to verify that $q_i \leqslant \ell(e)\, q_i \pmod{p_i} \leqslant p_i - q_i$.

Next we consider the case that $\ell(e) = p_t - 1$, $p_t - 2$ or $p_t - 5$ for some $t \leqslant i - 1$. If $t = i - 1$ then since $p_i q_{i-1} - p_{i-1} q_i = 1$, we have

$$p_t q_i \pmod{p_i} = p_i - 1.$$

It follows that

$$(p_t - 1)\, q_i \pmod{p_i} = p_i - q_i - 1,$$

$$(p_t - 2)\, q_i \pmod{p_i} = p_i - 2q_i - 1,$$

and

$$(p_t - 5)\, q_i \pmod{p_i} = 2p_i - 5q_i - 1 = (p_i - q_i) - (4q_i + 1 - p_i).$$

Since $3q_i < p_i < 4q_i$, we conclude that $q_i \leqslant \ell(e)\, q_i \pmod{p_i} \leqslant p_i - q_i$.

If $1 \leqslant t \leqslant i - 2$, then by Lemma 3.2,

$$p_t q_i \pmod{p_i} = p_i - \Lambda_{t+2,\, i}.$$

By $(***)$ and Lemma 3.3, we have

$$p_i - 3q_i = \Lambda_{2,\, i}$$

and

$$2 \leqslant \alpha_i = \Lambda_{i,\, i} \leqslant \Lambda_{t+2,\, i} < \Lambda_{2,\, i}.$$

It follows that

$$(p_t - 1)\, q_i \pmod{p_i} = p_i - q_i - \Lambda_{t+2,\, i},$$

$$(p_t - 2)\, q_i \pmod{p_i} = p_i - 2q_i - \Lambda_{t+2,\, i},$$

and

$$\begin{aligned} (p_t - 5)\, q_i \pmod{p_i} &= 2p_i - 5q_i - \Lambda_{t+2,\, i} \\ &= (p_i - q_i) - (4q_i + \Lambda_{t+2,\, i} - p_i). \end{aligned}$$

By using the inequality $2 \leqslant \Lambda_{t+2,\, i} < p_i - 3q_i$, we conclude that $q_i \leqslant \ell(e)\, q_i \pmod{p_i} \leqslant p_i - q_i$. Therefore $\Delta$ is a $(p_i, q_i)$-colouring of $G_i$. ▮

Suppose $\chi_c(G_i) = p_i / q_i$ and $\Delta$ is an $(p_i, q_i)$-colouring of $G_i$. It follows from Lemma 5.1 that there is a directed cycle of $D_\Delta(G_i)$ of length at least $p_i$. Since $|G_i| = p_i$, we conclude that there is a Hamilton cycle, say $(c_1, c_2, ..., c_{p_i})$, of $G_i$ such that $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$.

Suppose $X=(x_1, x_2, ..., x_s)$ is a cycle (resp. a path) of a graph $Q$. Let $p_i/q_i$ be a rational number between 3 and 4 with Farey sequence $3/1=p_0/q_0<p_1/q_1<\cdots<p_i/q_i$. We call $X$ a *good cycle* (resp. a *good path*), with respect to $p_i/q_i$, if for any edge $e=x_kx_s$ of $Q$ and for any $0\leqslant t\leqslant i-1$, we have $|s-k|\neq p_t, p_t+1, p_t-3, p_t-4$.

Lemma 5.4. *Suppose $\chi_c(G_i)=p_i/q_i$ and $\Delta$ is an $(p_i, q_i)$-colouring of $G_i$. Let $X=(x_1, x_2, ..., x_{p_i}, x_1)$ be the Hamilton cycle of $G_i$ such that $\Delta(c_j)-\Delta(c_{j-1})=q_i \pmod{p_i}$. Then $X$ is a good Hamilton cycle of $G_i$ with respect to $p_i/q_i$.*

*Proof.* Let $C_i=\{q_i, q_i+1, ..., p_i-q_i\}$. Since $\Delta$ is a $(p_i, q_i)$-colouring of $G_i$, it follows that for any edge $e=c_sc_k$ of $G_i$, we have $|s-k|\, q_i \pmod{p_i}\in C_i$. Therefore to prove Lemma 5.4, it suffices to show that for any $0\leqslant t\leqslant i-1$,

$$p_tq_i \pmod{p_i}\notin C_i, \qquad (p_t+1)\, q_i \pmod{p_i}\notin C_i,$$
$$(p_t-3)\, q_i \pmod{p_i}\notin C_i, \quad \text{and} \quad (p_t-4)\, q_i \pmod{p_i}\notin C_i.$$

If $t=i-1$, then $p_tq_i=p_iq_t-1$. The required inequalities follow easily. (Note that when $i\geqslant 2$, we have $p_i-3q_i\geqslant 2$.) If $1\leqslant t\leqslant i-2$, then $p_tq_i=p_iq_t-\Lambda_{t+2,i}$, by Lemma 3.2. Hence

$$p_tq_i \pmod{p_i}=p_i-\Lambda_{t+2,i}.$$

As

$$2\leqslant\alpha_i\leqslant\Lambda_{t+2,i}<\Lambda_{2,i}=p_i-3q_i,$$

straightforward calculations give the required inequalities. For $t=0$, we only need to show that $3q_i\notin C_i$ and that $4q_i \pmod{p_i}=4q_i-p_i\notin C_i$, and these simply follow from the condition that $3<p_i/q_i<4$. ∎

Lemma 5.5. *For a positive integer $t$, let $Q_t$ be the square of the path $12\cdots t$ on the $t$ vertices $1, 2, ..., t$ (so the edge set of $Q_t$ is $\{xy\colon 1\leqslant|x-y|\leqslant 2\}$). For any $t\geqslant 1$, the graph $Q_t$ has a unique good Hamilton path (with respect to any $3<p/q<4$), up to an automorphism of $Q_t$.*

*Proof.* It is straightforward to verify that $X=(1, 2, ..., t)$ is a good Hamilton path of $Q_t$. We shall prove that this is the unique good Hamilton path of $Q_t$, up to an automorphism of $Q_t$. If $t\leqslant 3$, then $Q_t$ is a complete graph, and hence the lemma is true. Suppose $t\geqslant 4$ and $P=(x_1, x_2, ..., x_t)$ is a good Hamilton path of $Q_t$. Then for any $i\leqslant t-3$, $x_ix_{i+3}$ is not an edge of $Q_t$, i.e., $|x_i-x_{i+3}|\geqslant 3$. Assume that $x_1<x_4$ (the case that $x_1>x_4$ is symmetric, and can be treated similarly). Then $x_1\leqslant x_4-3$.

Since $x_2 \leqslant x_1 + 2$ and $x_5 \geqslant x_4 - 2$ (because $x_1 x_2$ and $x_4 x_5$ are edges of $Q_t$), we conclude that $x_2 \leqslant x_5 + 1$. However, $|x_2 - x_5| \geqslant 3$. Therefore $x_2 \leqslant x_5 - 3$. Repeating this argument, we can prove that $x_i \leqslant x_{i+3} - 3$ for all $i \leqslant t-3$. This implies that $\{1, 2, 3\} = \{x_1, x_2, x_3\}$, for otherwise there exist $j \leqslant 3$ and an $i \geqslant 1$ such that $x_{i+3} = j$. But then $1 \leqslant x_i \leqslant x_{i+3} - 3 = j - 3 \leqslant 0$, an obvious contradiction.

