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Ricci-corrected derivatives and invariant differential operators ☆

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Abstract

We introduce the notion of Ricci-corrected differentiation in parabolic geometry, which is a modification of covariant differentiation with better transformation properties. This enables us to simplify the explicit formulae for standard invariant operators given in [A. Čap, J. Slovák, V. Souček, Invariant operators on manifolds with almost hermitian symmetric structures, III. Standard operators, Differential Geom. Appl. 12 (2000) 51–84], and at the same time extend these formulae from the context of AHS structures (which include conformal and projective structures) to the more general class of all parabolic structures (including CR structures).

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Introduction

A fundamental part of differential geometry is the study of differential invariants of geometric structures. Our concern in this paper is the explicit construction of such invariants. More specifically, we wish to construct invariant differential operators for a class of structures known as *parabolic geometries*. These geometries have attracted attention in recent years for at least two reasons: first, they include examples of long-standing interest in differential geometry, such as conformal structures, projective structures and CR structures; second, they have a rich algebraic theory, due to their intimate relation with the representation theory of parabolic subgroups P of semisimple Lie groups G.

A great deal of progress in our understanding of invariant differential operators in parabolic geometry has been made through the efforts of many people. The key idea, pioneered by Eastwood and Rice [19] and Baston [2] is that (generalized) Bernstein–Gelfand–Gelfand (BGG) complexes of parabolic Verma module homomorphisms are dual to complexes of invariant linear differential operators on generalized flag varieties G/P, and these complexes should admit 'curved analogues', that is, there should exist sequences of invariant linear differential operators on curved manifolds modelled on these homogeneous spaces. This is now known to hold for all regular BGG complexes [16].

The prototypical parabolic geometry is conformal geometry, whose model is the generalized flag variety $\mathrm{SO}_0(n+1,1)/\mathrm{CO}(n) \ltimes \mathbb{R}^{n*}$, namely the n-sphere S^n with its standard flat conformal structure. In this case another, more explicit, approach to the study of differential invariants has been fruitfully applied: one first chooses a representative Riemannian metric (or, more generally, a compatible Weyl structure); then one invokes the well-known results of invariant theory in Riemannian (or Weyl) geometry; finally one studies how the differential invariants depend on the choice of Riemannian metric (or Weyl structure [25,33]). The differential invariants which are independent of the choice are invariants of the conformal structure. The advantage of these methods is that they give explicit formulae for the differential conformal invariants in familiar Riemannian terms. They have been particularly successful in the construction of first and second order linear differential operators [6,21,23], but have been extended to higher order operators and certain other parabolic structures (variously known as the $|1|$-graded, abelian or AHS structures) in [13–15]. In the first order case, the approach of Fegan [21] has also been extended to arbitrary parabolic geometries [31].

In this paper we build upon these explicit constructions of invariant differential operators in terms of a compatible Weyl structure. The natural geometries for such constructions are parabolic geometries, because we have a good notion of Weyl structure [12], similar to the conformal case, and we can be sure than many invariant operators exist because we have the BGG sequences [16]. Our results are three-fold: first, we simplify the formulae for standard linear differential operators given in [15]; second, we extend these formulae to arbitrary parabolic structures; third, we uncover a fundamental object in parabolic geometry, the Weyl jet operator, and its components, the Ricci-corrected Weyl derivatives of the title of our paper (cf. [33]). Since it will take us a little while in the body of the text to reach these results, we shall spend some time now explaining what the Ricci-corrected Weyl derivatives are, in the case of conformal geometry.

Let M be an n-dimensional manifold with a conformal structure c. A compatible Weyl connection D on M is a torsion-free conformal connection on the tangent bundle of M. It therefore induces a connection on the conformal frame bundle of M, and hence covariant derivatives on any vector bundle V associated to the frame bundle via a representation λ of the conformal group $\mathrm{CO}(n)$ on a vector space $\mathbb{V}$.

The most familiar Weyl connections are the Levi-Civita connections of representative Riemannian metrics for c. However, the broader context of Weyl connections has a few advantages in conformal geometry:

- Weyl connections form an affine space, modelled on the space of 1-forms γ and we write $D \mapsto D+\gamma$ for this affine structure;
- the construction of the Levi-Civita connection from a Riemannian metric involves taking a derivative, so that differential invariants have one order higher in the metric than in the connection;
- a choice of Riemannian metric reduces the structure group to $\mathrm{SO}(n)$, making it easy to forget the 1-dimensional representations of the conformal group, an omission which comes back with a vengeance in the form of conformal weights.

