Note

Some Results Concerning \(((q + 1)(n - 1); n)\)-Arcs and \(((q + 1)(n - 1) + 1; n)\)-Arcs in Finite Projective Planes of Order q

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In a finite projective plane \(\pi(q)\) of order \(q\), any nonvoid set of \(k\) points may be described as a \(\{k; n\}\)-arc, where \(n(n \neq 0)\) is the greatest number of collinear points in the set. \(\{k; 2\}\)-arcs may be more simply called \(k\)-arcs. In a given plane a \(\{k; n\}\)-arc is said to be complete if there exists no \(\{k'; n\}\)-arc, \(k' > k\), which contains it. For given \(q\) and \(n(n \neq 0)\), \(k\) can never exceed \((n - 1)(q + 1) + 1\), and an arc with that number of points will be called a maximal arc [1]. Equivalently, a maximal arc may be defined as a nonvoid set of points meeting every line in just \(n\) points or in none at all. Evidently a maximal arc is complete. We remark that \(\pi(q)\) is a maximal \(\{q^2 + q + 1; q + 1\}\)-arc and that \(\pi(q)^*\) line \(L\) is a maximal \(\{q^2; q\}\)-arc.

If \(K\) is a \(\{qn - q + n; n\}\)-arc (i.e., a maximal arc) of a projective plane \(\pi(q)\) of order \(q\), where \(n \leq q\), then it is easy to prove that the set \(K' = \{\text{lines } L \text{ of } \pi(q) \parallel L \cap K = \emptyset\}\) is a \(\{q(q - n + 1)/n; q/n\}\)-arc (i.e., a maximal arc) of the dual projective plane \(\pi^*(q)\) of \(\pi(q)\). It follows immediately that, if the desarguesian projective plane \(PG(2, q)\) over the Galois field \(GF(q)\) contains a \(\{qn - q + n; n\}\)-arc, \(n \leq q\), then it also contains a \(\{q(q - n + 1)/n; q/n\}\)-arc.

From the preceding it follows that a necessary condition for the existence of a maximal arc (as a proper subset of a given plane \(\pi(q)\)) is that \(n\) should be a factor of \(q\). But the condition is not sufficient; Cosssu [2] has proved that, in the desarguesian plane of order 9, there is no \(\{21, 3\}\)-arc. In [3], Dennistone proves that the condition does suffice in the case of any desarguesian plane of order \(2^a\). Recently the author [5] has proved that
there exist \(\{2^m - 2^{2m} + 2^m; 2^m\}\)-arcs \((m = 2s + 1, s \geq 1)\) in the projective Lüneburg-plane of order \(2^{2m}\).

Finally we mention the following two theorems, which are due to Barlotti [1].

(a) If the projective plane \(\pi(q)\) contains a \((q + 1)(n - 1); n\)-arc \(K, 2 < n < q + 1\), then \(n\) is a factor of \(q\);  
(b) In a plane \(\pi(q)\) every \((q + 1)(n - 1); n\)-arc \(K, n > 2\), is incomplete, and can be completed in only one way to form a \(\{qn - q + n; n\}\)-arc.

**Theorem 1.** Let \(p_1, p_2, p_3\) be three noncollinear points of a \(\{qn - q + n; n\}\)-arc \(K, n \leq q\), of the desarguesian plane \(PG(2, q)\), and suppose that \(p_1p_2 \cap K = \{p_1, p_2, p_{k,1}, p_{k,2}, \ldots, p_{k,n-2}\}; \{i,j,k\} = \{1,2,3\}\). If the points \(p_1, p_2, p_3\) are chosen as fundamental points \((1,0,0), (0,1,0), (0,0,1)\) and if the equations of the lines \(p_ip_{i,j}\) \((i = 1, 2, 3; j = 1, 2, \ldots, n - 2)\) are written in the form \(p_1p_{1,j}: x_2 = k_{1,j}x_3; p_2p_{2,j}: x_3 = k_{2,j}x_1; p_3p_{3,j}: x_1 = k_{3,j}x_2\) \((j = 1, 2, \ldots, n - 2)\), then

\[
\prod_{j=1}^{n-2} k_{1,j}k_{2,j}k_{3,j} = 1.
\]