Suppose $x_i = 4$. Then $i \geqslant 4$, and $x_{i-3} \leqslant x_i - 3 = 1$. Hence $x_{i-3} = 1$. If $x_j = 5$, then $x_{j-3} \leqslant 2$ and hence $x_{j-3} = 2$. Repeating this argument, it can be proved that $x_{k-3} = x_k - 3$ for all $4 \leqslant k \leqslant t$. Now we shall show that $x_i < x_{i+1}$ for all $i \leqslant t-4$. Assume to the contrary that $x_i > x_{i+1}$. Then $0 < x_{i+4} - x_i = x_{i+1} + 3 - x_i < 3$, and hence $x_i x_{i+4}$ is an edge of $Q_t$, contrary to the assumption that $P$ is a good Hamilton path. For $i \geqslant t-3$, we also have $x_i < x_{i+1}$, because $x_j = x_{j-3} + 3$. In case $t \geqslant 5$, this implies that $P = (1, 2, ..., t)$. In case $t = 4$, this implies that $P = (1, x, y, 4)$, where $\{x, y\} = \{2, 3\}$. Since in $Q_4$ the two vertices 2 and 3 are symmetric, there is an automorphism of $Q_t$ which maps the path $P$ to the path $(1, 2, 3, 4)$. This completes the proof. ∎

We observe that the first vertex of $F_i$ and the last vertex of $H_i$ form a 2-vertex cut of $G_i$. It follows that any good Hamilton cycle of $G_i$ must be the union of a good Hamilton path, say $P$, of $F_i$ and a good Hamilton path, say $P'$, of $H_i$. Moreover, the initial vertex of $P$ (resp. $P'$) must be the first (resp. last) vertex of $F_i$ (resp. $H_i$).

Lemma 5.6. *Suppose $P$ (resp. $P'$) is a good Hamilton path of $F_i$ (resp. $H_i$) which can be extended to a good Hamilton cycle of $G_i$. If $i = 1$, then the paths $P$ and $P'$ are unique. If $i \geqslant 2$, then the first and the last* 5 *vertices of $P$ (resp. $P'$) are the first and the last* 5 *vertices of $F_i$ (resp. $H_i$), in that order.*

*Proof.* We shall prove it by induction on $i$. When $i = 1$, then the conclusion follows from Lemma 5.5. If $i = 2$, then $P$ is unique by Lemma 5.5.

The graph $H_2$ is constructed by hooking copies of $F_1$ (which is a singleton) to copies of $H_1$. By the observation above (previous to the statement of Lemma 5.6), $P'$ is a concatenation of the good Hamilton paths of these copies of $F_1$ and $H_1$. By Lemma 5.5, $F_1$ and $H_1$ have unique good Hamilton paths. However, there is a possibility that after the concatenation, the path is not unique any more, because of the asymmetry of the hook edges. Indeed, when the good Hamilton path of the first copy of $F_1$ (which is a singleton) is concatenated with the good Hamilton path of the first copy of $H_1$, we could have started from the first vertex of the good Hamilton path of $H_1$, instead of the last vertex of $H_1$. Because of the asymmetry of the hook, different choices of the starting vertex of $H_1$ will result in non-isomorphic paths. However, we observe here that in order that the good Hamilton path $P'$ be extendable to a good Hamilton cycle of $G_2$, the second vertex of $P'$ must be the last vertex of the first copy of $H_1$ in the concatenation. This is because the first vertex of $F_2$ is adjacent to the second vertex of $H_2$ (which is the last vertex of the first copy of $H_1$ in $H_2$). If the path $P'$ is concatenated in the wrong way, i.e., if the second vertex of $P'$ is the first vertex of the first copy of $H_1$, then when $P'$ is extended to a Hamilton cycle, say $X$, of $G_2$, the edge $x_{2,1}y_{2,2}$ would have distance $p_1$ along $X$, contrary to the definition of a good Hamilton cycle.

Therefore the second vertex of $P'$ must be the second vertex of $H_2$, and this implies that the first 5 vertices of $P'$ must be the first 5 vertices of $H_2$ (indeed the first $3\alpha_1+2$ vertices must be the first $3\alpha_1+2$ vertices of $H_2$), in the same order.

Similarly we can prove that the last 5 vertices of $P'$ must be the last 5 vertices of $H_2$, in the same order.

The same argument applies to $F_3$.

For $i \geqslant 3$, the graph $H_i$ is obtained from copies of $F_{i-1}$ and $H_{i-1}$ by appropriately hooking them together. Any good Hamilton path of $H_i$ must be the concatenation of the good Hamilton paths of these copies of $F_{i-1}$ and $H_{i-1}$. Unlike the case for $i=2$, none of the graphs $F_{i-1}$ and $H_{i-1}$ is a singleton (because $\alpha_1 \geqslant 3$). Hence once the good Hamilton paths of the copies of $F_{i-1}$ and $H_{i-1}$ are chosen, there is a unique way of concatenating these paths together. Now the conclusion of the lemma follows easily from the induction hypothesis. ∎

Lemma 5.7. *Suppose $\chi_c(G_i) = p_i/q_i$ for some $i$. Let $\Delta$ be any $(p_i, q_i)$-colouring of $G_i$. If $i=1$, then the colour of the single vertex of $F_1$ is determined by the colours of the first and the last vertex of $H_1$. If $i \geqslant 2$, then the colours of the first and last vertex of $F_i$ uniquely determine the colours of the first and the last 5 vertices of $H_i$. Conversely, the colours of the first and the last vertex of $H_i$ uniquely determine the colours of the first and the last 5 vertices of $F_i$.*

*Proof.* The case that $i=1$ is trivial. Assume now that $i \geqslant 2$. Let $C$ be the good Hamilton cycle determined by $\Delta$ (cf. Lemma 5.4). Then the colours of a vertex is determined by the position of that vertex in the Hamilton cycle. The Hamilton cycle $C$ is the union of a good Hamilton path $P$ of $F_i$ and a good Hamilton path $P'$ of $H_i$. By Lemma 5.6, the first and the last 5 vertices of $P$ are the first and the last 5 vertices of $F_i$. In other words, the position of the first and the last 5 vertices of $F_i$ is determined by the position of the first and the last vertex of $H_i$, and hence their colours are determined by the colours of the first and the last vertex of $H_i$. The same is true for the first and the last 5 vertices of $H_i$. This completes the proof of Lemma 5.6. ∎

To prove that $\chi_c(G_i) \geqslant p_i/q_i$ (and hence $\chi_c(G_i) = p_i/q_i$), we need another gadget. For $2 \leqslant i \leqslant n$, let $T_i$, $S_i$, $T'_i$ and $S'_i$ be graphs defined as follows:

- if $\alpha_1$ is odd, then $T_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 3 hook; $S_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 3 hook; $T'_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 4 hook; $S'_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 4 hook;
- if $\alpha_1$ is even and $i$ is even, then $T_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 1 hook; $S_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 3 hook; $T'_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 4 hook; $S'_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 2 hook;
- if $\alpha_1$ is even and $i$ is odd, then $T_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 3 hook; $S_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 1 hook; $T'_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 2 hook; $S'_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 4 hook.

An alternate way of defining $T_i$ and $T'_i$ is as follows.

The graph $G_i$ (resp. $G'_i$) contains $H_i$ as a subgraph, and $H_i$ consists of copies of $F_{i-1}$ and $H_{i-1}$. We identify the vertices of the first copy of $F_{i-1}$ with the corresponding vertices of the last copy of $F_{i-1}$, and delete all the other vertices of $H_i$. The resulting graph is $T_i$ (resp. $T'_i$).

An alternate way of defining $S_i$ and $S'_i$ is as follows.

The graph $T_{i+1}$ (resp. $T'_{i+1}$) contains $F_{i+1}$ as a subgraph, and $F_{i+1}$ consists of copies of $F_{i-1}$ and copies of $H_{i-1}$. We identify the vertices of the first copy of $F_{i-1}$ with the corresponding vertices of the last copy of $F_{i-1}$, and delete all the other vertices of $F_{i+1}$. The resulting graph is $S_i$ (resp. $S'_i$). Note that in case $F_{i+1}$ consists of a single copy of $F_{i-1}$ and no copy of $H_{i-1}$, then $S_i = T_{i+1}$ (resp. $S'_i = T'_{i+1}$).

Similarly, the graph $T_i$ (resp. $T'_i$) can also be obtained from $S_{i+1}$ (resp. $S'_{i+1}$) in the same way.

