Introduction

The problem of partitioning the set $S = \{1, 2, \ldots, 2m\}$ into differences $d, d+1, \ldots, d+m-1$ is today known as the problem of constructing a Langford sequence of defect $d$ and length $m$. Langford [3] posed this problem for $d=2$ and subsequently Frad [4] solved it, although slightly earlier Skolem [8] posed this problem for $d=1$ (and solved it: solutions for $d=1$ are called Skolem sequences). For arbitrary $d$ Bermond et al. [2] and Simpson [7] give two necessary and sufficient conditions for a solution to exist: (1) $m \geq 2d-1$, and (2) $m \equiv 0, 1 \pmod{4}$ if $d$ is odd and $m \equiv 0, 3 \pmod{4}$ if $d$ is even.

The partition $P = ((5, 3), (4, 1), (6, 2))$ is a Langford sequence of defect 2 and length 3, but equivalently it may be written 342324, where $j$ appears in positions $b_j, a_j$ with $b_j - a_j = j$ and $(b_j, a_j) \in P$. In what follows we shall use this latter form (due to Nickerson [6]). One feature of Langford sequences is that the differences and the set of positions are both in arithmetic progression. We will now find all other sequences that share this property.

Let $s_1 \leq s_2 \leq \cdots \leq s_{2n}$ and consider the problem of partitioning $S = \{s_1, s_2, \ldots, s_{2n}\}$ into differences $d_1, d_2, \ldots, d_n$. The following two conditions are necessary (see [5]) for a solution to exist:

\[
\sum_{i=1}^{2n} s_i \equiv \sum_{i=1}^{n} d_i \pmod{2},
\]

\[
\sum_{i=1}^{n} d_i \leq \sum_{i=n+1}^{2n} s_i - \sum_{i=1}^{n} s_i.
\]

They are called the parity and the density conditions (respectively).

In the case where the elements of $S$ and the differences are in arithmetic progression let $S = \{a, a+d, \ldots, a+(2n-1)d\}$ and the differences be $c, c+d_0, \ldots, c+(n-1)d_0$. Without loss of generality take $a = 0$, and to avoid trivialities let $n > 1$. Then $d|c$, since $c = kd - k'd = (k-k')d$, and hence $d|d_0$, since $c+d_0 = jd - j'd = (j-j')d$ and $d|c$. Dividing by $d$ reduces the problem to partitioning $[0, 1, 2, \ldots, 2n-1]$ into differences $e, e+f, \ldots, e+(n-1)f$, where $c = de$ and $d_0 = df$. The density condition is now

\[
n e + \frac{1}{2} n(n-1) f \leq n^2.
\]

If $f \geq 3$ then the left-hand side is $\geq \frac{1}{2}(3n^2 - n)$ which is larger than $n^2$ for $n > 1$, so $f = 1, 2$. When $f = 1$ the problem is that of constructing a Langford sequence. If $f = 2$ the density condition becomes $n(e-1) \leq 0$, hence $f = 2, e = 1$ holds the only possibility for a solvable problem. The solution is trivial, however, as $\{(n, n-1), (n+1, n-2), \ldots, (2n-1, 0)\}$

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is clearly the unique solution. The pattern is even more apparent when written as 75311357 \((n = 4)\). It is fair to say then that this type of sequence is a cousin of the Langford sequence, or even a neglected Langford sequence. We shall simply call it ‘the odd Langford sequence’.

We shall now consider a question for the odd Langford sequence that has already been taken up for the usual Langford sequences. A \(p\)-extended Langford sequence of defect \(d\) and length \(m\) is a partition of \([1, 2, \ldots, 2m + 1] - \{p\}\) into differences \(d, d + 1, \ldots, d + m - 1\). Thus \(34,3242\) is a 3-extended Langford sequence of defect 2 and length 3, the underscore indicating that \(d = 9\). Let \(\text{Largest odd difference} \ 9\), \(A\)

A calculation has period 3, as the existence of \(n\) next two lines indicate that the integer \(2n - 1\) has been overlooked.

