Invertibility and Spectrum Localization of Non-Self-Adjoint Operators

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This paper deals with Volterra perturbations of normal operators in a separable Hilbert space. Invertibility conditions and estimates for the norm of the inverse operators are established. In addition, bounds for the spectrum are suggested. Applications to integral, integro-differential, and matrix operators are discussed.

Key Words: linear operators; invertibility; spectrum; integral operators; integro-differential operators; infinite matrices.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

A lot of papers and books are devoted to the spectrum of linear operators. Mainly, the asymptotic distributions of the eigenvalues are considered; cf. the books by König [7], Pietsch [10], and Prössdorf [11] and the references therein. However, in many applications, for example, in numerical mathematics and stability analysis, bounds for the eigenvalues and invertibility conditions are very important. But the bounds and invertibility conditions are investigated considerably less than the asymptotic distributions. Below we investigate a class of non-self-adjoint operators in a Hilbert space. The new invertibility conditions and estimates for the norm of the inverse operator are derived. By these invertibility conditions, bounds for the spectrum are established. In addition, we suggest new estimates for the spectral radius which supplement the well-known results; cf. [8].

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Let $H$ be a separable Hilbert space with a scalar product $(\cdot, \cdot)$, the norm $\| \cdot \|$, and the unit operator $I$. For a linear operator $A$, $\sigma(A)$ is the spectrum, $\text{Dom}(A)$ is the domain, $r_s(A)$ denotes the spectral radius. A linear operator $V$ is a Volterra one if it is quasi-nilpotent (i.e., $\sigma(V) = \{0\}$) and compact; cf. [5]. Recall that a maximal resolution of the identity $P(t)$ ($-\infty \leq t \leq \infty$) is a left-continuous orthogonal resolution of the identity, such that any gap $P(t_0 + 0) - P(t_0)$ of $P(t)$ (if it exists) is one dimensional; cf. the books by Brodskii [1], Gohberg and Krein [5], and Gil’ [2, p. 69].

In this paper we consider a certain kind of “Volterra perturbation” of normal operators on a separable Hilbert space. Thus suppose $D : \text{Dom}(D) \subseteq H \to H$ is a densely defined, possibly unbounded, normal operator, with a maximal resolution of the identity $P(t)$:

$$ P(t)Dh = DP(t)h, \quad -\infty \leq t \leq \infty, \ h \in \text{Dom}(D). $$

We consider an operator of the form $D + V_+ + V_-$ with Volterra operators $V_+$ and $V_-$ belonging to a norm ideal $Y$ and “almost” commuting with $D$:

$$ V_+, V_- \in Y, \quad P(t)V_+P(t) = V_+P(t), \quad P(t)V_-P(t) = P(t)V_. $$  \hspace{1cm} (1.1)

We recall that a “norm ideal” $Y \subseteq B(H)$ is algebraically a two-sided ideal, which is complete in an auxiliary norm $|\cdot|_Y$ for which $|CB|_Y$ and $|BC|_Y$ are both dominated by $\|C\||B|_Y$, and note, in addition, that there are positive $\theta_k (k \in \mathbb{N})$, with $\theta_k^k \to 0$, for which, for arbitrary Volterra operators $V \in Y$,

$$ \|V^k\| \leq \theta_k|V|_Y^k. $$  \hspace{1cm} (1.2)

We write

$$ J_Y(V) = \sum_{k=0}^{\text{ni}(V)-1} \theta_k |V|^k_Y \leq \sum_{k=0}^{\infty} \theta_k |V|^k_Y = I_Y(V), $$  \hspace{1cm} (1.3)

where $\text{ni}(V)$ denotes the “nilpotency index” of a nilpotent operator $V \in Y$, so that $V^{\text{ni}(V)} = 0 \neq V^{\text{ni}(V)-1}$; if $V$ is quasi-nilpotent but not nilpotent, we write $\text{ni}(V) = \infty$. With this notation we have our main theorem.

**Theorem 1.1.** Suppose

$$ A = D + V_+ + V_-, $$  \hspace{1cm} (1.4)

where $D$ is normal and Volterra operators $V_\pm$ satisfy (1.1), where the ideal $Y$ has property (1.2). If, in addition, $D$ is boundedly invertible

$$ d_0 = \inf |\sigma(D)| > 0, $$  \hspace{1cm} (1.5)
and
\[ \xi_0(A) = \max \{ J^{-1}_Y(D^{-1}V_\pm) - \| D^{-1}V_+ \|, \\
J^{-1}_Y(D^{-1}V_-) - \| D^{-1}V_- \| \} > 0, \] (1.6)
then \( A = D + V_+ + V_- \) is also invertible, with
\[ \| A^{-1} \| \leq \frac{1}{d_0 \xi_0(A)}. \] (1.7)

Since \( D \) is normal and \( V_\pm \) are Volterra operators, condition (1.1) is enough [2, Lemma 3.2.4] to guarantee that \( W_\pm = D^{-1}V_\pm \) are also Volterra operators.

The proof of Theorem 1.1 is presented in the next section. Theorem 1.1 supplements the well-known results on the invertibility of linear operators; cf. [6] and the references therein.

Note that in Theorem 1.1, according to (1.3), one can replace \( J_Y(W_\pm) \) by \( I_Y(W_\pm) \).

2. PROOF OF THEOREM 1.1

We need the following simple result.