THEOREM 5.3. *For $i \geqslant 1$, we have $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$. For each $i \geqslant 2$, each of the graphs $T_i$, $S_i$, $T'_i$, $S'_i$ has circular chromatic number greater than $p_{i-1}/q_{i-1}$.*

*Proof.* First we show that $\chi_c(G_1) = p_1/q_1$. It is easy to verify that $\chi(G_1) = \chi(G'_1) = 4$. Hence $\chi_c(G_1) > 3$. Suppose $\chi_c(G_1) = k/d$. Then $k \leqslant |V(G_1)| = p_1$, which implies that $k/d \geqslant p_1/q_1$, as any fraction strictly between 3 and $p_1/q_1$ has its numerator greater than $p_1$ (cf. the definition of the Farey sequence.) By Lemma 5.2, we have $\chi(G_1) \leqslant p_1/q_1$. Therefore $\chi_c(G_1) = p_1/q_1$.

Now we show that $\chi_c(T_2) > p_1/q_1$. As before, we can verify that $\chi(T_2) = 4$, and hence $\chi_c(T_2) > 3$. Suppose that $\chi_c(T_2) = k/d$. Then $k \leqslant |V(T_2)| < p_1$ (note that $|V(T_i)| < |V(G_i)|$ for all $i$, because $|V(F_{i-1})| < |V(H_i)|$). This implies that $k/d > p_1/q_1$, by the construction of the Farey sequence. Similarly, we can show that $\chi_c(S_2) > p_1/q_1$, $\chi_c(T'_2) > p_1/q_1$ and $\chi_c(S'_2) > p_1/q_1$.

Suppose that for some $i \geqslant 1$ we have $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$, and that each of the graphs $T_{i+1}$, $S_{i+1}$, $T'_{i+1}$, $S'_{i+1}$ has circular chromatic number greater than $p_i/q_i$. We shall prove that $\chi_c(G_{i+1}) = p_{i+1}/q_{i+1}$ and that $\chi_c(G'_{i+1}) = p_{i+1}/q_{i+1}$.

Assume to the contrary that $\chi_c(G_{i+1}) < p_{i+1}/q_{i+1}$. Since $p_i/q_i$ is the largest fraction with the property that $p_i/q_i < p_{i+1}/q_{i+1}$ and $p_i \leqslant p_{i+1} = |V(G_{i+1})|$, we conclude that $\chi_c(G_{i+1}) \leqslant p_i/q_i$

Recall that $H_{i+1}$ contains $\alpha_{i+1}$ copies of $F_i$, say $F_i^1, ..., F_i^{\alpha_{i+1}}$, and $\alpha_{i+1} - 1$ copies of $H_i$, say $H_i^1, ..., H_i^{\alpha_{i+1}-1}$. For each $1 \leqslant j \leqslant \alpha_{i+1} - 1$, each of the subgraphs induced by the sets $F_i^j \cup H_i^j$ is isomorphic to $G_i$, and each of the subgraphs induced by the set $H_i^j \cup F_i^{j+1}$ is a isomorphic to $G'_i$. By the induction hypothesis, $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$. Therefore $\chi_c(G_{i+1}) = p_i/q_i$.

Let $\phi$ be a $(p_i, q_i)$-colouring of $G_{i+1}$. The restriction of $\phi$ to $F_i^j \cup H_i^j$ and $H_i^j \cup F_i^{j+1}$ are $(p_i, q_i)$-colourings of $G_i$ and $G'_i$. By Lemma 5.7, the colours of the first and the last vertex of $H_i^1$ uniquely determine the colours of the first and the last 5 vertices of $F_i^1$, as well as the first and the last 5 vertices of $F_i^2$. Therefore, the first and the last 5 vertices of $F_i^1$ are coloured the same way as the first and the last 5 vertices of $F_i^2$. Repeating this argument, we can prove that the first and the last 5 vertices of the first copy of $F_i$ are coloured the same way as the first and the last 5 vertices of the last copy of $F_i$.

Since in hooking $F_{i+1}$ to $F_i$ in the process of constructing $T_{i+1}$, the only vertices of $F_i$ that become adjacent to any vertices of $F_{i+1}$ are among the first and the last 5 vertices of $F_i$, we conclude that the restriction of the colouring $\phi$ to $F_i^1 \cup F_{i+1}$ is indeed a $(p_i, q_i)$-colouring of $T_{i+1}$, contrary to the assumption that $\chi_c(T_{i+1}) > p_i/q_i$.

The proof of $\chi_c(G'_{i+1}) = p_{i+1}/q_{i+1}$ is similar.

Now assume that for some $i \geqslant 2$, we have $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$, $\chi_c(T_i) > p_{i-1}/q_{i-1}$, $\chi_c(S_i) > p_{i-1}/q_{i-1} \chi_c(T_i) > p_{i-1}/q_{i-1}$ and $\chi_c(S'_i) > p_{i-1}/q_{i-1}$. We shall prove that each of the graphs $T_{i+1}$, $S_{i+1}$, $T'_{i+1}$ and $S'_{i+1}$ has circular chromatic number greater than $p_i/q_i$.

Assume to the contrary that $\chi_c(T_{i+1}) \leqslant p_i/q_i$. Since $|F_{i+1}| < |H_i|$, hence $|T_{i+1}| < |G_i| = p_i$. It follows from Corollary 5.1 that $\chi_c(T_{i+1}) = k/d \leqslant p_i/q_i$ for some integers $k$, $d$ with $k < p_i$. As $p_{i-1}/q_{i-1}$ is the largest fraction satisfying the property that $p_{i-1} < p_i$ and $p_{i-1}/q_{i-1} \leqslant p_i/q_i$, we conclude that $\chi_c(T_i) \leqslant p_{i-1}/q_{i-1}$.

We consider two cases:

*Case* 1. $\alpha_i = 2$. In this case $F_{i+1} = F_{i-1}$, and hence $T_{i+1} = S_i$, contrary to our assumption that $\chi_c(S_i) > p_{i-1}/q_{i-1}$.

*Case* 2. $\alpha_i > 2$. In this case $F_{i+1}$ consists of $\alpha_i - 1$ copies of $F_{i-1}$, say $F^1_{i-1}, ..., F^{\alpha_i - 1}_{i-1}$, and $\alpha_i - 2$ copies of $H_{i-1}$, say $H^1_{i-1}, ..., H^{\alpha_i - 2}_{i-1}$. For each $1 \leqslant j \leqslant \alpha_i - 2$, each of the subgraphs induced by the sets $F^j_{i-1} \cup H^j_{i-1}$ is isomorphic to $G_{i-1}$, and each of the subgraphs induced by the set $H^j_{i-1} \cup F^{j+1}_{i-1}$ is a isomorphic to $G'_{i-1}$. By the induction hypothesis, $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$. Therefore $\chi_c(T_i) = p_{i-1}/q_{i-1}$.

Let $\phi$ be a $(p_{i-1}, q_{i-1})$-colouring of $T_{i+1}$. Using the same argument as in the previous paragraphs, we can conclude that the first and the last 5 vertices of the first copy of $F_{i-1}$ are coloured the same way as the first and the last 5 vertices of the last copy of $F_{i-1}$ (provided that $F_{i-1}$ is not a singleton, and in case $F_{i-1}$ is a singleton, the colour of that single vertex of each copy of $F_{i-1}$ is coloured the same colour). Hence the restriction of the colouring $\phi$ to $F^1_{i-1} \cup F_i$ is indeed a $(p_{i-1}, q_{i-1})$-colouring of $S_i$, contrary to our assumption.

The proofs for $\chi_c(S_{i+1}) > p_i/q_i$, $\chi_c(T'_{i+1}) > p_i/q_i$ and $\chi_c(S'_{i+1}) > p_i/q_i$ are similar. ∎

## 6. THE CASE $\alpha_1 = 2$

In this section, we consider the case that $\alpha_1 = 2$. We shall define two types of special hooks, which will be applied to the case where $F_i$ is a triangle.

DEFINITION 6.1. Suppose $X$ and $Y$ are disjoint ordered graphs, where $X$ has three vertices $x_1$, $x_2$, $x_3$ in that order, and the vertex ordering of $Y$ is $(y_1, y_2, ..., y_t)$. To say *hook X to Y with*

- *Type* 1 *special hook* means to add the following set of edges connecting $X$ to $Y$:

$$x_1 y_1, \qquad x_3 y_t, \qquad x_3 y_{t-1}.$$

- *Type* 2 *special hook* means to add the following set of edges connecting $X$ to $Y$:

$$x_1 y_1, \qquad x_2 y_1, \qquad x_2 y_t, \qquad x_3 y_t.$$

We shall construct the graphs $F_i$, $H_i$, $G_i$, $G'_i$ in the case that $\alpha_1 = 2$ as follows:

As before, $F_1$ is a singleton., $H_1$ is a squared path with $3\alpha_1 = 6$ vertices, and $F_2$ is a squared path with $3\alpha_1 - 3 = 3$ vertices, i.e., $F_2$ is a triangle.