The affine structure of the space of Weyl connections provides a straightforward formulation of the well-known folklore that conformal invariance only needs to be checked infinitesimally: we regard a differential invariant F constructed using a Weyl connection as a function $F(D)$; F is said to be a *conformal invariant* if it is independent of D; by the fundamental theorem of calculus, this amounts to checking that $\partial_\gamma F(D) = 0$ for all Weyl connections D and 1-forms γ. More generally, if $F(D)$ is polynomial in D then this dependence can be computed using Taylor's Theorem. In particular, if s is a section of an associated bundle V and X is a vector field then $\partial_\gamma D_X s = [\gamma, X] \cdot s$ where $[\gamma, X] = \gamma(X)\,\mathrm{id} + \gamma \wedge X \in \mathfrak{co}(TM) = \mathbb{R}\,\mathrm{id}_{TM} \oplus \mathfrak{so}(TM)$ and $\cdot$ denotes the natural action of $\mathfrak{co}(TM)$ on V (induced by the representation λ of $\mathfrak{co}(n)$ on $\mathbb{V}$).

The Ricci-corrected derivatives have their origins in the observation that the explicit formulae for conformally invariant differential operators in terms of a Weyl connection appear to have a systematic form, essentially depending only on the order of the operator. For first order operators, this is quite straightforward [21,23]: conformally invariant first order linear operators are all of the form $\pi \circ D : \mathrm{C}^\infty(M, V) \to \mathrm{C}^\infty(M, W)$, where V and W are associated bundles, D the covariant derivative on V induced by a Weyl connection, and π is induced by an equivariant map $\mathbb{R}^{n*} \otimes \mathbb{V} \to \mathbb{W}$. Evidently $\partial_\gamma \pi(Ds) = \pi([\gamma, \cdot] \cdot s)$, so we obtain conformally invariant operators by letting π be the projection onto the zero eigenspace of the operator $\Psi \in \mathrm{End}(T^*M \otimes V)$ defined by $\Psi(\gamma \otimes s) = [\gamma, \cdot] \cdot s$. Since these eigenvalues can be shifted, by tensoring $\mathbb{V}$ with a one dimensional representation of $\mathrm{CO}(n)$, a large number of first order operators are obtained.

Ricci corrections make their first appearance at the level of second order operators. Here one finds that many conformally invariant operators are of the form $\pi(D^2 s + r^D \otimes s)$ where π is induced by an equivariant map $\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{V} \to S^2\mathbb{R}^{n*} \otimes \mathbb{V} \to \mathbb{W}$ (the first map being symmetrization), and r^D is the *normalized Ricci curvature* of the Weyl connection, which is a covector-valued 1-form on M constructed from the curvature of the Weyl connection. For present purposes, all we need to know about r^D is its dependence on D: $\partial_\gamma r^D = -D\gamma$.

Now compare this with the variation of the second derivative:

$$\begin{aligned}\partial_\gamma D^2_{X,Y} s &= [\gamma, X] \cdot D_Y s - D_{[\gamma,X]\cdot Y} s + D_X\big([\gamma, Y] \cdot s\big) - [\gamma, D_X Y] \cdot s \\ &= [D_X\gamma, Y] \cdot s + [\gamma, X] \cdot D_Y s + [\gamma, Y] \cdot D_X s - D_{[\gamma,X]\cdot Y} s.\end{aligned}$$

This formula means that we can use the Ricci curvature to make the second derivative *algebraic* in D.

Definition. The *Ricci-corrected second derivative* on sections s of an associated bundle V is defined by $D^{(2)}_{X,Y}s = D^2_{X,Y}s + [r^D(X), Y]\cdot s$.

Hence $\partial_\gamma D^{(2)}s = \gamma *_1 Ds$ where

$$\begin{aligned}(\gamma *_1 \phi)_{X,Y} &= [\gamma, X]\cdot\phi_Y + [\gamma, Y]\cdot\phi_X - \phi_{[\gamma,X]\cdot Y}\\ &= \big([\gamma, X]\cdot\phi\big)_Y + [\gamma, Y]\cdot\phi_X.\end{aligned}$$

It is now a purely algebraic matter to find projections π such that $\pi(D^{(2)}s)$ is a conformal invariant of s. As it turns out, these projections often have the property that $\pi([r^D(\cdot), \cdot]\cdot s) = \pi(r^D \otimes s)$. The simplest example is the conformal hessian $\mathrm{sym}_0 D^{(2)}s = \mathrm{sym}_0(D^2 s + r^D s)$ where s is a section of the weight 1 line bundle L.

The same ideas apply to higher order operators: we want to write these operators as $\pi \circ D^{(k)}$ for some projections induced by an equivariant map $(\otimes^k \mathbb{R}^{n*}) \otimes \mathbb{V} \to S^k\mathbb{R}^{n*} \otimes \mathbb{V} \to \mathbb{W}$, where $D^{(k)}$ is a Ricci-corrected kth power of the Weyl connection. Again we make the observation that if $\partial_\gamma(\pi \circ D^{(k)}) = 0$ then $\pi \circ D^{(k)}$ must certainly be algebraic in D, and we can in fact arrange for $D^{(k)}$ itself to be algebraic in D.