**Proof.** Let \(p\) be a point of \(K\backslash(p_1p_2 \cup p_2p_3 \cup p_3p_1) = K^*\). The equations of the lines \(p_1p, p_2p, p_3p\) can be written in the form \(x_2 = l_1x_3, x_3 = l_2x_1, x_1 = l_3x_2\). Evidently \(l_1l_2l_3 = 1\). If \(p\) describes \(K^*\), then \(l_i\) reaches \(n - 1\) times each element of \(GF(q)\backslash\{0, k_{i,1}, k_{i,2}, \ldots, k_{i,n-2}\}\) and \(n - 2\) times each element of \(\{k_{i,1}, k_{i,2}, \ldots, k_{i,n-2}\}\) \((i = 1, 2, 3)\). So we obtain

\[
\prod_{p \in K^*} l_1l_2l_3 = \frac{(a_1a_2 \cdots a_{q-1})^{n-1}}{k_{1,1}k_{1,2} \cdots k_{1,n-2}} \times \frac{(a_2a_3 \cdots a_{q-1})^{n-1}}{k_{2,1}k_{2,2} \cdots k_{2,n-2}} \times \frac{(a_3a_1 \cdots a_{q-1})^{n-1}}{k_{3,1}k_{3,2} \cdots k_{3,n-2}} = (a_1a_2 \cdots a_{q-1})^{3(n-1)} / \prod_{j=1}^{n-2} k_{1,j}k_{2,j}k_{3,j},
\]

where \(GF(q) = \{0, a_1, a_2, \ldots, a_{q-1}\}\). As \(a_1a_2 \cdots a_{q-1} = -1\) and \(l_1l_2l_3 = 1\), \(\forall p \in K^*\), there holds

\[
\prod_{j=1}^{n-2} k_{1,j}k_{2,j}k_{3,j} = (-1)^{3(n-1)}.
\]

From \(n \leq q\) it follows that \(n\) and \(q\) have the same parity (since \(n \mid q\)), and so

\[
\prod_{j=1}^{n-2} k_{1,j}k_{2,j}k_{3,j} = 1.
\]
**Theorem 2.** In $\text{PG}(2, q)$, $q = 3^h$ and $h > 1$, there are no $\{2q + 3; 3\}$-arcs and no $\{q(q - 2)/3; q/3\}$-arcs.

**Proof.** Let us suppose that there is a $\{2q + 3; 3\}$-arc $K$ in $\text{PG}(2, q)$, $q = 3^h$ and $h > 1$. Then a Steiner triple system $K^*(2, 3, 2q + 3) = K^*$ is defined as follows. The points of $K^*$ are the points of $K$; the blocks of $K^*$ are the nonvoid intersections of $K$ with the lines of $\text{PG}(2, q)$; the incidence is that of $\text{PG}(2, q)$. Now we shall prove that the Steiner triple subsystem $K_1^*$ of $K^*$, generated by any three noncollinear points $p_1, p_2, p_3$ of $K^*$, is the desarguesian affine plane $\text{AG}(2, 3)$. For that purpose $p_1, p_2, p_3$ are chosen as fundamental points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. If $p_1 p_2 \cap K = \{p_1, p_2, p_3\}$, $p_2 p_3 \cap K = \{p_2, p_3, p_1\}$, $p_3 p_1 \cap K = \{p_3, p_1, p_2\}$, then from Theorem 1 it follows that the equations of $p_1 p_1', p_2 p_2', p_3 p_3'$ can be written in the form $x_2 = k_1 x_3, x_3 = k_2 x_1, x_1 = k_3 x_2$, with $k_1 k_2 k_3 = 1$. Consequently the lines $p_1 p_1', p_2 p_2', p_3 p_3'$ all meet in one point $w$ (1). As is permissible, let us suppose that $w$ is the unit point $(1, 1, 1)$ of the reference system in $\text{PG}(2, q)$. Then we have: $p_1'(0, 1, 1), p_2'(1, 0, 1), p_3'(1, 1, 0)$. Applying Theorem 1 (or (1)) on the triangle $p_1 p_2 p_3$, we obtain $p_i'' \in K$, where $\{p_i''\} = p_1 p_1' \cap p_2 p_2', \{i, j, k\} = \{1, 2, 3\}$. Hence $p_1''(1, 2, 2) \in K, p_2''(1, 2, 2) \in K, p_3''(1, 2, 2) \in K$. Since the points $p_1(1, 0, 0), p_2(0, 1, 0), p_3(0, 0, 1), p_1'(1, 1, 1), p_2'(1, 1, 1), p_3'(1, 1, 1), p_1''(1, 1, 1), p_2''(1, 1, 1), p_3''(1, 1, 1)$ belong to $K_1^*$, it follows immediately that $K_1^*$ is the desarguesian affine plane $\text{AG}(2, 3) = \{p_1, p_2, p_3, p_1', p_2', p_3', p_1'', p_2'', p_3''\}$.