**Lemma 2.1.** Under conditions (1.1) and (1.5), let
\[ \psi_0 \equiv \|(D + V_-)^{-1}V_+\| < 1. \] (2.1)
Then the operator \( A \) represented by (1.4) is boundedly invertible. Moreover,
\[ \| A^{-1} \| \leq \frac{\|(D + V_-)^{-1}\|}{1 - \psi_0}. \] (2.2)

**Proof.** According to (1.4), we have
\[ A = (D + V_+)(I + (D + V_-)^{-1}V_+). \] (2.3)
Due to (1.1) and Lemma 3.2.12 from [2], we have
\[ \sigma(D + V_\pm) = \sigma(D). \] (2.4)
So, according to (1.5), \( D + V_\pm \) is invertible. Moreover, under condition (2.1), the operator \( I + (D + V_-)^{-1}V_+ \) is invertible and
\[ \|(I + (D + V_-)^{-1}V_+)^{-1}\| \leq \sum_{k=0}^{\infty} \|(D + V_-)^{-1}V_+\|^k \leq \sum_{k=0}^{\infty} \psi_0^k = (1 - \psi_0)^{-1}. \]
So, due to (2.3),
\[ \| A^{-1} \| \leq \|(I + (D + V_-)^{-1}V_+)^{-1}\| \|(D + V_-)^{-1}\|. \]
This proves the required result. Q.E.D.
Proof of Theorem 1.1. Due to (1.2),
\[
\|(D + V_-)^{-1}V_+\| = \|(I + D^{-1}V_-)^{-1}D^{-1}V_+\| = \|(I + W_-)^{-1}W_+\|
\leq \|W_+\| \sum_{k=0}^{\infty} \|W_-^k\| = \|W_+\| \sum_{k=0}^{n(W_)-1} \|W_-^k\|
\leq \|W_+\| \sum_{k=0}^{n(W_)-1} \theta_k |W_-|^k = \|W_+\| J_Y(W_-).
\]

Hence \(\psi_0 \leq \|W_+\| J_Y(W_-)\). But condition (1.6) implies that at least one of the inequalities
\[
\|W_+\| J_Y(W_-) < 1 \quad (2.5)
\]
or
\[
\|W_-\| J_Y(W_+) < 1 \quad (2.6)
\]
is valid. If condition (2.5) holds, then (2.1) is valid. Moreover, since \(D\) is a normal operator, \(\|D^{-1}\| = d_0^{-1}\). Thus
\[
\|(D + V_-)^{-1}\| = \|(I + W_-)^{-1}D^{-1}\| \leq \|D^{-1}\| \sum_{k=0}^{\infty} \|W_-^k\|
= d_0^{-1} \sum_{k=0}^{n(W_)-1} \|W_-^k\| \leq d_0^{-1} \sum_{k=0}^{n(W_)-1} \theta_k |W_-|^k = d_0^{-1} J_Y(W_-).
\]

Thus, under (2.5), Lemma 2.1 yields the inequality
\[
\|A_0^{-1}\| \leq \frac{J_Y(W_-)}{d_0(1 - \|W_+\| J_Y(W_-))} = \frac{1}{d_0(J_Y^{-1}(W_-) - \|W_+\|)} \quad (2.7)
\]
Interchanging \(W_-\) and \(W_+\), under condition (2.6), we get
\[
\|A_0^{-1}\| \leq \frac{1}{d_0(J_Y^{-1}(W_-) - \|W_+\|)}.
\]
This relation and (2.7) yield the required result. Q.E.D.

3. LOCALIZATION OF THE SPECTRUM

For a Volterra operator \(V \in Y\), under (1.2) denote
\[
\tilde{J}_Y(V; m, z) = \sum_{k=0}^{m-1} z^{-1-k} \theta_k |V|^k, \quad z > 0.
\]
So \( J_Y(V) = \tilde{J}_Y(V, \text{ni}(V), 1) \). Due to Lemma 3.2.4 from [2], \((D - \lambda I)^{-1}V\) is a quasi-nilpotent operator for any \( \lambda \notin \sigma(D) \). Put

\[ v_\pm(\lambda) \equiv \text{ni}((D - \lambda I)^{-1}V) \).

Everywhere below we can replace \( v_\pm(\lambda) \) by \( \infty \).

Let \( R_\lambda(A) = (A - \lambda I)^{-1} \) be the resolvent and let

\[ \rho(\lambda, D) = \inf_{z \in \sigma(D)} |s - z| \]

be the distance between a \( \lambda \in \mathbb{C} \) and \( \sigma(D) \). So \( d_0 = \rho(0, D) \).

**Lemma 3.1.** Under conditions (1.1) and (1.2), for a \( \lambda \notin \sigma(D) \), let

\[
\xi(A, \lambda) \equiv \max \left\{ \frac{1}{J_Y(V, \nu_-(\lambda), \rho(\lambda, D))} - \|V_+\|, \frac{1}{\tilde{J}_Y(V, \nu_+(\lambda), \rho(\lambda, D))} - \|V_-\| \right\} > 0. \tag{3.1}
\]

Then \( \lambda \) is a regular point of the operator \( A \) represented by (1.4). Moreover,

\[ \|R_\lambda(A)\| \leq \frac{1}{\xi(A, \lambda)\rho(\lambda, D)}. \tag{3.2} \]

**Proof.** Since \( D \) is a normal operator, \( \|(D - \lambda I)^{-1}\| = \rho^{-1}(\lambda, D) \). Thus

\[ |(D - \lambda I)^{-1}V_\pm|_Y \leq \|(D - \lambda I)^{-1}\| |V_\pm|_Y = \rho^{-1}(\lambda, D)|V_\pm|_Y \]