Definition. The *Ricci-corrected powers of the Weyl connection* on an associated bundle V are defined inductively by $D^{(0)}s = s$, $D^{(1)}s = Ds$ and

$$\iota_X D^{(k+1)}s = D_X D^{(k)}s + r^D(X) *_1 D^{(k-1)}s,$$

where

$$(\gamma *_1 \phi)_{X_1,\dots,X_k} = \sum_{j=1}^{k}\big([\gamma, X_i]\cdot\phi_{X_1,\dots,X_{i-1}}\big)_{X_{i+1},\dots,X_k}.$$

A calculation shows that $\partial_\gamma D^{(k)}s = \gamma *_1 D^{(k-1)}s$.

The inductive formula can easily be summed to give

$$\begin{aligned}D^{(k)} = \sum_{\ell+m=k}\ \sum_{1\leqslant i_1<\dots<i_\ell\leqslant m} & D^{m-i_\ell}\circ\big(r^D(\cdot)*_1\big)\circ D^{i_\ell-i_{\ell-1}-1}\circ\cdots\\ &\circ\big(r^D(\cdot)*_1\big)\circ D^{i_2-i_1-1}\circ\big(r^D(\cdot)*_1\big)\circ D^{i_1-1}.\end{aligned}$$

Thus the search for explicit invariant operators reduces to an algebraic problem, and we shall find that a large class of projections π annihilating $\gamma *_1 D^{(k-1)}s$ produce universal numbers when applied to the terms of $D^{(k)}$, essentially because the action $\gamma *_1$ on $(\otimes^j T^*M)\otimes V$ for $j<k$ is closely related to the action on $(\otimes^k T^*M)\otimes V$.

This theory of Ricci-corrected derivatives in conformal geometry was developed over several years by the first two authors, and described, in part, in [18]. In the homogeneous case, i.e., on S^n, the second author explained the action $\gamma *_1$ in terms of the second order part of the action of a conformal vector field on sections of a homogeneous vector bundle [18]. It became clear however, that there was a systematic underlying principle behind these formulae, even in the curved case, which should also generalize to arbitrary parabolic geometries. More precisely, the action $\gamma *_1$ is related to part of the action of the nilradical of the parabolic subalgebra $\mathfrak{p}$ on certain *semiholonomic jet modules* $\hat{J}^k_0\mathbb{V}$ associated to a module $\mathbb{V}$ [20,30]. This action is considerably more complicated in general than it is in the conformal case, and we are forced to consider the representation theory of the entire parabolic subgroup P, not just its Levi factor P_0 as we did in the conformal case (where $P_0 = \mathrm{CO}(n)$ and $P = \mathrm{CO}(n) \ltimes \mathbb{R}^{n*}$). Hence we must work with P-modules $\mathbb{V}$, and the corresponding vector bundles V are associated to a larger principal P-bundle, which in the conformal case is the Cartan bundle with its normal Cartan connection. This development ultimately provides a simple conceptual explanation for the formulae we obtain.

The structure of the paper is as follows. We begin by defining Cartan geometries and invariant differentiation in Section 1: this is a standard way to treat parabolic geometries [8,10,30], although in practice a geometry is defined by more primitive data, which must be differentiated to obtain the Cartan connection [11]. The key point is that a Cartan connection determines semiholonomic jet operators taking values in bundles associated to semiholonomic jet modules.

In Sections 2–3 we begin the study of parabolic geometries and Weyl structures. We adopt a novel approach to Weyl structures (which we relate to the approach of Čap and Slovak [12] in Appendix A) in order to emphasise the relationship between the geometry of Weyl structures and some elementary representation theory which we exploit throughout our treatment. At the algebraic level, a Weyl structure is a lift ε of a certain ‘grading element’ ε_0 in the Levi factor $\mathfrak{p}_0$ to the parabolic Lie algebra $\mathfrak{p}$. The grading element induces a filtration of a P-module $\mathbb{V}$ and a lift ε splits this filtration, i.e., determines an isomorphism $\varepsilon_{\mathbb{V}}$ of $\mathbb{V}$ with its associated graded module $\operatorname{gr}\mathbb{V}$. Geometric Weyl structures for parabolic geometries are given simply by applying the same procedure pointwise on the underlying manifold: a geometric Weyl structure E then determines a splitting $E_V : V \to \operatorname{gr} V$ of any associated filtered P-bundle V. Now if V is a filtered P-bundle, so is its semiholonomic k-jet bundle $\hat{J}^k V$, and splitting this bundle allows us to project out Ricci corrected derivatives as components of the k-jet. This simple construction, which we present in Section 4, makes Ricci corrected differentiation easy to study from a theoretical point of view. On the other hand, it is also easily related to covariant differentiation: explicit formulae are obtained as soon as one understands the action of the nilradical of $\mathfrak{p}$ on jet modules.