### 2. Extended Odd Langford Sequences

Let \(S\) be a finite set of integers. The shape of \(S\) is \(a_1a_2a_3\ldots a_{k-1}a_k\), where \(a_i\) denotes a block of \(i\) consecutive integers in \(S\) and \(\hat{a}_j\) denotes a block of \(a_j\) consecutive integers omitted from \(S\). The shape of \(S = \{1, 2, 4, 5, 6, 7, 9, 10\}\) is \(214\hat{1}\hat{2}\), and if \(S\) is partitioned into differences \(1, 3, 5, 7\) the shape is clearly visible in the sequence: \(57,1153,73\). Let \(A(n)\) denote any sequence with shape \(11(2n - 2)1\hat{1}\) and odd differences \(1, 3, \ldots, 2n - 1, n \geq 1\). We prove its existence by performing a periodicity calculation as follows:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Largest odd difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{1}(2n - 2)\hat{1})</td>
<td>(2n - 1)</td>
</tr>
<tr>
<td>(\hat{2}(2n - 3))</td>
<td>(2n - 3)</td>
</tr>
<tr>
<td>(2n - 6)\hat{1}\hat{2}</td>
<td>(2n - 5)</td>
</tr>
<tr>
<td>(\hat{1}(2n - 8)\hat{1})</td>
<td>(2n - 7)</td>
</tr>
</tbody>
</table>

Thus the first two lines indicate that the integer \(2n - 1\) occupies positions 3 and \(2n + 2\), the next two lines indicate that the integer \(2n - 3\) occupies positions 1 and \(2n - 2\), and so on. The calculation has period 3, as the existence of \(A(n)\) implies that of \(A(n + 3)\). To prove \(A(n)\) exists for \(n = 3\) and \(n \geq 5\) it is enough to construct \(A(3) = 5,113,53, A(5) = 7,95113573, 9\), and \(A(7) = b_d9357115b97, d\) as \(A(6)\) comes from \(A(3)\) and then \(A(5), A(6), A(7)\) are three consecutive sequences of the required shape. One easily checks there are no \(A(1), A(2)\) or \(A(4)\) sequences.

We also require sequences \(B(n)\) of shape \(2\hat{1}(2n - 4)\hat{2}\) with differences \(1, 3, \ldots, 2n - 1\), for which we perform another periodicity calculation:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Largest odd difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2\hat{1}(2n - 4)\hat{1}\hat{2})</td>
<td>(2n - 1)</td>
</tr>
<tr>
<td>(\hat{1}(2n - 4)\hat{2}\hat{1})</td>
<td>(2n - 3)</td>
</tr>
<tr>
<td>((2n - 6)\hat{1}\hat{2}\hat{1})</td>
<td>(2n - 5)</td>
</tr>
<tr>
<td>((2n - 7)\hat{4})</td>
<td>(2n - 7)</td>
</tr>
<tr>
<td>(4\hat{1}(2n - 12))</td>
<td>(2n - 9)</td>
</tr>
<tr>
<td>(\hat{1}\hat{2}(2n - 13))</td>
<td>(2n - 11)</td>
</tr>
<tr>
<td>(\hat{2}(2n - 16)\hat{1})</td>
<td>(2n - 13)</td>
</tr>
</tbody>
</table>
One easily checks no $B(2), B(3)$ sequences exist. As the period is 6 this number of sequences is needed to start the induction: $B(4) = 57_{115373}, B(5) = 79_{113573_95}, B(6) = 9b_{73113597}, B(7) = bd_{951137539b}, B(8) = df_{b751193573db}, B(9) = fh_{d953713597f}$. 

Lemma 2.1. An $A(n)$ sequence exists iff $n \geq 5$ or $n = 3$; a $B(n)$ sequence exists iff $n \geq 4$.

We now introduce the two sequences of interest, each with differences 1, 3, 4, 7, 9, 12, ... and form two sequences simultaneously. Note that by the parity condition $p$ is odd for any $E^p$ or $F^p$ sequence. In what follows the symbol $`\epsilon$' stands for $`\ast$'.

An $H$-fragment of defect $d$ and length $m$ denotes the sequence $H_{d,m} = d + 2m - 2, \ldots, d + 4, d + 2, d, \epsilon^{d-1}, d, d + 2, d + 4, \ldots, d + 2m - 2$. It uses differences all of the same parity, and there is no bound on the length as a function of $d$. An example is $H_{4,6} = ceca8646ae$ ace. The union of two sequences is their superposition when aligned at their starts, for example $2_{211} \cup 4e_{3453_35} = 24211453_{35}$. 

Lemma 2.2. If an $F^p(n)$ exists for all odd $p$ then an $E^p(n)$ exists for all odd $p$.

Proof. We have $E^p(n + 1) = F^p(n) \cup \epsilon(2n + 3)\epsilon^{2n+2}(2n + 3)$ for all odd $1 \leq p \leq 2n + 4$, and the construction $\epsilon^{2n+5}(n + 1) = H_{1,n+2\epsilon}$ completes the range of admissible $p$ for an $E^p(n + 1)$. 

We are now ready to prove the main result.