Hence

\[
\| (D - \lambda I)^{-1}V_+ \| J_Y((D - \lambda I)^{-1}V_-) \\
\leq \|V_+\| \rho^{-1}(\lambda, D) \sum_{k=0}^{\nu_-(\lambda)-1} \theta_k |(D - \lambda I)^{-1}V_-|^k_Y \\
\leq \|V_+\| \sum_{k=0}^{\nu_-(\lambda)-1} \theta_k \rho^{-1-k}(\lambda, D)|V_-|^k_Y = \|V_+\| \|\tilde{J}_Y(V, \nu_-(\lambda), \rho(\lambda, D))\|.
\]

Similarly,

\[
\| (D - \lambda I)^{-1}V_- \| J_Y((D - \lambda I)^{-1}V_+) \leq \|V_-\| \|\tilde{J}_Y(V, \nu_+(\lambda), \rho(\lambda, D))\|.
\]

Now Theorem 1.1 with \( A - \lambda I = D + V_+ + V_- \lambda I \) instead of \( A \) yields the required result. \( \text{Q.E.D.} \)

Lemma 3.1 implies the following result.
COROLLARY 3.2. Under conditions (1.1), (1.2), and (1.4), for any \( \mu \in \sigma(A) \), there is a \( \mu_0 \in \sigma(D) \), such that either \( \mu = \mu_0 \) or both inequalities
\[
\|V_+\| \tilde{J}_Y(V_-, \nu_-(\mu), |\mu - \mu_0|) \geq 1,
\]
\[
\|V_-\| \tilde{J}_Y(V_+, \nu_+(\mu), |\mu - \mu_0|) \geq 1
\]
(3.3)
are true.

This result is exact in the following sense: If either \( V_- = 0 \) or (and) \( V_+ = 0 \), then, due to the latter corollary,
\[
\sigma(A) = \sigma(D).
\]
(3.4)

Now put
\[
\tilde{\nu}_\pm = \sup_{\lambda \in \sigma(D)} \nu_\pm(\lambda) = \sup_{\lambda \in \sigma(D)} \text{ni}((D - \lambda I)^{-1}V_\pm).
\]

In the remainder of this paper, one can replace \( \tilde{\nu}_\pm \) by \( \infty \).

THEOREM 3.3. Under conditions (1.1), (1.2), and (1.4), let \( V_+ \neq 0 \), \( V_- \neq 0 \). Then each of the following equations:
\[
\|V_+\| \tilde{J}_Y(V_-, \tilde{\nu}_-, z) = 1 \quad \text{and} \quad \|V_-\| \tilde{J}_Y(V_+, \tilde{\nu}_+, z) = 1
\]
(3.5)
has a unique positive root \( z_{\text{up}}(Y) \) and \( z_{\text{down}}(Y) \), respectively. Moreover, for any \( \mu \in \sigma(A) \), there is a \( \mu_0 \in \sigma(D) \), such that
\[
|\mu - \mu_0| \leq \min\{z_{\text{up}}(Y), z_{\text{down}}(Y)\}.
\]

Proof. Comparing equations (3.5) with inequalities (3.3), we arrive at the result. Q.E.D.

To estimate \( z_{\text{up}}(Y) \) and \( z_{\text{down}}(Y) \), consider the equation
\[
\sum_{k=1}^{\infty} a_k z^k = 1,
\]
(3.6)
where the coefficients \( a_k \) are nonnegative and have the property
\[
\gamma_0 = 2 \max_k \sqrt[2]{a_k} < \infty.
\]

LEMMA 3.4. The unique nonnegative root \( z_0 \) of (3.6) satisfies the estimate
\[
z_0 \geq 1/\gamma_0.
\]
Proof. In (3.6) set \( z = x \gamma^{-1} \). We have

\[
1 = \sum_{k=1}^\infty a_k \gamma^{-k} x^k. 
\]

But

\[
\sum_{k=1}^\infty a_k \gamma^{-k} \leq \sum_{k=1}^\infty 2^{-k} = 1,
\]

and therefore the unique positive root \( x_0 \) of (3.7) satisfies the inequality \( x_0 \geq 1 \). Hence \( z_0 = \gamma^{-1} x_0 \geq \gamma^{-1} \), as claimed. Q.E.D.

Note that Lemma 3.4 generalizes the well-known result for algebraic equations; cf. [9, p. 277].

Lemma 3.4 gives us the inequalities

\[
z_{\text{up}}(Y) \leq 2 \max_{j=1,2,\ldots} \sqrt{\theta_{j-1} |V_{-Y}^{j-1}| V_+}
\]

and

\[
z_{\text{down}}(Y) \leq 2 \max_{j=1,2,\ldots} \sqrt{\theta_{j-1} |V_{+Y}^{j-1}| V_-}
\]

with \( \theta_0 = 1 \).

Now Theorem 3.3 implies the following result.

**Corollary 3.5.** Under conditions (1.1), (1.2), and (1.4), for any \( \mu \in \sigma(A) \), there is a \( \mu_0 \in \sigma(D) \), such that

\[
|\mu - \mu_0| \leq \psi_Y(A) \equiv 2 \min \left\{ \max_{j=1,2,\ldots} \sqrt{\theta_{j-1} |V_{-Y}^{j-1}| V_+}, \right. \\
\left. \max_{j=1,2,\ldots} \sqrt{\theta_{j-1} |V_{+Y}^{j-1}| V_-} \right\}. \tag{3.8}
\]

Theorem 3.3 and Corollary 3.5 yield the following.