Let $t$ be the smallest odd integer (if it exists) such that $\alpha_t \geqslant 3$. First we shall construct graphs $F_{i+1}$, $H_i$ and $G_i$, $G'_i$ for $i \leqslant t-1$. The construction is similar to the construction in Section 4.

Suppose $i \leqslant t-2$. The graph $H_{i+1}$ is obtained by taking $\alpha_{i+1}$ copies of $F_i$ and $\alpha_{i+1}-1$ copies of $H_i$. For $j=1, 2, ..., \alpha_{i-1}-1$, the $j$th copy of $F_i$ is hooked to the $j$th copy of $H_i$ with a type 3 hook, the $(j+1)$th copy of $F_i$ to the $j$th copy of $H_i$ with a type 4 hook. The linear order of the vertices of $H_{i+1}$ is also defined in the same way, i.e., the vertices of $F_i^1$ in order, followed by the vertices of $H_i^1$ in the reverse order, etc.. The graph $F_{i+2}$ is constructed the same way as $H_{i+1}$, just with one fewer copy of $F_i$ and $H_i$. In particular, for even $i \leqslant t$, $F_i$ is a triangle.

For $1 \leqslant i \leqslant t-1$, let $G_i$ be obtained by hooking $F_i$ to $H_i$ with a type 3 hook, and let $G'_i$ be obtained by hooking $F_i$ to $H_i$ with type 4 hook. Moreover, for even $i \leqslant t-1$. We also define another graph $G''_i$ which is obtained by hooking $F_i$ to $H_i$ with a type 2 special hook. (Note that if $i \leqslant t-1$ is even, then $F_i$ is a triangle.)

Next, we define graphs $T_i$, $S_i$, $T'_i$, $S'_i$ (for $2 \leqslant i \leqslant t$) as before, i.e., $T_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 3 hook; $S_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 3 hook; $T'_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 4 hook; $S'_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 4 hook; For $2 \leqslant i \leqslant t$, we shall define another graph $T''_i$ as follows: when $i$ is even, $T''_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 2 special hook; when $i$ is odd, $T''_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 2 special hook.

Lemma 6.1. *For $1 \leqslant i \leqslant t-1$, the graphs $G_i, G'_i$ and $G''_i$ are planar graphs, and have circular chromatic number $p_i/q_i$. For $2 \leqslant i \leqslant t$, each of the graphs $T_i$, $S_i$, $T'_i$, $S'_i$ and $T''_i$ has circular chromatic number greater than $p_{i-1}/q_{i-1}$.*

The proof of Lemma 6.1 is similar to the proofs of Theorems 5.1 and 5.3. It makes use of the following lemma, which is proved in the same way as Lemma 5.6.

Lemma 6.2. *Suppose that $1 \leqslant i \leqslant t-1$, and that $C$ is a good Hamilton cycle of $G_i$ (resp. $G'_i$, $G''_i$). Then $C$ is the union of a good Hamilton path $P$ of $F_i$ and a good Hamilton path $P'$ of $H_i$. Moreover, the first and the last vertex of $P'$ are the first and the last vertex of $H_i$ respectively; and in case $F_i$ is not a singleton, the first and the last two vertices of $P$ are the first and the last two vertices of $F_i$, respectively, in that order.*

We shall omit the details of the proofs of Lemmas 6.1 and 6.2

Now we shall construct the graphs $H_t$, $F_{t+1}$, $G_t$ and $G'_t$. In order to construct $H_t$, we first construct another ordered graph $H'_t$. Take $\alpha_t$ copies of $F_{t-1}$ (which are triangles), denoted by $F^1_{t-1}, F^2_{t-1}, ..., F^{\alpha_t}_{t-1}$, and $\alpha_t-1$ copies of $H_{t-1}$, denoted by $H^1_{t-1}, H^2_{t-1}, ..., H^{\alpha_t-1}_{t-1}$. Then hook them as follows:

- hook $F^1_{t-1}$ to $H^1_{t-1}$ with a type 3 hook;
- for $j=2, ..., \alpha_t-1$, hook $F^j_{t-1}$ to $H^j_{t-1}$ with a type 2 special hook;
- hook $F^{\alpha_t}_{t-1}$ to $H^{\alpha_t-1}_{t-1}$ with a type 4 hook;
- for $j=1, 2, ..., \alpha_t-2$, hook $F^{j+1}_{t-1}$ to $H^j_{t-1}$ with a type 1 special hook.

The linear order of the vertices of $H'_t$ is defined similarly, i.e., the vertices of $F^1_{t-1}$ in order, followed by the vertices of $H^1_{t-1}$ in the reverse order, followed by the vertices of $F^2_{t-1}$ in order, etc.

LEMMA 6.3. *The graph $H'_t$ is a planar graph. Moreover, there is a plane embedding of $H'_t$ such that for any $1 \leqslant j \leqslant \alpha_t-2$, the four vertices $y^j_{t-1, h_{t-1}}$, $y^j_{t-1, 2}$, $x^{j+1}_{t-1, 3}$, $x^{j+1}_{t-1, 1}$ (in this cyclic ordering) form a face of the embedding.*

The proof of Lemma 6.3 is easy and similar to the proof of Theorem 5.1, and we shall omit the details.

Now we construct the graphs $H_t$ from $H'_t$ by adding vertices and edges as follows: For each $1 \leqslant j \leqslant \alpha_t-2$, let $S_j$ be the face formed by the four vertices

$$y^j_{t-1, h_{t-1}},\ y^j_{t-1, 2},\ x^{j+1}_{t-1, 3},\ x^{j+1}_{t-1, 1},$$

we put two new vertices $a_j$, $b_j$ into the interior of the face $S_j$ and add the edges

$$a_j b_j,\quad a_j y^j_{t-1, h_{t-1}},\quad b_j y^j_{t-1, h_{t-1}},\quad a_j x^{j+1}_{t-1, 3},\quad b_j x^{j+1}_{t-1, 3},\quad a_j x^{j+1}_{t-1, 1}.$$

The resulting graph is $H_t$.

It follows from the construction that $H_t$ is planar. The graph $H_t$ is not an ordered graph. However, for convenience, we shall refer the $j$th vertex of $H'_t$ as the $j$th vertex of $H_t$. Intuitively, we may regard $H_t$ as an ordered graph with some extra vertices.

LEMMA 6.4. *The graph $H_t$ has circular chromatic number $\chi_c(H_t) \geqslant p_{t-1}/q_{t-1}$, and if $\phi$ is a $(p_{t-1}, q_{t-1})$-colouring of $H_t$, then the first copy of $F_{t-1}$ is coloured the same way as the last copy of $F_{t-1}$.*

*Proof.* Since $H_t$ contains $G_{t-1}$ as a subgraph, and $\chi_c(G_{t-1}) = p_{t-1}/q_{t-1}$, we conclude that $\chi_c(H_t) \geqslant p_{t-1}/q_{t-1}$. Indeed, it is not difficult to show that $\chi_c(H_t) = p_{t-1}/q_{t-1}$, however, we shall not need that. Suppose $\phi$

is a $(p_{t-1}, q_{t-1})$-colouring of $H_t$. Then the restriction of $\phi$ to $H'_t$ is a $(p_{t-1}, w_{t-1})$-colouring of $H'_t$.