In Section 5, we pave the way for the construction of invariant operators by studying special types of projections from jet modules, which have the effect of killing most of the complicated terms in the jet module action. For irreducible modules, the remaining terms of the jet module action reduce to the projection of a scalar action, which we compute using Casimirs. In Section 6 we give our construction of a large class of invariant operators, and write out the formulae for operators up to order 8. We illustrate the scope of the constructions in Section 7 and give some examples in conformal geometry.

Finally let us mention further potential applications of Ricci-corrected differentiation. Although in this paper we have applied Ricci-corrected derivatives to the construction of invariant *linear* differential operators, the same ideas can be expected to yield explicit formulae for multilinear differential operators, such as the operators of [8]. Indeed, this was our original motivation to study Ricci-corrected differentiation in conformal geometry: one approach to construct (say) bilinear differential operators is to combine terms constructed from pairs of *noninvariant* linear differential operators; to do this one needs noninvariant operators which nevertheless depend on the choice of Weyl structure in a simple way—projections of Ricci-corrected derivatives onto irreducible components have this property.

1. Invariant derivatives in Cartan geometry

Parabolic geometries are geometries modelled on a generalized flag variety G/P, i.e., G is a semisimple Lie group and P is a parabolic subgroup. A standard way to define 'curved versions' of homogeneous spaces is as Cartan geometries. In this section, we recall the basic calculus of such geometries, following [8,10,20,29,30].

Fix a Lie algebra $\mathfrak{g}$ with a Lie group P acting by automorphisms such that $\mathfrak{p}$ is a P-equivariant subalgebra of $\mathfrak{g}$, and the derivative of the P-action on $\mathfrak{g}$ is the adjoint action of $\mathfrak{p}$ on $\mathfrak{g}$. (These technical conditions are simply those that arise when P is a subgroup of a Lie group G with Lie algebra $\mathfrak{g}$.)

Definition 1.1. Let M be a manifold with dimension $\dim M = \dim \mathfrak{g} - \dim \mathfrak{p}$. A *Cartan connection* of type $(\mathfrak{g}, P)$ on M is a principal P-bundle $\pi : \mathcal{G} \to M$, together with a P-invariant $\mathfrak{g}$-valued 1-form $\theta : T\mathcal{G} \to \mathfrak{g}$ such that for each $y \in \mathcal{G}$, $\theta_y : T_y\mathcal{G} \to \mathfrak{g}$ is an isomorphism which sends each generator ζ_ξ of the P-action to the corresponding $\xi \in \mathfrak{p}$, i.e., $\zeta_{\xi,y} = \theta_y^{-1}(\xi)$. Here P-invariance means that $\mathrm{Ad}(p) \cdot r_p^*\theta = \theta$ for any $p \in P$, where r_p denotes the right P-action on $\mathcal{G}$. We refer to $(M, \mathcal{G}, \theta)$ as a *Cartan geometry*.

Note that Cartan connections form an *open subset* of an affine space modelled on the space of horizontal P-invariant $\mathfrak{g}$-valued 1-forms. However, one can freely add horizontal P-invariant $\mathfrak{p}$-*valued* 1-forms to a Cartan connection, without losing invertibility.

Associated to any P-module $\mathbb{V}$ is a vector bundle $V = \mathcal{G} \times_P \mathbb{V}$, defined to be the quotient of $\mathcal{G} \times \mathbb{V}$ by the action $(y, v) \mapsto (yp^{-1}, p \cdot v)$. This induces an action $(p \cdot f)(y) = p \cdot f(yp)$ on functions $f \in \mathrm{C}^\infty(\mathcal{G}, \mathbb{V})$ which identifies sections φ of $V = \mathcal{G} \times_P \mathbb{V}$ over M with P-invariant functions $f : \mathcal{G} \to \mathbb{V}$:

$$\mathrm{C}^\infty(M, V) = \mathrm{C}^\infty(\mathcal{G}, \mathbb{V})^P.$$

Similarly P-invariant horizontal $\mathbb{V}$-valued forms on $\mathcal{G}$ are identified with forms on M with values in V. In particular the 1-form $T\mathcal{G} \to \mathfrak{g}/\mathfrak{p}$ induced by the Cartan connection θ is P-invariant and horizontal, corresponding to a bundle map $TM \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. The open condition on the Cartan connection means that this is an isomorphism and henceforth we identify TM with $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ in this way. We let $\mathfrak{g}_M = \mathcal{G} \times_P \mathfrak{g}$ and observe that there is a surjective bundle map from $\mathfrak{g}_M$ to TM, with kernel $\mathfrak{p}_M = \mathcal{G} \times_P \mathfrak{p}$.