**Corollary 3.6.** Under conditions (1.1) and (1.2), let \( D \) be bounded. Then the spectral radius of the operator \( A \) represented by (1.4) satisfies the inequalities

\[
r_s(A) \leq \min \{ z_{\text{up}}(Y), z_{\text{down}}(Y) \} + r_s(D) \leq \psi_Y(A) + r_s(D).
\]

For a linear operator \( A \), put

\[
\alpha(A) \equiv \sup \Re \sigma(A).
\]

We will say that a linear operator \( A \) is stable if \( \alpha(A) \leq 0 \).

**Corollary 3.7.** Under conditions (1.1) and (1.2), the operator \( A \) represented by (1.4) is stable provided \( \alpha(D) + \psi_Y(A) \leq 0 \).
4. OPERATORS WITH HILBERT–SCHMIDT NILPOTENT PARTS

4.1. Invertibility Conditions

Let $C_2$ be the ideal of the Hilbert–Schmidt operators (HSO’s) with the Hilbert–Schmidt norm

$$N_1(K) \equiv [\text{Trace } K^*K]^{1/2}, \quad K \in C_2.$$ 

Throughout this section it is assumed that

$$V_\pm \in C_2.$$ 

(4.1)

Let $D$ be boundedly invertible. Then, under (4.1), $W_\pm = D^{-1}V_\pm$ are HSO’s, as well. For a Volterra HSO $V$, put

$$J_2(V) = \sum_{k=0}^{m(V)-1} \frac{N_1^k(V)}{\sqrt{k!}}.$$ 

(4.2)

**Theorem 4.1.** Let relations (1.1) with $Y = C_2$ and (1.5) hold. In addition, let

$$\tilde{\zeta}_H(A) \equiv \max\{J_2^{-1}(W_+) - \|W_+\|, J_2^{-1}(W_-) - \|W_-\|\} > 0.$$ 

(4.3)

Then the operator $A$ represented by (1.4) is boundedly invertible. Moreover, the inverse operator satisfies the inequality

$$\|A^{-1}\| \leq \frac{1}{d_{0\tilde{\zeta}_H}(A)}.$$ 

(4.4)

**Proof.** Due to Lemma 2.3.1 from [2], we have

$$\|V^j\| \leq \frac{N_1^j(V)}{\sqrt{j!}}, \quad j = 1, 2, \ldots,$$ 

(4.5)

for any Volterra HSO $V$. Now the result follows from Theorem 1.1. Q.E.D.

Furthermore, for a constant $a > 0$, the Schwarz inequality implies

$$\left(\sum_{k=0}^{\infty} \frac{a^k}{\sqrt{k!}}\right)^2 = \left(\sum_{k=0}^{\infty} \frac{2^{k/2}a^k}{2^{k/2}\sqrt{k!}}\right)^2 \leq \sum_{j=0}^{\infty} 2^{-j} \sum_{k=0}^{\infty} \frac{2^k a^{2k}}{k!} = 2 \exp[2a^2].$$

Hence it follows that

$$J_2(V) \leq \sqrt{2} e^{N_1^2(V)}$$ 

(4.6)

for an arbitrary Volterra HSO $V$. Due to relation (4.6), the previous theorem yields the following result.
Corollary 4.2. Let relations (1.1) with \( Y = C_2 \) and (1.5) hold. In addition, let at least one of the inequalities
\[
\sqrt{2} \|W_+\| e^{N_1^2(W_-)} < 1
\] (4.7)
and
\[
\sqrt{2} \|W_-\| e^{N_1^2(W_+)} < 1
\] (4.8)
be valid. Then the operator \( A \) represented by (1.4) is boundedly invertible. Moreover,
\[
\|A^{-1}\| \leq \frac{\sqrt{2} e^{N_1^2(W_-)}}{(1 - \sqrt{2}\|W_+\| e^{N_1^2(W_-)})d_0}
\]
provided (4.7) holds and
\[
\|A^{-1}\| \leq \frac{\sqrt{2} e^{N_1^2(W_+)} }{(1 - \sqrt{2}\|W_-\| e^{N_1^2(W_+)})d_0}
\]
provided (4.8) holds.

4.2. Localization of the Spectrum

For a Volterra HSO \( V \), put
\[
\tilde{J}_2(V, m, z) = \sum_{k=0}^{m} \frac{N_1^k(V)}{z^{k+1} \sqrt{k!}}, \quad z > 0.
\]
So \( J_2(V) = \tilde{J}_2(V, n_i(V), 1) \). Recall that \( \nu_{\pm}(\lambda) = ni((D - \lambda)^{-1}V_{\pm}) \).

Lemma 4.3. Under condition (1.1) with \( Y = C_2 \), for a \( \lambda \notin \sigma(D) \), let
\[
\xi_H(A, \lambda) \equiv \max \left\{ \frac{1}{J_Y(V_-, \nu_-(\lambda), \rho(\lambda, D))} - \|V_+\|, \frac{1}{J_Y(V_+, \nu_+(\lambda), \rho(\lambda, D))} - \|V_-\| \right\} > 0.
\] (4.9)

Then \( \lambda \) is a regular point of the operator \( A \) represented by (1.4). Moreover,
\[
\|R_\lambda(A)\| \leq \frac{1}{\xi_H(A, \lambda) \rho(\lambda, D)}.
\] (4.10)

This result is due to Lemma 3.1 and estimate (4.5). Lemma 4.3 implies the following result.
Corollary 4.4. Under conditions (1.1) with \( Y = C_2 \) and (1.4), for any \( \mu \in \sigma(A) \), there is a \( \mu_0 \in \sigma(D) \), such that either \( \mu = \mu_0 \) or both inequalities
\[
\| V_+ \| J_2^t(V_-, \nu_-(\mu), |\mu - \mu_0|) \geq 1 \quad \text{and} \quad \| V_- \| J_2^t(V_+, \nu_+(\mu), |\mu - \mu_0|) \geq 1
\]
are true.