In graph $H'_t$, the union of $F^1_{t-1}$ and $H^1_{t-1}$ induces a copy of $G_{t-1}$; the union of $F^{\alpha_t}_{t-1}$ and $H^{\alpha_t-1}_{t-1}$ induces a copy of $G'_{t-1}$; and for $j=2, 3, ..., \alpha_t-1$, the union of $F^j_{t-1}$ and $H^j_{t-1}$ induces a copy of $G''_{t-1}$. Each of the graphs $G_{t-1}$, $G'_{t-1}$ and $G''_{t-1}$ has circular chromatic number $p_{t-1}/q_{t-1}$ by previous results. Applying Lemmas 6.2 and 5.4, we conclude that the colours of the first and the last vertex of $H^{\alpha_t-1}_{t-1}$ determine the colours of the vertices of $F^{\alpha_t}_{t-1}$ and $F^{\alpha_t-1}_{t-1}$. Therefore, the last two copies of $F_{t-1}$ are coloured the same way.

Similarly, for $j=1, 2, ..., \alpha_{t-2}$, the colours of the first and the last vertex of $H^j_{t-1}$ determine the colours of the vertices $F^j_{t-1}$. It remains to show that for $j=1, 2, ..., \alpha_t-2$, the colour of the first and the last vertex of $H^j_{t-1}$ determine the colours of the vertices of $F^{j+1}_{t-1}$, and hence the $j$th copy of $F_{t-1}$ is coloured the same way as the $(j+1)$st copy of $F_{t-1}$. This is subtly different from the argument in the proof of Theorem 5.3, because for $j=1, 2, ..., \alpha_t-2$, the union of the $j$th copy of $H_{t-1}$ and the $(j+1)$th copy of $F_{t-1}$ does not induce a subgraph of circular chromatic number $p_{t-1}/q_{t-1}$.

We now prove that $F^1_{t-1}$ and $F^2_{t-1}$ are coloured the same way.

Let the three vertices of $F^1_{t-1}$ (resp. $F^2_{t-1}$) be $(x^1_{t-1,1}, x^1_{t-1,2}, x^2_{t-1,3})$ (resp. $(x^2_{t-1,1}, x^2_{t-1,2}, x^2_{t-1,3})$). And let the first and the last vertex of $H^1_{t-1}$ be $y^1_{t-1,1}$ and $y^1_{t-1,h_{t-1}}$, respectively.

Without loss of generality, we may assume that $\phi(x^1_{t-1,1})=0$. By Lemmas 5.4 and 6.2, we may assume that

$$\phi(x^1_{t-1,2})=q_{t-1}, \phi(x^1_{t-1,3})=2q_{t-1}, \phi(y^1_{t-1,1}) \\ =3q_{t-1}, \phi(y^1_{t-1,h_{t-1}})=p_{t-1}-q_{t-1}.$$

Next we consider the restriction of $\phi$ to the union of $F^2_{t-1}$ and $H^2_{t-1}$, which induces a copy of $G_{t-1}$ as well. Assume that $\phi(x^2_{t-1,1})=b$. Applying Lemma 6.2 and Lemma 5.4, we conclude that depending on the direction of the good Hamilton cycle, we have either

$$\phi(x^2_{t-1,2})=b+q_{t-1} \qquad (\text{mod } p_{t-1}),$$

and

$$\phi(x^2_{t-1,3})=b+2q_{t-1} \qquad (\text{mod } p_{t-1}),$$

or

$$\phi(x^2_{t-1,2})=b-q_{t-1} \qquad (\text{mod } p_{t-1}),$$

and

$$\phi(x^2_{t-1,3}) = b - 2q_{t-1} \qquad (\text{mod } p_{t-1}).$$

Recall that $F^2_{t-1}$ is hooked to $H^1_{t-1}$ with a type 1 special hook. This means that the following pairs are edges of $H'_t$:

$$e_1 = x^2_{t-1,1} y^1_{t-1,h_{t-1}}, \qquad e_2 = x^2_{t-1,3} y^1_{t-1,1}, \qquad e_3 = x^2_{t-1,3} y^1_{t-1,2}.$$

The edge $e_1$ gives the inequality

$$0 \leqslant b \leqslant p_{t-1} - 2q_{t-1},$$

for otherwise $\phi$ is not a $(p_{t-1}, q_{t-1})$-colouring of $G_t$.

First we assume that $\phi(x^2_{t-1,3}) = b + 2q_{t-1}$ (mod $p_{t-1}$). Then the edge $e_2$ gives the inequality

$$b + 2q_{t-1} \leqslant 2q_{t-1},$$

which implies that $b = 0$, and hence the second copy and the first copy of $F_{t-1}$ are coloured the same way.

Next we assume that $\phi(x^2_{t-1,3}) = b - 2q_{t-1}$ (mod $p_{t-1}$). Since $0 \leqslant b \leqslant p_{t-1} - 2q_{t-1}$, we have

$$b - 2q_{t-1} \leqslant p_{t-1} - 4q_{t-1} < 0$$

and hence

$$\phi(x^2_{t_1,3}) = b - 2q_{t-1} \qquad (\text{mod } p_{t-1}) = p_{t-1} + b - 2q_{t-1} \geqslant p_{t-1} - 2q_{t-1}.$$

Then the edge $e_2$ gives the inequality

$$p_{t-1} - 2q_{t-1} \leqslant \phi(x^2_{t-1,3}) \leqslant 2q_{t-1}.$$

Now we shall show that in order to extend the colouring of $\phi(y^1_{t-1,h_{t-1}})$, $\phi(x^2_{t-1,1})$, $\phi(x^2_{t-1,3})$ to a proper $(p_{t-1}, q_{t-1})$-colouring of the two vertices $a_1, b_1$ of $H_t$, we must have $\phi(x^3_{t-1,3}) = 2q_{t-1}$, and that $\phi(x^2_{t-1,1}) = b = 0$. This is contrary to the assumption that $\phi(x^2_{t-1,3}) = b - 2q_{t-1}$ (mod $p_{t-1}$).

Since $\phi(y^1_{t-1,h_{t-1}}) = p_{t-1} - q_{t-1}$ and $p_{t-1} - 2q_{t-1} \leqslant \phi(x^2_{t-1,3}) \leqslant 2q_{t-1}$, and that each of $a_1$, $b_1$ is adjacent to both $y^1_{t-1,h_{t-1}}$ and $x^2_{t-1,3}$, it follows that

$$0 \leqslant \phi(a_1) \leqslant q_{t-1}$$

and

$$0 \leqslant \phi(b_1) \leqslant q_{t-1}.$$

As $a_1$, $b_1$ are adjacent, it follows that one of $a_1$, $b_1$ has colour 0 and the other has colour $q_{t-1}$. This implies that $\phi(x^2_{t-1,3}) = 2q_{t-1}$.

However, $a_1$ is adjacent to $x^2_{t-1,1}$. Then the condition that

$$0 \leqslant \phi(x^2_{t-1,1}) \leqslant 4q_{t-1} - p_{t-1}$$

forces $b = \phi(x^2_{t-1,1}) = 0$, contrary to the assumption that $\phi(x^2_{t-1,3}) = b - 2q_{t-1} \pmod{p_{t-1}}$. This proves that under the colouring $\phi$, the vertices of $F^1_{t-1}$ and $F^2_{t-1}$ are coloured the same way.

The same argument can be used to show that the vertices $F^j_{t-1}$ and the vertices of $F^{j+1}_{t-1}$ are coloured the same way, for all $j = 1, 2, ..., \alpha_t - 2$. As we have already shown that the vertices of $F^{\alpha_t - 1}_{t-1}$ and the vertices of $F^{\alpha_t}_{t-1}$ are coloured the same way, Lemma 6.4 is proved. ∎

The graph $F_{t+1}$ is constructed in the same way as $H_t$, however, with one fewer copy of $F_{t-1}$ and one fewer copy of $H_{t-1}$. The graph $G_t$ is obtained by hooking $F_t$ to $H_t$ by a type 3 hook; the graph $G'_t$ is obtained by hooking $F_t$ to $H_t$ with a type 4 hook. The graph $T_{t+1}$ is obtained by hooking $F_{t+1}$ to $F_t$ with a type 3 hook; $S_{t+1}$ is obtained by hooking $F_t$ to $F_{t+1}$ with a type 3 hook; $T'_{t+1}$ is obtained by hooking $F_{t+1}$ to $F_t$ with a type 4 hook; $S'_{t+1}$ is obtained by hooking $F_t$ to $F_{t+1}$ with a type 4 hook.