Cartan connections do not in general induce covariant derivatives on associated bundles, but there is a way of differentiating sections of such bundles using $\mathfrak{g}_M$ instead of TM.

Definition 1.2. Let $(\mathcal{G}, \theta)$ be a Cartan connection of type $(\mathfrak{g}, P)$ on M, and let $\mathbb{V}$ be a P-module with associated vector bundle $V = \mathcal{G} \times_P \mathbb{V}$. Then the linear map defined by

$$\nabla^\theta : \mathrm{C}^\infty(\mathcal{G}, \mathbb{V}) \to \mathrm{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V}),$$
$$\nabla^\theta_\xi f = df\big(\theta^{-1}(\xi)\big)$$

(for all ξ in $\mathfrak{g}$) is P-equivariant. The restriction to $\mathrm{C}^\infty(\mathcal{G}, \mathbb{V})^P$, or equivalently the induced linear map $\nabla^\theta : \mathrm{C}^\infty(M, V) \to \mathrm{C}^\infty(M, \mathfrak{g}_M^* \otimes V)$, is called the *invariant derivative* on V.

The *curvature* $K : \wedge^2 T\mathcal{G} \to \mathfrak{g}$ of a Cartan geometry is defined by

$$K(X, Y) = d\theta(X, Y) + \big[\theta(X), \theta(Y)\big].$$

It induces a curvature function $\kappa : \mathcal{G} \to \wedge^2\mathfrak{g}^* \otimes \mathfrak{g}$ via

$$\kappa_y(\xi, \chi) = K_y\big(\theta^{-1}(\xi), \theta^{-1}(\chi)\big) = [\xi, \chi] - \theta_y\big[\theta^{-1}(\xi), \theta^{-1}(\chi)\big],$$

where $y \in \mathcal{G}$ and the latter bracket is the Lie bracket of vector fields on $\mathcal{G}$.

The $\mathfrak{p}$-invariance of $\chi \mapsto \theta^{-1}(\chi)$ for $\chi \in \mathfrak{g}$ means that $[\theta^{-1}(\xi), \theta^{-1}(\chi)] = \theta^{-1}[\xi, \chi]$ for any $\xi \in \mathfrak{p}$, and hence that $\kappa(\xi, \cdot) = 0$ for $\xi \in \mathfrak{p}$ so that $\kappa \in \mathrm{C}^\infty(\mathcal{G}, \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})^P$. In other words K is a horizontal 2-form and induces $K_M \in \mathrm{C}^\infty(M, \wedge^2 T^*M \otimes \mathfrak{g}_M)$.

Lemma 1.3. *Let $(\mathcal{G}, \theta)$ be a Cartan geometry of type $(\mathfrak{g}, P)$ on M.*

(i) *For $f \in \mathrm{C}^\infty(\mathcal{G}, \mathbb{V})^P$, we have $(\nabla^\theta_\xi f)(y) + \xi \cdot (f(y)) = 0$ for all $\xi \in \mathfrak{p}$ and $y \in \mathcal{G}$.*
(ii) *We also have $\nabla^\theta_\xi(\nabla^\theta_\chi f) - \nabla^\theta_\chi(\nabla^\theta_\xi f) = \nabla^\theta_{[\xi,\chi]} f - \nabla^\theta_{\kappa(\xi,\chi)} f$ for all $\xi, \chi \in \mathfrak{p}$.*

Proof. (i) Differentiate the P-invariance condition $p \cdot (f(yp)) = f(y)$.

(ii) Both sides are equal to $df([\theta^{-1}(\xi), \theta^{-1}(\chi)])$. $\square$

These facts enable us to define a semiholonomic jet operator $\hat{j}^k_\theta$ identifying the semiholonomic jet bundle $\hat{J}^k V$ with an associated bundle $\mathcal{G} \times_P \hat{J}^k_0 \mathbb{V}$ [8,13,30]. Recall that the semiholonomic jet bundles are defined inductively by $\hat{J}^1 V = J^1 V$ and $\hat{J}^{k+1} V$ is the subbundle of $J^1 \hat{J}^k V$ on which the two natural maps to $J^1 \hat{J}^{k-1} V$ agree. The advantage of semiholonomic jets is that they depend only on the 1-jet functor, the natural transformation $J^1 V \to V$ and some abstract nonsense.