Theorem 3.3 and relation (4.5) yield the following result.

Theorem 4.5. Under conditions (1.1) with \( Y = C_2 \), let \( V_+ \neq 0 \), \( V_- \neq 0 \).

Then each of the equations
\[
\| V_+ \| J_2^t(V_-, \tilde{\nu}_-, z) = 1 \quad \text{and} \quad \| V_- \| J_2^t(V_+, \tilde{\nu}_+, z) = 1
\]
(4.11)
has a unique positive root \( z_{\text{up}}(\text{HS}) \) and \( z_{\text{down}}(\text{HS}) \), respectively. Moreover, for any points of the spectrum of the operator \( A \) represented by (1.4), there is a \( \mu_0 \in \sigma(D) \), such that
\[
|\mu - \mu_0| \leq \min\{z_{\text{up}}(\text{HS}), z_{\text{down}}(\text{HS})\}.
\]

Clearly, in Lemma 4.1, Corollary 4.2, and Theorem 4.3 we can also replace \( \nu_\pm(\lambda) \) and \( \tilde{\nu}_\pm \) by \( \infty \).

According to Lemma 3.4,
\[
z_{\text{up}}(\text{HS}) \leq 2 \max_{j=1, 2, \ldots} \frac{\| V_+ \| N_1^{j-1}(V_-)}{\sqrt{(j-1)!}}
\]
(4.12)
\[
z_{\text{down}}(\text{HS}) \leq 2 \max_{j=1, 2, \ldots} \frac{\| V_- \| N_1^{j-1}(V_+)}{\sqrt{(j-1)!}}
\]

We need the following simple result.

Lemma 4.6. The unique positive root \( z_0 \) of the equation
\[
ze^{z_0} = a, \quad a = \text{const} > 0,
\]
(4.13)
satisfies the estimate
\[
z_0 \geq \ln\left[ \frac{1}{2} + \sqrt{\frac{1}{4} + a} \right].
\]
(4.14)
If, in addition, the condition \( a \geq e \) holds, then
\[
z_0 \geq \ln a - \ln \ln a.
\]
(4.15)
Proof. Since \( z \leq e^z - 1 \) (\( z \geq 0 \)), we arrive at the relation \( a \leq e^{z_0} - e^{z_0} \). Hence \( e^{z_0} \geq r_{1,2} \), where \( r_{1,2} \) are the roots of the polynomial \( y^2 - y - a \). This proves inequality (4.14).

Furthermore, if the condition \( a \geq e \) holds, then \( z_0 e_0^z \geq e \) and \( z_0 \geq 1 \). Now (4.13) yields \( e^{z_0} \leq a \) and \( z_0 \leq \ln a \). So

\[
a = z_0 e^{z_0} \leq e^{z_0} \ln a.
\]

Hence inequality (4.15) follows. Q.E.D.

According to (4.6) and (4.11), we have \( z_{up}(HS) \leq \tilde{z}_{up} \) and \( z_{down}(HS) \leq \tilde{z}_{down} \), where \( \tilde{z}_{up} \) and \( \tilde{z}_{down} \) are unique positive roots of

\[
\sqrt{2} \Vert V_+ \Vert z^{-1} \exp\{z^{-2}N_1^2(V_-)\} = 1
\]

and

\[
\sqrt{2} \Vert V_- \Vert z^{-1} \exp\{z^{-2}N_1^2(V_+)\} = 1,
\]

respectively. Clearly, (4.16) is equivalent to the equation

\[
2 \Vert V_+ \Vert^2 z^{-2} \exp\{2z^{-2}N_1^2(V_-)\} = 1.
\]

Denote

\[
a_{up} \equiv \frac{N_1^2(V_-)}{\Vert V_+ \Vert^2} \quad \text{and} \quad a_{low} \equiv \frac{N_1^2(V_+)}{\Vert V_- \Vert^2}.
\]

(4.17)

Substitute \( z^2 = 2N_1^2(V_-)x^{-1} \). Then we have \( xe^x = a_{up} \). Now Lemma 4.6 implies

\[
\tilde{z}_{up}^2 \leq \frac{2N_1^2(V_-)}{\ln\{1/2 + \sqrt{1/4 + a_{up}}\}}.
\]

Similarly,

\[
\tilde{z}_{down}^2 \leq \frac{2N_1^2(V_+)}{\ln\{1/2 + \sqrt{1/4 + a_{low}}\}}.
\]

Denote

\[
\delta(A) \equiv \sqrt{2} \min\left\{ \frac{N_1(V_-)}{\ln^{1/2}\{1/2 + \sqrt{1/4 + a_{up}}\}}, \frac{N_1(V_+)}{\ln^{1/2}\{1/2 + \sqrt{1/4 + a_{low}}\}} \right\}.
\]
That is,
\[
\delta(A) \equiv \sqrt{2} \min \left\{ \frac{N_1(V_-)}{\ln^{1/2} \left[ 1/2 + \sqrt{1/4 + \|V_\|^{-2}N_1^2(V_-)} \right]}, \frac{N_1(V_+)}{\ln^{1/2} \left[ 1/2 + \sqrt{1/4 + \|V_\|^{-2}N_1^2(V_+)} \right]} \right\},
\]
(4.18)

Then
\[
\min \{z_{up}(HS), z_{down}(HS)\} \leq \min \{\hat{z}_{up}, \hat{z}_{down}\} \leq \delta(A).
\]

Clearly, \(\delta(A) \to 0\) if either \(N_1(V_-) \to 0\) or \(N_1(V_+) \to 0\).