Theorem 2.3. There is no $E^3(1)$, and there are no $F^3(0)$, $F^3(1)$, $F^3(2)$ or $F^3(3)$ sequences. For all other $n \geq 1$ and odd $p$ the sequences $E^p(n)$ and $F^p(n)$ exist.

Proof. The exceptions are readily verified. By Lemma 2.2 it suffices to prove the existence of $F^p(n)$ sequences, but what is most convenient is to prove the existence of a $E^p(n)$, which is just a $E^p(n)$ sequence that has the difference $2n + 1$ in the second position and $1 \leq p \leq 2n + 1$. This is the inductive hypothesis.

To start the induction construct $F^p(n)$ and $E^p(n)$ sequences for $n = 0, 1, 2$ and odd $p$ not in the list of exceptions. For $n \geq 3$ write $n = 3u + r$, $0 \leq r \leq 2$ and apply the appropriate construction below to get a $E^p(n)$. Note that $u \geq 1$, and that often an (unstarred) $E^p(n)$ sequence suffices as an ingredient.

(i) $n \equiv 0 \pmod{3}$. Write $n = 3u$ and form

$$(2u + 1)\epsilon^{2u}(2u + 1) \cup \epsilon H_{2u+1,2u} \cup \epsilon^{2u+2}E^p(u - 1).$$

When $u = 2$, $p = 3$ the construction fails, in that case we may use $E^p_u(6) = bd_{539735_{11b79d}}$. Otherwise note where the difference $2n + 1$ occurs, and that by varying $p$ above we obtain all admissible holes in the interval $[2u + 3, 4u + 3]$. For $1 \leq j \leq 2u - 1$ form

$$(2u + 1 + 2j)\epsilon^{2u+2j}(2u + 1 + 2j) \cup \epsilon H_{2u+3+2j,2u+j} \cup \left\{ \begin{array}{ll} \epsilon^{2u+1-j}E^3j+1(u + j - 1) & \text{j even} \\ \epsilon^{2u+2-j}E^3j(u + j - 1) & \text{j odd} \end{array} \right.$$
When $u = j = 1$ the construction fails, indeed no construction can produce a $E^3_2(3)$ as it does not exist. In this case we construct $E^3(3) = 73_{-531175}$ for later use. Otherwise note that we obtain holes $2u + 2 - j$ for odd $j \in [1, 2u - 1]$ and holes $4u + 3 + j$ for even $j \in [1, 2u - 1]$. Altogether we get all odd holes in $[3, 2u + 1] \cup [4u + 5, 6u + 1]$, which together with holes in $[2u + 3, 4u + 3]$ above gives all holes in $[3, 6u + 1]$. The hole in the first position is obtained by $\mathcal{E}H_{1,n+1}$, and now we have all required holes in $[1, 6u + 1]$ and the difference $2n + 1$ in the correct position.

(ii) $n \equiv 2 \pmod{3}$. Write $n = 3u + 2$ and form the following sequence:

$$(2u + 3)e^{2u+2}(2u + 3) + e\mathcal{H}_{2u+5,2u+1} + e^{2u+2}F^p(u).$$

When $F^p(u)$ does not exist we give direct constructions:

(a) $u = 1, p = 5$: $\mathcal{E}^3_2(6) = \mathcal{E}^4_2(6) = 7b911537_{-559b}$

(b) $u = 2, p = 3$: $\mathcal{E}^3_2(8) = bhfd5793_{-53b7119dfh}$

(c) $u = 3, p = 9$: $\mathcal{E}^7_{17}(11) = blnjhfdf973b5117_{-59dfhjln}$.

Otherwise, by varying $p$, we obtain all admissible holes in the interval $[2u + 3, 4u + 6]$.

For $1 \leq j \leq 2u$ form the following sequences:

$$(2u + 3 + 2j)e^{2u+2+2j}(2u + 3 + 2j) + e\mathcal{H}_{2u+5+2j,2u+1+j} + 
\begin{cases}
  e^{2u+2-j}\mathcal{E}^3_{j+1}(u+j) & j \text{ odd} \\
  e^{2u+2-j}\mathcal{E}^3_{j+1}(u+j) & j \text{ even}.
\end{cases}$$

This gives holes $2u + 3 - j$ for even $j \in [1, 2u]$ and holes $4u + 6 + j$ for odd $j \in [1, 2u]$. Altogether we get all odd holes in $[3, 2u+2] \cup [4u+7, 6u+6]$, which together with the above gives all holes in $[3, 6u+1]$. The hole in the first position is easily obtained by $\mathcal{E}H_{1,n+1}$, and now we have all required holes in $[1, 6u+5]$ and the difference $2n + 1$ in the correct position.