Proposition 1.4. *Let $(\mathcal{G}, \theta)$ be a Cartan geometry of type $(\mathfrak{g}, P)$ on M and $\mathbb{V}$ a P-module.*

(i) *The map $j^1_\theta : \mathrm{C}^\infty(M, V) \to \mathrm{C}^\infty(M, V \oplus (\mathfrak{g}^*_M \otimes V))$ sending φ to $(\varphi, \nabla^\theta \varphi)$ defines an injective bundle map, from the 1-jet bundle $J^1 V$ to $V \oplus (\mathfrak{g}^*_M \otimes V)$, whose image is $\mathcal{G} \times_P J^1_0 \mathbb{V}$ where $J^1_0 \mathbb{V} = \{(\phi_0, \phi_1) \in \mathbb{V} \oplus (\mathfrak{g}^* \otimes \mathbb{V})\colon \phi_1(\xi) + \xi \cdot \phi_0 = 0$ for all $\xi \in \mathfrak{p}\}$.*
(ii) *Similarly the map $\hat{j}^k_\theta$ sending a section φ to $(\varphi, \nabla^\theta \varphi, (\nabla^\theta)^2 \varphi, \ldots, (\nabla^\theta)^k \varphi)$ defines an isomorphism between the semiholonomic jet bundle $\hat{J}^k V$ and the subbundle $\mathcal{G} \times_P \hat{J}^k_0 \mathbb{V}$ of $\bigoplus_{j=0}^k ((\otimes^j \mathfrak{g}^*_M) \otimes V)$, where $\hat{J}^k_0 \mathbb{V}$ is the set of all $(\phi_0, \phi_1, \ldots, \phi_k)$ in $\bigoplus_{j=0}^k ((\otimes^j \mathfrak{g}^*) \otimes \mathbb{V})$ satisfying (for $1 \leqslant i < j \leqslant k$) the equations*

$$\phi_j(\xi_1, \ldots, \xi_i, \xi_{i+1}, \ldots, \xi_j) - \phi_j(\xi_1, \ldots, \xi_{i+1}, \xi_i, \ldots, \xi_j) = \phi_{j-1}\big(\xi_1, \ldots, [\xi_i, \xi_{i+1}], \ldots, \xi_j\big),$$
$$\phi_i(\xi_1, \ldots, \xi_i) + \xi_i \cdot \big(\phi_{i-1}(\xi_1, \ldots, \xi_{i-1})\big) = 0$$

for all $\xi_1, \ldots, \xi_j \in \mathfrak{g}$ with $\xi_i \in \mathfrak{p}$.

Proof. (i) Certainly the map on smooth sections only depends on the 1-jet at each point, and it is injective since the symbol of ∇^θ is the inclusion $T^*M \otimes V \to \mathfrak{g}^*_M \otimes V$. It maps into $\mathcal{G} \times_P J^1_0 \mathbb{V}$ by vertical triviality, but this has the same rank as $J^1 V$.

(ii) Similar: the equations are those given by the vertical triviality and the Ricci identity, bearing in mind that κ is horizontal. The (semiholonomic) symbols of the iterated invariant derivatives are still given by inclusions $(\otimes^j T^*M) \otimes V \to (\otimes^j \mathfrak{g}^*_M) \otimes V$. $\square$

2. Parabolic subalgebras and algebraic Weyl structures

Parabolic geometries are Cartan geometries of type $(\mathfrak{g}, P)$ where $\mathfrak{g}$ is semisimple and the Lie algebra $\mathfrak{p}$ of P is a parabolic subalgebra. In this section we develop a few basic facts about parabolic subalgebras and their representations, emphasising the relation between filtered and graded modules. The key feature of parabolic subalgebras is the presence of 'algebraic Weyl structures' which split filtered modules. Such splittings, carried out pointwise, equip parabolic geometries with covariant derivatives on associated bundles.

A parabolic subalgebra of a semisimple Lie algebra is a subalgebra containing a Borel (i.e., maximal solvable) subalgebra. However, to keep our treatment as self-contained as possible, with minimal use of structure theory, we find the following equivalent and elementary definition more convenient. We refer to [4,12,30] for an alternative approach.

Definition 2.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. For a subspace $\mathfrak{u}$ of $\mathfrak{g}$ we let $\mathfrak{u}^\perp$ be the orthogonal subspace with respect to the Killing form $(\cdot,\cdot)$. Then a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is *parabolic* iff $\mathfrak{p}^\perp$ is the *nilradical* of $\mathfrak{p}$, i.e., its maximal nilpotent ideal. It follows that the quotient $\mathfrak{p}_0 := \mathfrak{p}/\mathfrak{p}^\perp$ is a reductive Lie algebra, called the *Levi factor*.