Furthermore, Theorem 4.5 implies the following.

**Corollary 4.7.** Under conditions (1.1) with \(Y = C_2\) and (1.4), for any \(\mu \in \sigma(A)\), there is a \(\mu_0 \in \sigma(D)\), such that \(|\mu - \mu_0| \leq \delta(A)\).

So the operator \(A\) is stable, provided \(\alpha(D) + \delta(A) \leq 0\).

If, in addition, \(D\) is bounded, then \(r_s(A) \leq \delta(A) + r_s(D)\).

Furthermore, if \(a_{up} \geq e\), then, due to Lemma 4.6,
\[
\hat{z}_{up}^2 \leq \frac{2N_1^2(V_-)}{\ln a_{up} - \ln \ln a_{up}}.
\]
Similarly, if \(a_{low} \geq e\), then
\[
\hat{z}_{down}^2 \leq \frac{2N_1^2(V_+)}{\ln a_{low} - \ln \ln a_{low}}.
\]

Put
\[
\Delta(A) \equiv \sqrt{2} \min \left\{ \frac{N_1(V_-)}{[\ln a_{up} - \ln \ln a_{up}]^{1/2}}, \frac{N_1(V_+)}{[\ln a_{low} - \ln \ln a_{low}]^{1/2}} \right\},
\]
(4.19)

Then, under the condition
\[
\min \{a_{up}, a_{low}\} \geq e,
\]
(4.20)
we have
\[
\min \{z_{up}(HS), z_{down}(HS)\} \leq \min \{\hat{z}_{up}, \hat{z}_{down}\} \leq \Delta(A).
\]

Now Theorem 4.5 implies the following result.

**Corollary 4.8.** Under conditions (1.1) with \(Y = C_2\), (1.4), and (4.20), for any \(\mu \in \sigma(A)\), there is a \(\mu_0 \in \sigma(D)\), such that \(|\mu - \mu_0| \leq \Delta(A)\).

So the operator \(A\) is stable, provided \(\alpha(D) + \Delta(A) \leq 0\).

If, in addition, \(D\) is bounded, then \(r_s(A) \leq \Delta(A) + r_s(D)\).
5. OPERATORS WITH VON NEUMANN–SCHATTEN NILPOTENT PARTS

In this section it is assumed that \( V \pm \) belong to the von Neumann–Schatten ideal \( C_{2p} \) with some integer \( p > 1 \). That is,

\[
N_p(V_{\pm}) = \left[ \text{Trace}(V_{\pm}^* V_{\pm})^p \right]^{1/2p} < \infty.
\] (5.1)

Denote

\[
\theta_k^{(p)} = 1, \ k < 2p, \quad \text{and} \quad \theta_k^{(p)} = \frac{1}{\sqrt{\Gamma(k/p)}}, \ k \geq 2p,
\]

where \( \Gamma(\cdot) \) is the Euler Gamma function.

For a Volterra operator \( V \in C_{2p} \), put

\[
J_{2p}(V) = \sum_{k=0}^{n(V)-1} \theta_k^{(p)} N_k^p(V).
\]

**Lemma 5.1.** Let relations (1.1) and (1.5) hold with \( Y = C_{2p} \). In addition, let

\[
\tilde{\zeta}_{2p}(A) = \max \left\{ J_{2p}^{-1}(W_-) - \|W_+\|, J_{2p}^{-1}(W_+) - \|W_-\| \right\} > 0.
\] (5.2)

Then the operator \( A \) represented by (1.4) is boundedly invertible. Moreover,

\[
\|A^{-1}\| \leq \frac{1}{\tilde{\zeta}_{2p}(A) \eta(D)}.
\] (5.3)

**Proof.** For any Volterra operator \( V \in C_{2p} \), we have \( V^p \in C_2 \). According to (4.5),

\[
\|V^p\| \leq \frac{N_p^j(V^p)}{\sqrt{j!}} \leq \frac{N_p^p(V)}{\sqrt{j!}}, \quad V \in C_{2p}, \quad j = 1, 2, \ldots.
\]

Hence, for any \( k = m + jp(m = 0, \ldots, p-1; j = 0, 1, 2, \ldots) \), we have

\[
\|V^k\| \leq \frac{\|V^m\| N^j(V^p)}{\sqrt{j!}} \leq \frac{N^{m+jp}(V)}{\sqrt{j!}}.
\] (5.4)

But \( j = (k-m)/p \) and

\[
J! = [(k-m)/p]! \geq [k/p - 1]! = \Gamma(k/p), \quad m < p.
\]

Hence

\[
\|V^k\| \leq \theta_k^{(p)} N^k_p(V), \quad k = 1, 2, \ldots.
\] (5.5)

Now the required result is due to Theorem 1.1. Q.E.D.
Put
\[ I_p(K) = \sum_{j=0}^{p-1} \sum_{k=1}^{\infty} \frac{N_p^{j+p}(K)}{\sqrt{k!}} \]
for an arbitrary operator \( K \in C_{2p} \). According to (5.4), one can replace \( J_{2p}(V_{\pm}) \) in (5.2) by \( I_p(V_{\pm}) \).

For a Volterra \( V \in C_{2p}(p > 1) \), put
\[ J_2p(V, m, z) = \sum_{k=0}^{m-1} \frac{\theta_k^{(p)} N_p^k(V)}{z^{k+1}}, \quad z > 0. \]

So \( J_2p(V) = J_2p(V, \text{ni}(V), 1) \). Recall that \( \nu_\pm(\lambda) \equiv \text{ni}((D - \lambda)^{-1}V_\pm) \leq \infty \).