Figure 3 illustrates the construction of $F_i$, $H_i$, $G_i$, for the fraction 26/7, whose alpha sequence is (2, 2, 3).

LEMMA 6.5. *The graphs $G_t$ and $G'_t$ have circular chromatic number at most $p_t/q_t$.*

*Proof.* First we consider the subgraph $R$ of $G_t$ which is obtained from $G_t$ by deleting the vertices $a_j$, $b_j$ for $j = 1, 2, ..., \alpha_t - 2$. The graph $R$ is also obtained by hooking $F_t$ to $H'_t$ with a type 3 hook. Each of the graph $F_t$ and $H'_t$ is an ordered graph. The graph $R$ is also an ordered graph, where the order being the vertices of $F_t$ in order, followed by the vertices of $H'_t$ in the reverse order. Note that the number of vertices of $R$ is $p_t$ (cf. Lemma 5.2). We rename the vertices of $R$ so that the vertices are $(c_1, c_2, ..., c_{p_t})$ in that

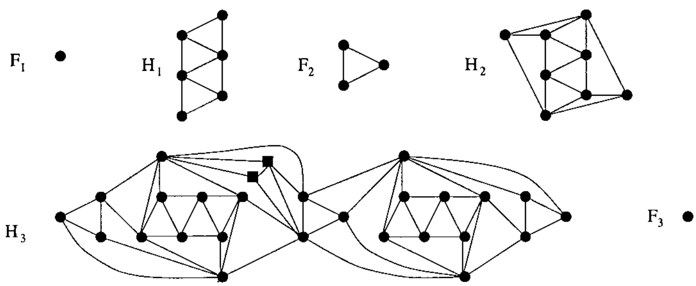


**FIG. 3.** $F_i$ and $H_i$ for the alpha sequence (2, 2, 3).

order. Then we define a colouring $\phi$ as $\phi(c_j) = jq_t \pmod{p_t}$. The same argument as the proof of Theorem 5.2 shows that $\phi$ is a proper colouring of $R$.

Now we extend $\phi$ to a colouring of $G_t$ as follows: For each $1 \leqslant j \leqslant \alpha_t - 2$, we let $\phi(a_j) = \phi(x^{j+1}_{t-1,2})$ and let $\phi(b_j) = \phi(x^{j+1}_{t-1,1})$. Here we abuse the names of the vertices, and let $x^{j+1}_{t-1,i}$ denote the $i$th vertex of the $(j+1)$th copy of $F_{t-1}$ in $H_t$. It is straightforward to verify that this extension of $\phi$ is a $(p_t, q_t)$-colouring of $G_t$.

The same argument shows that $\chi_c(G'_t) \leqslant p_t/q_t$. ∎

The proof of Lemma 6.1 can also be done as follows: we identify $a_j$ with $x^{j+1}_{t-1,2}$, and $b_j$ with $x^{j+1}_{t-1,1}$ in $G_t$. Then using the same argument as in the proof of Theorem 5.2, we can show that the resulting graph (which unfortunately is non-planar) has circular chromatic number at most $p_t/q_t$.

Lemma 6.6. *The circular chromatic number of the graph $G_t$ and $G'_t$ are strictly greater than $p_{t-1}/q_{t-1}$.*

*Proof.* Assume to the contrary that $\chi_c(G_t) \leqslant p_{t-1}/q_{t-1}$. Since $G_t$ contains $G_{t-1}$ is a subgraph and $\chi_c(G_{t-1}) = p_{t-1}/q_{t-1}$, we conclude that $\chi_c(G_t) = p_{t-1}/q_{t-1}$.

Let $\phi$ be a $(p_{t-1}, q_{t-1})$-colouring of $G_t$. By Lemma 6.4, the first copy of $F_{t-1}$ and the last copy of $F_{t-1}$ are coloured the same way.

We note that in $H_t$, the two vertices $x^1_{t-1,2}$ and $x^1_{t-1,3}$ have the same neighbours, so that they are not distinguishable in $H_t$. However, they are distinguishable in $G_t$, as $x^1_{t-1,2}$ is adjacent to $x^2_{t,f_t}$, while $x^1_{t-1,3}$ is not. Therefore if we are simply colouring $H_t$, the colours of $x^1_{t-1,2}$ and $x^1_{t-1,3}$ could be interchanged, however, if we colour the whole graph $G_t$, it can be shown that their colours cannot be exchanged (cf. proof of Lemma 5.4).

Since the first and the last copy of $F_{t-1}$ in $H_t$ are coloured the same way, we conclude that the restriction of $\phi$ to the union of $F_t$ and the first copy of $F_{t-1}$ (in $H_t$) is a $(p_{t-1}, q_{t-1})$-colouring of $T_t$, contrary to the previous result $\chi(T_t) > p_{t-1}/q_{t-1}$. Therefore $\chi_c(G_t) > p_{t-1}/q_{t-1}$. The proof for $\chi_c(G'_t) > p_{t-1}/q_{t-1}$ is the same. ∎

Theorem 6.1. *The graphs $G_t$ and $G'_t$ have circular chromatic number $p_t/q_t$.*

*Proof.* By Lemmas 6.5 and 6.6, we have

$$p_{t-1}/q_{t-1} < \chi_c(G_t) \leqslant p_t/q_t.$$

Assume to the contrary that $\chi_c(G_t) = k/d \neq p_t/q_t$. By the construction of the Farey sequence, it follows that $k > p_t$. (Indeed, straightforward calculation shows that $k \geqslant p_t + p_{t+1}$.)

Let $\phi$ be a $(k, d)$-colouring of $G_t$. By Corollary 5.1, $G_t$ has a cycle of length $k$ (note that the number of vertices of $G_t$ is certainly less than $2p_t < 2k$, and hence $G_t$ cannot have a cycle of length $2k$ or more), say, $C = (c_1, c_2, ..., c_k)$, such that $\phi(c_i) = id \pmod{k}$. This cycle must be a good cycle (although not Hamilton) with respect to the fraction $k/d$ (cf. proof of Lemma 5.4).

It is straightforward to verify that for two fractions $a/b$ and $a'/b'$ between 3 and 4, $a/b < a'/b'$ if and only if the alpha sequence of $a/b$ is greater than the alpha sequence of $a'/b'$ under the lexicographic order. As $p_{t-1}/q_{t-1} < k/d < p_t/q_t$, and that the initial part of the alpha sequence of $p_t/q_t$ is the alpha sequence of $p_{t-1}/q_{t-1}$, we conclude that the initial part of the alpha sequence of $k/d$ is the alpha sequence of $p_{t-1}/q_{t-1}$. Therefore $(p_0/q_0, p_1/q_1, ..., p_{t-1}/q_{t-1})$ is an initial part of the Farey sequence of $k/d$. So far any edge $c_k c_s$ of $G_t$, we have $|s-k| \neq q_j + 1$, $q_j + 2$, $q_j - 3$, $p_j - 4$ for any $0 \leqslant j \leqslant t-1$.

Consider the intersection of $C$ with $H_t$. We regard $H_t$ as the union of $F^j_{t-1}$ for $j = 1, 2, ..., \alpha_t$, and $H^j_{t-1}$ for $j = 1, 2, ..., \alpha_t - 1$, and $A_j = \{a_j, b_j\}$ for $j = 1, 2, ..., \alpha_t - 2$.

By the pigeonhole principle, we know that there exists $1 \leqslant j \leqslant \alpha_t - 2$, such that the intersection $C \cap (A_j \cup F^{j+1}_{t-1} \cup H^{j+1}_{t-1})$ has at least $f_{t-1} + h_{t-1} + 1 = g_{t-1} + 1$ vertices (cf. the proof of Lemma 5.2).

However, we should show that this is impossible. To be precise, we shall prove the following claim (which will be used in later proofs as well):

*Claim* 1. For any good path $B$ of $H_t$ (with respect to the fraction $p_{t-1}/q_{t-1}$), if the initial vertex of $B$ is contained in $F^1_{t-1}$ and the terminal vertex is contained in $F^{\alpha_t}_{t-1}$, then the intersection $B \cap (A_j \cup F^{j+1}_{t-1} \cup H^{j+1}_{t-1})$ has at most $f_{t-1} + h_{t-1} = g_{t-1}$ vertices.