Let $\mathfrak{p}^\perp, (\mathfrak{p}^\perp)^2 = [\mathfrak{p}^\perp, \mathfrak{p}^\perp], \dots, (\mathfrak{p}^\perp)^{j+1} = [\mathfrak{p}^\perp, (\mathfrak{p}^\perp)^j], \dots$ be the descending central series of $\mathfrak{p}^\perp$. Since $\mathfrak{p}^\perp$ is nilpotent there is an integer $k \geqslant 0$, called the *depth* of $\mathfrak{p}$, such that $(\mathfrak{p}^\perp)^{k+1} = 0$ but $(\mathfrak{p}^\perp)^k \neq 0$. Thus $\mathfrak{p}^\perp$ has a k-step filtration ($k = 0$ is the trivial case $\mathfrak{p}^\perp = 0$ and $\mathfrak{p} = \mathfrak{g}$). We obtain from this a filtration of $\mathfrak{g}$ by setting $\mathfrak{g}_{(-j)} = (\mathfrak{p}^\perp)^j$ and $\mathfrak{g}_{(j-1)} = \mathfrak{g}^\perp_{(-j)}$ for $j \geqslant 1$ so that

$$\mathfrak{g} = \mathfrak{g}_{(k)} \supset \mathfrak{g}_{(k-1)} \supset \cdots \supset \mathfrak{g}_{(1)} \supset \mathfrak{g}_{(0)} = \mathfrak{p} \supset \mathfrak{p}^\perp = \mathfrak{g}_{(-1)} \supset \cdots \supset \mathfrak{g}_{(-k)} \supset \mathfrak{g}_{(-k-1)} = 0.$$

It is easily verified that $[\mathfrak{g}_{(i)}, \mathfrak{g}_{(j)}] \subseteq \mathfrak{g}_{(i+j)}$, so that $\mathfrak{g}$ is a filtered Lie algebra. The associated graded Lie algebra is $\operatorname{gr}\mathfrak{g} = \bigoplus_{j=-k}^{k} \mathfrak{g}_j$ where $\mathfrak{g}_j = \mathfrak{g}_{(j)}/\mathfrak{g}_{(j-1)}$ and is said to be $|k|$-graded. Note in particular that $\mathfrak{p}_0 = \mathfrak{g}_0$.

An important fact about parabolic subalgebras is that this filtration of $\mathfrak{g}$ is split.

Lemma 2.2. *There are* (*non-canonical*) *splittings of the exact sequences*

$$0 \to \mathfrak{g}_{(j-1)} \to \mathfrak{g}_{(j)} \to \mathfrak{g}_j \to 0 \tag{2.1}$$

which induce a Lie algebra isomorphism between $\mathfrak{g}$ *and* $\operatorname{gr}\mathfrak{g}$.

Proof. Any semisimple Lie algebra admits a Cartan involution, i.e., an automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma^2 = \mathrm{id}$ and $h(\xi, \chi) := (\sigma(\xi), \chi)$ is positive definite. We split (2.1) for each j by identifying $\mathfrak{g}_j$ with the h-orthogonal complement to $\mathfrak{g}_{(j-1)}$ in $\mathfrak{g}_{(j)}$. Suppose $\xi \in \mathfrak{g}_{(i)}$ is h-orthogonal to $\mathfrak{g}_{(i-1)}$, i.e., $\sigma(\xi) \in \mathfrak{g}^\perp_{(i-1)} = \mathfrak{g}_{(-i)}$, and $\chi \in \mathfrak{g}_{(j)}$ is h-orthogonal to $\mathfrak{g}_{(j-1)}$. Then $[\xi, \chi] \in \mathfrak{g}_{(i+j)}$ and $\sigma[\xi, \chi] = [\sigma(\xi), \sigma(\chi)] \in \mathfrak{g}_{(-i-j)}$ so $[\xi, \chi]$ is h-orthogonal to $\mathfrak{g}_{(i+j-1)}$. Hence the splittings defined by σ induce a Lie algebra isomorphism. □

We refer to such a splitting of $\mathfrak{g}$ as an *algebraic Weyl structure*: it is not unique, but we can obtain very good control over the possible splittings thanks to the following.

Lemma 2.3. *There is a unique element ε_0 in the centre of $\mathfrak{p}_0 = \mathfrak{p}/\mathfrak{p}^\perp$ such that $[\varepsilon_0, \xi] = j\xi$ for all $\xi \in \mathfrak{g}_j$ and all j.*

Proof. Since $\operatorname{gr}\mathfrak{g}$ is semisimple, the derivation defined by $\xi \mapsto j\xi$ for $\xi \in \mathfrak{g}_j$ must be inner, i.e., equal $\operatorname{ad}\varepsilon_0$ for $\varepsilon_0 \in \operatorname{gr}\mathfrak{g}$ (which is unique since $Z(\mathfrak{g}) = 0$). Now $[\varepsilon_0, \varepsilon_0] = 0$ and $[\varepsilon_0, \xi] = 0$ for all $\xi \in \mathfrak{g}_0$, so ε_0 is in the centre of $\mathfrak{g}_0 = \mathfrak{p}_0$. $\square$