**Theorem 5.2.** Under condition (1.1) with \( Y = C_{2p} \), for a \( \lambda \notin \sigma(D) \), let
\[ \xi(\lambda, A) \equiv \max \left\{ \frac{1}{J_2p(V_-, \nu_-(\lambda), \rho(\lambda, D))} - \| V_+ \|, \frac{1}{J_2p(V_+, \nu_+(\lambda), \rho(\lambda, D))} - \| V_- \| \right\} > 0. \]

Then \( \lambda \) is a regular point of the operator \( A \) represented by (1.4). Moreover,
\[ \| R_\lambda(A) \| \leq \frac{1}{\rho(\lambda, D) \xi(\lambda, A)}. \]

Theorem 3.3 and relation (5.5) yield the following result.

**Theorem 5.3.** Under conditions (1.1) with \( Y = C_{2p} \) and (1.4), let \( V_+ \neq 0 \) and \( V_- \neq 0 \). Then each of the equations
\[ \| V_+ \| J_2p(V_-, \nu_-, z) = 1 \quad \text{and} \quad \| V_- \| J_2p(V_+, \nu_+, z) = 1 \]
has a unique positive root \( z_{\text{up}}(C_{2p}) \) and \( z_{\text{down}}(C_{2p}) \), respectively. Moreover, for any \( \mu \in \sigma(A) \), there is a \( \mu_0 \in \sigma(D) \), such that
\[ |\mu - \mu_0| \leq \min\{z_{\text{up}}(C_{2p}), z_{\text{down}}(C_{2p})\}. \]

To estimate \( z_{\text{up}}(C_{2p}) \) and \( z_{\text{down}}(C_{2p}) \), we can apply Lemma 3.4. Note that, according to (5.4), one can replace (5.6) by
\[ \| V_+ \| \sum_{j=0}^{p-1} \sum_{k=1}^{\infty} \frac{N_p^{j+p}(V_-)}{z^{j+p}k+1 \sqrt{k!}} = 1 \quad \text{and} \quad \| V_- \| \sum_{j=0}^{p-1} \sum_{k=1}^{\infty} \frac{N_p^{j+p}(V_+)}{z^{j+p}k+1 \sqrt{k!}} = 1. \]
Due to Lemma 3.4 and the previous theorem under conditions (1.1) with \(Y = C_{2p}\) and (3.1), for any \(\mu \in \sigma(A)\), there is a \(\mu_0 \in \sigma(D)\), such that 
\(|\mu - \mu_0| \leq \psi_{2p}(A)\), where

\[
\psi_{2p}(A) = 2 \min \left\{ \max_{j=1,2,...,p-1; k=0,1,...} \sqrt{\frac{\|V_+\|N^{j+k,p}(V_-)}{\sqrt{k!}}} \right\},
\]

where \(\psi_{2p}(A)\) is a real bounded measurable scalar-valued function. The real kernel \(K\) satisfies the following condition: Its \(p\) iteration is a Hilbert–Schmidt kernel. So condition (5.1) holds.

6. EXAMPLES

Let us consider briefly some operators which can be represented by (1.4) under condition (1.1).

6.1. Integral Operators

Consider an integral operator \(A\) defined in \(H = L^2[0, 1]\) by

\[
(Au)(x) = a(x)u(x) + \int_0^1 K(x, s)u(s) \, ds, \quad u \in L^2[0, 1], \ x \in [0, 1], \ (6.1)
\]

where \(a(\cdot)\) is a real bounded measurable scalar-valued function. The real kernel \(K\) satisfies the following condition: Its \(p\) iteration is a Hilbert–Schmidt kernel. So condition (5.1) holds.

For \(0 \leq t \leq 1\), define \(P(t)\) by

\[
(P(t)u)(x) = 0 \quad \text{for} \ t < x \leq 1 \quad \text{and} \quad (P(t)u)(x) = u(x) \quad \text{for} \ 0 \leq x < t.
\]

In addition, put

\[
P(t) = 1 \quad \text{for} \ t > 1 \quad \text{and} \quad P(t) = 0 \quad \text{for} \ t < 0.
\]

Then, in this case, conditions (1.1) and (1.4) are valid with \((Du)(x) = a(x)u(x)\),

\[
(V_+ u)(x) = \int_0^1 K(x, s)u(s) \, ds,
\]

\[
(V_- u)(x) = \int_0^x K(x, s)u(s) \, ds, \quad u \in L^2[0, 1], \ x \in [0, 1].
\]

Now we can directly apply the results of the previous section. Clearly, these results allow us to consider also operator (6.1) with an unbounded function \(a(x)\). For example, one can take \(a(x) = x^{-1}\). In this case,

\[
\Dom(D) = \{ h \in L^2[0, 1] : x^{-1} h(x) \in L^2[0, 1] \}. 
\]
For instance, let $K$ be a real Hilbert–Schmidt kernel. Then

$$N^2_1(V_+) = \int_0^1 \int_x^1 K^2(x, s) \, ds \, dx, \quad N^2_1(V_-) = \int_0^1 \int_x^1 K^2(x, s) \, ds \, dx.$$ 

Due to Corollary 4.7, the spectrum of the operator (6.1) is included in the set

$$\{ z \in C : |a(x) - z| \leq \delta(A), \ 0 \leq x \leq 1 \},$$

where $\delta(A)$ is defined by (4.18). So the operator (6.1) is stable, provided

$$a(x) + \delta(A) \leq 0, \quad 0 \leq x \leq 1.$$ 

In particular, if $a(x) \equiv 0$, then

$$r_s(A) \leq \delta(A). \quad (6.2)$$

Inequality (6.2) in the case of the operator (6.1) with $a(x) \equiv 0$ improves the well-known estimate

$$r_s(A) \leq \tilde{\delta}_0(A) \equiv \sup_x \int_0^1 |K(x, s)| \, ds \quad (6.3)$$

from [8, Theorem 16.2] if $\tilde{\delta}_0(A) > \delta(A)$. That is, (6.2) improves the estimate (6.3) for operators which are “close” to Volterra ones, since $\delta(A) \to 0$ if either $N_1(V_-) \to 0$ or $N_1(V_+) \to 0$.