Assume to the contrary that $B$ is a good path of $H_t$ such that

$$|B \cap (A_j \cup F^{j+1}_{t-1} \cup H^{j+1}_{t-1})| \geqslant g_{t-1} + 1.$$

Figure 4 below shows the union $A_j \cup F^{j+1}_{t-1} \cup H^{j+1}_{t-1}$, together with $H^j_{t-1}$. For simplicity, we shall refer to the vertices by the names in Fig. 4, which are different from the names we used before.

By noting that $y_1$, $y_2$ form a cut set, and that $h_{t-1} \geqslant 8$, it is easy to see that $B \cap H^{j+1}_{t-1}$ is a path of $H^{j+1}_{t-1}$ whose two end vertices are $y_1$ and $y_2$ respectively. Moreover, this path of $H^{j+1}_{t-1}$ is either a Hamilton path, or a path of length $h_{t-1} - 1$, i.e., missing at most one vertex of $H^{j+1}_{t-1}$, for otherwise the intersection $B \cap (A_j \cup F^{j+1}_{t-1} \cup H^{j+1}_{t-1})$ could not have more than $g_{t-1}$ vertices.

One of the end vertices, say $y_i$ ($i = 1$ or 2), of the path $B \cap H^{t+1}_{t-1}$ is adjacent to some $x_s$ in the path $B$. First we show $s \neq 2$. This is because

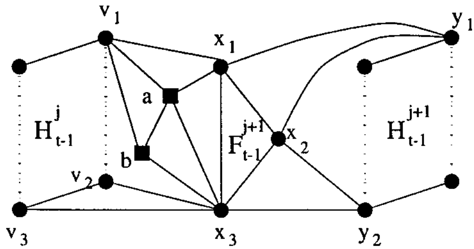


**FIG. 4.** For the proof of Claim 1.

$x_2 y_{3-i}$ is an edge of $H_t$. If $x_2$ is the vertex preceding $y_i$ in the path $B$, then the positive difference of the positions of $x_2$ and $y_{3-i}$ in $B$ is equal to either $p_{t-1}-3$ or $p_{t-1}-4$ (according to whether $|B \cap V(H_{t-1}^{j+1})| = h_{t-1}$, or $|B \cap V(H_{t-1}^{j+1})| = h_{t-1}-1$), contrary to the assumption $B$ is a good path with respect to $k/d$. Therefore $s=1$ or 3.

First we consider the case $s=3$, i.e., $x_3 y_2$ is an edge of the path $B$. If we traverse along the path $B$ backwards, and starting at $x_3$, then the path will pass through 3 or 4 of the vertices $x_1$, $x_2$, $a$, $b$ and then reach the vertex $v_1$. (If $B$ contains less than 3 of the vertices $x_1$, $x_2$, $a$, $b$, then the intersection $B \cap (A_j \cup F_{t-1}^{j+1} \cup H_{t-1}^{j+1})$ could not have more than $g_{t-1}$ vertices.) Let $u$ be the vertex among $x_1$, $x_2$, $a$, $b$ which is adjacent to $v_1$ in $B$. Then $x_3 u$ is an edge of $H_t$, and the positive difference of their positions in $B$ is 3 or 4, contrary to the assumption that $B$ is a good path.

Next we consider the case that $s=1$, i.e., $x_1 y_1$ is an edge of $B$. If the vertex preceding $x_1$ in $B$ is $v_1$, then the path $B$ must pass through both the vertices $a$ and $b$, and then reach the vertex $x_3$ (if we traverse $B$ backwards). However $x_1 x_3$ is an edge of $H_t$, which implies that the positive difference of their positions in $B$ cannot be 3. Assume now that the vertex preceding $x_1$ in $B$ is not $v_1$. Then the path will pass through 3 or 4 of the vertices $x_2$, $x_3$, $a$, $b$ and then reach either the vertex $v_1$ or the vertex $v_2$. If the path reaches $v_2$, then it is easy to see that the positive difference of the positions of $x_1$ and $x_3$ in $B$ is 3, which is a contradiction. Assume now that the path will pass through 3 or 4 of the vertices $x_2$, $x_3$, $a$, $b$ and then reach the vertex $v_1$. If it passes through only 3 of the vertices $x_2$, $x_3$, $a$, $b$, then the positive difference of the positions of $x_1$ and $v_1$ in $B$ is 4, which is a contradiction. Therefore $B$ passes through all the 4 vertices $x_2$, $x_3$, $a$, $b$. However, it is easy to see that in this case, the positive difference of the positions of $x_1$ and $a$ in $B$ is either 3 or 4, again contrary to the assumption that $B$ is a good path. This completes the proof of Claim 1, as well as the proof of Theorem 6.1. ∎

LEMMA 6.7. *The graphs $T_{t+1}$, $S_{t+1}$, $T'_{t+1}$ and $S'_{t+1}$ have circular chromatic number greater than $p_t/q_t$.*

*Proof.* The proof of this lemma is similar to the proof of the fact that $\chi_c(G_t) \geqslant p_t/q_t$. First of all, we show that $\chi_c(T_{t+1}) > p_{t-1}/q_{t-1}$, by applying the previous result that $\chi_c(S_t) > p_t/q_t$ and Lemma 6.4. Then we prove that for any fraction $p_{t-1}/q_{t-1} < k/d \leqslant p_t/q_t$, there is no $(k, d)$-colouring of $T_{t+1}$, by applying Claim 1. We shall omit the details.

Now we shall construct graphs $F_{i+1}$, $H_i$, $G_i$, $G'_i$, $T_i$, $S_i$, $T'_i$, $S'_i$ for $i \geqslant t+1$.

Suppose $i \geqslant t+1$. Then the graph $H_i$ is constructed by taking $\alpha_i$ copies of $F_{i-1}$ and $\alpha_i - 1$ copies of $H_{i-1}$. For $j = 1, 2, ..., \alpha_i - 1$, the $j$th copy of $F_{i-1}$ is hooked to the $j$th copy of $H_{i-1}$ with a type 3 hook; the $(j+1)$th copy of $F_i$ is hooked to the $j$th copy of $H_i$ with a type 4 hook. The graph $F_{i+1}$ is constructed the same way as $H_i$, but with one fewer copy of $F_{i-1}$ and one fewer copy of $H_{i-1}$.

Similarly to the graph $H_t$, the graphs $H_i$ and $F_{i+1}$, for $i \geqslant t+1$, are not ordered graphs. However, the subgraphs of $H_i$ and $F_{i+1}$, obtained by deleting those $a_j$'s and $b_j$'s added in the construction of $H_t$ and $F_{t+1}$, are ordered graphs. We shall regard $H_i$ and $F_{i+1}$ as ordered graphs with some extra vertices. It is in this sense that we add the hooking edges between these graphs. (Note that the types of hooks are only defined between ordered graphs.)

For $i \geqslant t+1$, the graph $G_i$ is obtained by hooking $F_i$ to $H_i$ with a type 3 hook; and $G'_i$ is obtained by hooking $F_i$ to $H_i$ with a type 4 hook.

For $i \geqslant t+1$, the graph $T_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 3 hook; $S_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 3 hook; $T'_i$ is obtained by hooking $F_i$ to $F_{i-1}$ with a type 4 hook; $S'_i$ is obtained by hooking $F_{i-1}$ to $F_i$ with a type 4 hook.

LEMMA 6.8. *For any $i \geqslant t+1$, any good path of $H_i$ (resp. $F_i$) has length at most $h_i$ (resp. $f_i$). Moreover, for any good path $P$ of $H_i$ (resp. $F_i$) of length $h_i$ (resp. $f_i$), the first and the last two vertices of $P$ are the first and the last two vertices of $H_i$ (resp. $F_i$) in that order.*

The proof of Lemma 6.8 is similar to the proof of Lemma 5.6, by using induction, and by applying Claim 1. We shall omit the details.