Next we introduce a parallelism in the following way: two blocks of $K^*(2, 3, 2q + 3)$ are called parallel if they are parallel blocks of a Steiner triple subsystem $K_1^*(2, 3, 9)$ of $K^*$. If two blocks $L$ and $M$ of $K^*$ are parallel, then we write $L \parallel M$. We shall prove that parallelism is an equivalence relation on the set of the blocks of $K^*$. Evidently the relation is reflexive and symmetric. So we have to show that it is also transitive. For that purpose we consider three blocks $L, M, N$ of $K^*$, where $L \parallel M$ and $M \parallel N$. Let $t \in L$, and let $K_1^*(2, 3, 9)$ (resp. $K_2^*(2, 3, 9)$) be the Steiner triple subsystem of $K^*$ defined by $M$ and $t$ (resp. $N$ and $t$). We remark that $L$ is the unique block of $K_1^*$ through $t$, which is parallel to $M (2)$. As the blocks $M$ and $N$ are parallel, they are parallel blocks of a Steiner triple subsystem $K_2^*(2, 3, 9)$ of $K^*$. From the first part of the proof it follows that, without loss of generality, we may assume that $M = \{p_1(0, 1, 0), p_2(0, 1, 0), p_3(1, 1, 0)\}$ and $N = \{p_1'(0, 1, 1), p_2'(0, 1, 1), p_3'(1, 1, 1)\}$. The coordinates of $t$ are denoted by $t_1, t_2, t_3$. If $p_1 t \cap K = \{p_1, t, u_1\}$ and $p_2 t \cap K = \{p_2, t, u_2\}$, then the lines $tp_3', u_1 p_2, u_2 p_1$ all meet in one point $v(t_1, v_2, v_3)$ (see (1)). From $v \in tp_3'$, it follows that $t_3(v_2 - v_1) + v_3(t_1 - t_2) = 0$ (3). Also we have $u_1 (v_1 t_3, v_3 t_2, v_2 t_3)$ and $u_2 (v_3 t_1, v_2 t_3, v_3 t_2)$. 
If $p_1u_2 \cap K = \{p_1, u_2, r\}$, then, applying Theorem 1 on the triangle $p_1u_2r$, we obtain that the lines $rt, u_2t, p_1p_2$ are concurrent. After some calculations it follows that the coordinates of $r$, resp., are $-v_3t_1 - v_1t_3$, $v_2t_3$, $v_3t_2$. We remark that the point $r$ belongs to $L \setminus \{t\}$ (see (2)). If $tp'_2 \cap K = \{t, p'_2, s\}$ then, applying Theorem 1, on the triangle $tp'_2s$, we obtain that the lines $tp'_2, p_2s, u_2p'_2$ are concurrent. After some calculations it follows that the coordinates of $s$, resp., are $v_3t_1 + v_3t_2 - v_2t_3$, $v_3t_2$, $v_3t_2 - v_2t_3 + v_3t_3$. Since

\[
\begin{array}{ccc}
0 & 1 & 1 \\
-v_3t_1 - v_1t_3 & v_2t_3 & v_3t_3 \\
v_3t_1 + v_3t_2 - v_2t_3 & v_2t_3 - v_2t_3 + v_3t_3 \\
\end{array}
\]

\[= t_3(v_2 - v_3)(t_3(v_2 - v_1) + v_3(t_1 - t_2)) = 0\] (see (3)),

the set $\{r, s, p'_1\}$ is a block of $K_2^*(2, 3, 9)$. Consequently the block $rt = L$ is the block of $K_2^*$ through $t$ which is parallel to $N$, and so $L \parallel N$. So we conclude that parallelism is an equivalence relation on the set of the blocks of $K^*$.