We can also apply Corollary 4.8 to the operator (6.1). For other bounds for the spectrum of integral operators, see also [3, 4].

### 6.2. Matrix Operators

Let $\{e_k\}_{k=1}^\infty$ be an orthogonal normed basis in $H$. Let $A$ be a linear operator in $H$ represented by a matrix with the entries

$$a_{jk} = (Ae_k, e_j), \quad j, k = 1, 2, \ldots, \quad (6.4)$$

where $(\cdot, \cdot)$ is the scalar product. Take $P(t) = \{P_k\}_{k=1}^\infty$, where $P_k$ are defined by

$$P_k = \sum_{j=1}^k (e_j, e_j) e_j.$$ 

In this case, $V_-, V_+$, and $D$ are the upper triangular, lower triangular, and diagonal parts of $A$, respectively:

$$(V_+ e_k, e_j) = a_{jk} \quad \text{for} \ j < k, \quad (V_+ e_k, e_j) = 0 \quad \text{for all} \ j > k, \quad (6.5)$$

$$(V_- e_k, e_j) = a_{jk} \quad \text{for} \ j > k, \quad (V_- e_k, e_j) = 0 \quad \text{for all} \ j < k,$$

$$(De_k, e_j) = a_{kk}, \quad (De_k, e_j) = 0 \quad \text{for} \ j \neq k, \ j, k = 1, 2, \ldots.$$
Clearly, condition (1.1) holds. In addition, if $D$ is unbounded, then

$$\text{Dom}(D) = \left\{ h = (h_k) \in H : \sum_{k=1}^{\infty} |a_{kk}h_k|^2 < \infty, \right.\left. h_k = (h, e_k), \ k = 1, 2, \ldots \right\}.$$  

Again, it is assumed that $V_\pm$ are compact. Due to Lemma 3.2.2 from [2], they are Volterra operators. So we can directly apply the previously derived results to matrix operators.

For example, let $V_\pm$ be Hilbert–Schmidt matrices. That is,

$$N_1^2(V_-) = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |a_{jk}|^2 < \infty, \quad N_1^2(V_+) = \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} |a_{jk}|^2 < \infty.$$  

Due to Corollary 4.7, the spectrum of the operator (6.4) is included in the set

$$\{ z \in \mathbb{C} : |a_{kk} - z| \leq \delta(A), \ k = 1, 2, \ldots, \}, \quad (6.6)$$

where $\delta(A)$ is defined by (4.18). So the considered matrix operator is stable, provided

$$\text{Re} a_{kk} + \delta(A) \leq 0, \quad k = 1, 2, \ldots.$$  

In particular, if $\|D\| = \sup_k |a_{kk}| < \infty$, then

$$r_s(A) \leq \sup_k |a_{kk}| + \delta(A). \quad (6.7)$$

Inequality (6.7) improves the well-known estimate

$$r_s(A) \leq \delta_1(A) \equiv \sup_j \sum_{k=1}^{\infty} |a_{jk}| \quad (6.8)$$

[8, inequality (16.2)], provided

$$\delta_1(A) > \sup_k |a_{kk}| + \delta(A).$$

That is, (6.7) improves the estimate (6.8) for matrices which are “close” to triangular ones.

Similarly, we can consider infinite matrices with off-diagonals belonging to a von Neumann–Schatten ideal $C_{2p}$ with $p > 1$. 

6.3. An Integro-differential Operator

In space $H = L^2[0, 1]$, let us consider the operator

$$(Au)(x) = \frac{d^2u(x)}{dx^2} + \int_0^1 K(x, s)u(s) \, ds,$$

where $K$ is a scalar Hilbert–Schmidt kernel defined on $[0, 1]^2$. The Dirichlet boundary conditions $u(0) = u(1) = 0$ are imposed. In addition,

$$\text{Dom}(A) = \{h \in L^2[0, 1] : h'' \in L^2[0, 1], \ h(0) = h(1) = 0\}.$$ 

With the orthogonal normed basis

$$e_k(x) = \sin \pi k x, \quad k = 1, 2, \ldots,$$

let

$$K(x, s) = \sum_{j,k=0}^{\infty} b_{jk} e_k(x) e_j(s),$$

be the Fourier expansions of $K$. Obviously,

$$a_{j,k} \equiv (Ae_k, e_j) = b_{j,k}, \quad j \neq k,$$

$$a_{k,k} \equiv (Ae_k, e_k) = -\pi^2 k^2 + b_{k,k}, \quad j, k = 1, 2, \ldots.$$ 

(6.10)

Define operators $D$ and $V_\pm$ according to (6.5). Since $K$ is a Hilbert–Schmidt kernel, $V_\pm$ are HSO’s. Thus, with the notation (6.10), the spectrum of the operator $A$ defined by relation (6.9) lies in the set (6.6). So the considered integro-differential operator is stable, provided

$$-\pi^2 k^2 + \text{Re} \ b_{k,k} + \delta(A) \leq 0, \quad k = 1, 2, \ldots.$$ 

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