THEOREM 6.2. *For each $i \geqslant t+1$, the graphs $F_i, H_i, G_i, G'_i$ are planar graphs with $\chi_c(G_i) = \chi_c(G'_i) = \chi_c(G''_i) = p_i/q_i$, and each of the graphs $T_{i+1}$, $S_{i+1}$, $T'_{i+1}$, $S'_{i+1}$ has circular chromatic number greater than $p_i/q_i$.*

*Proof.* First, we prove that $\chi_c(G_i) \leqslant p_i/q_i$ and $\chi_c(G'_i) \leqslant p_i/q_i$. The proof is similar to the proof of Lemma 6.5.

In $H_t$ (resp. $F_{t+1}$), we identify the vertex $a_j$ with $x^{j+1}_{t-1,2}$, and identify $b_j$ with $x^{j+1}_{t-1,1}$, for $j = 1, 2, ..., \alpha_t - 2$ (resp. for $j = 1, 2, ..., \alpha_t - 3$). Here we consider $H_t$ as the union of copies of $F_{t-1}$ and $H_{t-1}$ and refer the $i$th vertex of the $j$th copy of $F_{t-1}$ as $x^j_{t-1,i}$. We shall denote the resulting graph by $H^*_t$ (resp. $F^*_{t+1}$).

In the later constructions of $G_i$, $G'_i$, $F_i$, $H_i$, we may replace any copy of $H_t$ (resp. $F_{t+1}$) by $H^*_t$ (resp. $F^*_{t+1}$), and denote the resulting graphs by $G^*_i$, $G'^*_i$, $F^*_i$, $H^*_i$, etc. Then the graphs $G^*_i, G'^*_i, F^*_i, H^*_i$ are ordered graphs (although they are not planar any more), and $G^*_i$ has $p_i$ vertices (cf. proof of Lemma 5.2). Suppose the vertices of $G^*_i$ are $(c_1, c_2, ..., c_{p_i})$ in that order. Then define $\phi(c_j) = jq_i \pmod{p_i}$. The same argument as in the proof of Theorem 5.2 shows that $\phi$ is a $(p_i, q_i)$-colouring of $G^*_i$. Therefore $\phi$ induces a $(p_i, q_i)$-colouring of $G_i$. Similarly, we can prove that $\chi_c(G'_i) \leqslant p_i/q_i$.

Next, we shall prove that $\chi_c(G_i) = p_i/q_i$ and that each of the graphs $T_{i+1}$, $S_{i+1}$, $T'_{i+1}$, $S'_{i+1}$ has circular chromatic number greater than $p_i/q_i$, by induction. The argument is similar to the proof of Theorem 5.3, only instead of applying Lemma 5.6, we shall use Lemma 6.8. We shall omit the details. ▮

## 7. OPEN PROBLEMS

In this section, we ask a few questions motivated by the result of this paper.

A graph $H$ is a *minor* of a graph $G$ if $H$ is isomorphic to a graph obtained from a subgraph of $G$ by contracting edges. We say $G$ is *$H$-minor free* if $H$ is not a minor of $G$. It is well-known that planar graphs are $K_5$-minor free. Therefore, the result in this paper implies that for any rational number $2 \leqslant r \leqslant 4$, there exists a $K_5$-minor free graph whose circular chromatic number is $r$. A natural question is:

QUESTION 7.1. *What are the possible values of the circular chromatic numbers of $K_n$-minor free graphs?*

If Hadwiger conjecture is true, then any $K_n$-minor free graph has circular chromatic number at most $n-1$. A question parallel to that for the planar graphs is this:

QUESTION 7.2. *Is it true that for any rational number $2 \leqslant r \leqslant n-1$, there exists a $K_n$-minor free graph whose circular chromatic number is equal to $r$?*

A surprising negative answer for the case $n=4$ is recently obtained by P. Hell and X. Zhu [7]. It is shown in [7] that for any $K_4$-minor free graph $G$, we have either $\chi_c(G) = 3$ or $\chi_c(G) \leqslant 8/3$.

For $n \geqslant 6$, it is proved in [18] that for every rational number $2 \leqslant r \leqslant n-2$, there exists a $K_n$-minor free graph with circular chromatic number $r$. It remains unknown whether for every $n-2 < r < n-1$, there is a $K_n$-minor free graph with circular chromatic number $r$.

Another direction for generalizing the result of this paper is to consider graphs embedded on other surfaces.

QUESTION 7.3. *For an integer $n \geqslant 1$, what are the possible values of the circular chromatic numbers of graphs embeddable on the surface of (orientable) genus $n$?*

In particular, the following question seems non-trivial:

QUESTION 7.4. *Does there exists an $\varepsilon > 0$ and an integer $n$ such that for every rational number $4 \leqslant r \leqslant 4 + \varepsilon$, there exists a graph $G$ with $\chi_c(G) = r$, embeddable on the surface of (orientable) genus $n$?*

It seems to the author that the answer to this question is more likely to be negative.

An alternate way of asking Question 7.3 is this:

QUESTION 7.5. *Given a rational number $r$, what is the minimum $n$ such that there exists a graph $G$ embeddable on the surface of (orientable) genus $n$ and $\chi_c(G) = r$?*

If $r > 4$, it is possible that the number $n$ is somehow related to the length of the alpha sequence of $r$.

The last question is about flows of graphs, and is due to L. Goddyn, and was raised in a discussion [9] about the result in this paper. Colourings of graphs and flows of graphs are dual concepts. Goddyn *et al.* [8] defined the circular chromatic number of a graph $G$ by using the concept of flow of the cocyclic matroid of $G$. Given an oriented matroid $M$, define the circular flow number $F_c(M)$ of $M$ as the infimum of the ratios $k/d$ such that $M$ has an integer flow $f$ satisfying the condition that $d \leqslant |f(e)| \leqslant k - d$ for every element $e$ of $M$. It was shown in [8] that the circular chromatic number of a graph $G$ is equal to the circular flow number of the cocyclic matroid of $G$. An interesting problem is to determine the possible values of the circular flow number of a graph. To be precise, we define the circular flow number of a graph $G$ as follows:

Suppose $k$ and $d$ are integers such that $k \geqslant 2d$. A $(k, d)$-*flow* of an oriented (2-edge connected) graph $G$ is an assignment $f$ of integers to the edges of $G$ such that

- For any vertex $v$ of $G$, we have

$$\sum_{e \in N^+(v)} f(e) = \sum_{e \in N^-(v)} f(e).$$

Here $N^+(v)$ denotes the set of edges with $v$ as their tail and $N^-(v)$ denotes the set of edges with $v$ as their head.

- For every edge $e$ of $G$ we have

$$d \leqslant |f(e)| \leqslant k - d.$$

We define the *circular flow number* $F_c(G)$ of a 2-edge connected graph $G$ as the infimum of the ratios $k/d$ such that an (arbitrary) orientation of $G$ has a $(k, d)$-flow.

Just like the circular chromatic number of a graph, the circular flow number of a graph is a refinement of the *flow number* $\phi(G)$ of a graph [5], which is defined to be the minimum integer $n$ such that $G$ has a no-where-zero $n$-flow. It can be proved that $\phi(G) = \lceil F_c(G) \rceil$ for any graph $G$. Also similar to the circular chromatic number, the circular flow number of a finite graph is always a rational number.

It follows from Seymour's 6-flow Theorem that for any graph $G$ we have $F_c(G) \leqslant 6$. If Tutte's 5-flow conjecture is true, then for any graph $G$ we have $F_c(G) \leqslant 5$.

Like the problem we asked for planar graphs, a natural question for the circular flow numbers of graphs is the following:

QUESTION 7.6. *What are the possible values of the circular flow numbers of graphs?*

Since the circular chromatic number of a graph is the circular flow number of its cocyclic matroid, we conclude that the circular chromatic number of a planar graph is equal to the circular flow number of its dual graph. Therefore we have the following corollary:

COROLLARY 7.1. *For every rational number $2 \leqslant r \leqslant 4$, there exists a planar graph $G$ whose circular flow number is equal to $r$.*

However, the answer to the following question remains unknown:

QUESTION 7.7. *It is true that every rational number between* 4 *and* 5 *is the circular flow number of a graph?*

Recently, Steffen [12] proved that there are graphs whose circular flow numbers are greater than 4 but arbitrarily close to 4.

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