Now we shall prove that $K^*(2, 3, 2q + 3)$ is an affine space.

(a) Any two points of $K^*$ are contained in one and only one block.

(b) For each point $p$ and each block $L$ there is one and only one block $M$ such that $p \in M$ and $L \parallel M$ (the case $p \in L$ is trivial; if $p \notin L$, then $M$ is the block of $K_1^* (2, 3, 9)$ through $p$ which is parallel to $L$, where $K_1^*$ is the Steiner triple subsystem of $K^*$ defined by $p$ and $L$).

(c) Suppose that $L \parallel M$ ($L \neq M$), $p_1 \in L$, $p_2 \in M$, $p_3$ is the third point of the block $p_1p_2, p_2' \in M \setminus \{p_2\}$. Then $p_1, p_2, p_3, p_2', L, M$ belong to the Steiner triple system $K_1^*(2, 3, 9)$ defined by the parallel blocks $L$ and $M$. Consequently the blocks $L$ and $p_2'p_3$ intersect.

(d) Each block of $K^*$ has exactly three points.

(e) Consider a block $L$ and a point $p \notin L$. Then through $p$ there pass $q - 3 > 0$ blocks $M$ for which $L \cap M = \emptyset$ and $L \parallel M$.

From (a)–(e) it follows immediately that $K^*(2, 3, 2q + 3)$ is an affine space $AG(m, 3), m > 2$ [4]. Consequently $|K| = 2q + 3 = 2 \cdot 3^h + 2 = |AG(m, 3)| = 3^m$, which is only possible if $h = 1$ and $m = 2$, a contradiction. It follows that in $PG(2, q)$, $q = 3^h$ and $h > 1$, there are no $\{2q + 3; 3\}$-arcs.

As the existence of a $\{q(q - 2)/3; q/3\}$-arc in $PG(2, q)$, $q = 3^h$ and $h > 1$, implies the existence of a $\{2q + 3; 3\}$-arc in $PG(2, q)$, we conclude that there are no $\{q(q - 2)/3; q/3\}$-arcs in $PG(2, q)$, $q = 3^h$ and $h > 1$. 

Remark. Define a design $D$ in the following way. The points of $D$ are the points of $K^*$; the blocks of $D$ are the Steiner triple subsystems $K_1^*(2, 3, 9)$ of $K^*$; the incidence is that of $K^*$. Evidently $D$ is a $2 - (2q + 3, 9, q/3)$ design. If $r$ is the number of blocks of $D$, which pass through a given point of $D$, then $r = q(q + 1)/12$. Consequently 12 is a factor of $3^h(3^h + 1)$, and so $h$ must be an odd integer. So we have proved, very briefly, the following

**Theorem.** In $PG(2, q)$, $q = 3^h$ and $h$ even, there are no $\{2q + 3; 3\}$-arcs and no $\{q(q - 2)/3; q/3\}$-arcs.

**Corollary 1.** In $PG(2, 9)$ there are no $\{21; 3\}$-arcs (theorem of Cossu [2]).

**Corollary 2.** In $PG(2, q)$, $q > 3$, there are no $\{2q + 3; 3\}$-arcs.

**Corollary 3.** In $PG(2, q)$, $q > 3$, there are no $\{2q + 2; 3\}$-arcs; in $PG(2, q)$, $q = 3^h$ and $h > 1$, there are no $\{(q - 3)(q + 1)/3; q/3\}$-arcs.

This follows immediately from Introductory Remarks and Theorem 2.

**Corollary 4.** In $PG(2, q)$, $q > 3$, every $\{2q + 1; 3\}$-arc is complete; in $PG(2, q)$, $q = 3^h$ and $h > 1$, every $\{(q^2 - 2q - 6)/3; q/3\}$-arc is complete.

**Conjecture.** In $PG(2, q)$, $q$ odd, the only maximal arcs are $PG(2, q)$ and $PG(2, q) \setminus$ line $L = AG(2, q)$.

**References**