(iii) $n \equiv 1 \pmod{3}$. Write $n = 3u + 1$. Let

$$\begin{align*}
\mathcal{Y}_1 &= \mathcal{E}(2u+1)\mathcal{E}(2u+3)\mathcal{E}(2u+7)\mathcal{E}(2u+11) \cdots \\
&= (2u + 7)\mathcal{E}(2u + 1)\mathcal{E}(2u + 3)\mathcal{E}(2u + 7)\mathcal{E}(2u + 11) \\
\mathcal{Y}_2 &= (2u+1)\mathcal{E}(2u+3)\mathcal{E}(2u+7)\mathcal{E}(2u+11) \cdots \\
&= (2u + 5)\mathcal{E}(2u + 3)\mathcal{E}(2u+7)\mathcal{E}(2u+11) \cdots
\end{align*}$$

Form the sequence

$$\mathcal{Y}_2 \cup \mathcal{Y}_1 \cup e^{2u+1}(2u+3) + e^{2u+2}(2u+3) + e^{4u+2}(2u+1) + e^{2u+2}F^p(u-1).$$

When $F^p(u-1)$ does not exist we give direct constructions:

(a) $u = 2, p = 5$: $\mathcal{E}^3_2(7) = 7f69537_{-35119bd}$

(b) $u = 3, p = 3$: $\mathcal{E}^3_2(10) = bhfd97_{11b537935dfhjln}$

(c) $u = 4, p = 9$: $\mathcal{E}^7_{17}(13) = brpnlhfdf95b7115_{3973dfhjlnpr}$.

Otherwise, by varying $p$ we obtain all admissible holes in the interval $[2u + 1, 4u + 2]$. For $1 \leq j \leq 2u - 1$ form the following sequences:

$$(2u+3 + 2j)e^{2u+2+2j}(2u + 3 + 2j) + e\mathcal{H}_{2u+5+2j,2u-j} + 
\begin{cases}
  e^{2u+2-j}\mathcal{E}^3_{j+1}(u+j) & j \text{ even} \\
  e^{2u+2-j}\mathcal{E}^3_{j+1}(u+j) & j \text{ odd}.
\end{cases}$$

This gives holes $4u + 5 + j$ for even $j \in [1, 2u - 1]$ and holes $2u + 2 - j$ for odd $j \in [1, 2u - 1]$. Altogether we get all odd holes in $[3, 2u+1] \cup [4u+7, 6u+3]$, which together with the above gives all holes in $[3, 4u+2] \cup [4u+7, 6u+3]$; and again the hole $p = 1$ is easy to get. We still have to obtain the holes $p = 4u + 3, 4u + 5$. 

\[82 \quad V. \text{Linek}\]
To obtain \( p = 4u + 5 \) form the sequence
\[
Y_2 \cup Y_1 \cup \epsilon^{2u+1}(2u+1) \epsilon^{2u}(2u+1) \cup \epsilon^{4u}(2u+3) \epsilon^{2u+2}(2u+3) \cup \epsilon^{2u}A(u),
\]
and use the following in case of failure:

\[ (u = 1) : \quad E^9(4) = 93573115_{97} \text{ (no } E^9 \text{ exists)} \]
\[ (u = 2) : \quad E^{13}(7) = df9b51137539_{db7f} \]
\[ (u = 4) : \quad E^4(7) = drpnljhf97b15d537935_{bfhjhlpr}. \]

To obtain \( p = 4u + 3 \) form the sequence
\[
\epsilon^H_{2u+5,2u} \cup (2u+3) \epsilon^{2u+2}(2u+3) \cup \epsilon^{2u+1}B(u+1),
\]
using \( E^7(4) = 791153_{7359} \) when \( u = 1 \) and \( E^{11}(7) = 9f_{db117539_{357bfd}} \) when \( u = 2 \).

This completes the proof by induction.

\[ \square \]

3. Conclusion

Consider a problem involving two holes, such as partitioning \( \{1, 2, \ldots, 2n+2\} - \{p, q\} \) into differences \( 1, 2, \ldots, n \). Our result is relevant, as the sequence \( 6789ab627849_{a4b5113513} \) shows. The holes in \( 6789ab6789_{a,b} \) are filled with \( 2,2,4,4 \), which is the double of an extended Langford, and the subsequence at the end is a \( E^3(2) \). Clearly the boldened hole moves as the doubled extended Langford subsequence varies, as does the hole at the end when \( p \) varies in \( E^p(2) \). However, a complete solution to the problem of a \( p, q \)-extension seems to require more extended odd sequences than just the shapes \( E^p \) and \( F^p \), as well as other recursive constructions.

References


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