Definition 2.4. The element ε_0 is called the *grading element*. Let $\mathfrak{w} = \{\varepsilon \in \mathfrak{p}\colon\ \pi_0(\varepsilon) = \varepsilon_0\}$ be the set of all lifts of ε_0 to $\mathfrak{p}$ with respect to the exact sequence

$$0 \to \mathfrak{p}^\perp \to \mathfrak{p} \xrightarrow{\pi_0} \mathfrak{p}_0 \to 0. \tag{2.2}$$

The elements of $\mathfrak{w}$ are precisely the algebraic Weyl structures: the isomorphism of $\mathfrak{g}$ with $\operatorname{gr}\mathfrak{g}$ is given by the eigenspace decomposition of $\operatorname{ad}\varepsilon$ for a lift of ε_0 to $\varepsilon \in \mathfrak{p} \subseteq \mathfrak{g}$. The space of algebraic Weyl structures is therefore $\mathfrak{w}$, an affine space modelled on $\mathfrak{p}^\perp$.

Let P be a Lie group acting on $\mathfrak{g}$ with Lie algebra $\mathfrak{p}$ as in the previous section, and suppose additionally that the quotient group $P_0 = P/\exp\mathfrak{p}^\perp$ stabilizes $\varepsilon_0 \in \mathfrak{p}_0$ (which is automatic if P_0 is connected) so that the adjoint action of P on $\mathfrak{p}$ preserves $\mathfrak{w}$.

Lemma 2.5. $\exp\mathfrak{p}^\perp \leqslant P$ *acts freely and transitively on* $\mathfrak{w}$.

Proof. If $\gamma \in \mathfrak{p}^\perp$, $(\operatorname{Ad}\exp\gamma)\varepsilon = \exp(\operatorname{ad}\gamma)\varepsilon = \varepsilon + [\gamma, \varepsilon] + \cdots$. The result follows because $\operatorname{ad}\gamma$ is nilpotent on $\mathfrak{p}$, and $\gamma \mapsto [\gamma, \varepsilon]$ is a bijection on $\mathfrak{p}^\perp$. $\square$

The stabilizer of ε is thus a subgroup of P projecting isomorphically onto P_0, so that an algebraic Weyl structure splits the quotient group homomorphism $\pi_0 : P \to P_0$.

The fundamental vector fields ζ_γ ($\gamma \in \mathfrak{p}^\perp$) generating the action of $\exp\mathfrak{p}^\perp$ on $\mathfrak{w}$ give rise to a Maurer–Cartan form $\eta : T\mathfrak{w} \to \mathfrak{p}^\perp$ with $\eta(\zeta_\gamma) = \gamma$. If we identify $T\mathfrak{w}$ with $\mathfrak{w} \times \mathfrak{p}^\perp$ using the affine space structure then $\zeta_{\gamma,\varepsilon} = [\gamma, \varepsilon]$, so η_ε is the inverse of $\gamma \mapsto [\gamma, \varepsilon]$ on $\mathfrak{p}^\perp$.

2.1. Filtered and graded modules

We say that a P_0-module is *semisimple* if it is completely reducible and the grading element ε_0 acts by a scalar on irreducible components (the latter condition is automatic for complex modules). The eigenvalues of ε_0 will be called the *geometric weights* of the module.

There is a one to one correspondence between P_0-modules and P-modules on which $\exp\mathfrak{p}^\perp$ acts trivially. We say such a P-module is semisimple if the corresponding P_0-module is. More generally, we shall consider P-modules $\mathbb{V}$ with a P-invariant filtration

$$\mathbb{V} = \mathbb{V}_{(\lambda)} \supset \mathbb{V}_{(\lambda-1)} \supset \mathbb{V}_{(\lambda-2)} \supset \cdots \supset \mathbb{V}_{(\lambda-\ell)} \supset 0 \tag{2.3}$$

(for a scalar λ and an integer ℓ) such that the associated graded module $\operatorname{gr}\mathbb{V}$ is a semisimple P-module graded by geometric weight. We refer to such modules as *filtered P-modules*. We extend the definition in a straightforward way to direct sums of such modules.

An algebraic Weyl structure ε splits any filtered P-module $\mathbb{V}$ into the eigenspaces of ε, giving a vector space isomorphism $\varepsilon_{\mathbb{V}} : \mathbb{V} \to \operatorname{gr}\mathbb{V}$. This isomorphism is not P-equivariant (though it is tautologically

P_0-equivariant using the splitting of $\pi_0 : P \to P_0$ defined by ε). However, by the naturality of the construction, the map $\varepsilon \mapsto \varepsilon_{\mathbb{V}}$ is P-equivariant. More precisely, if $\tilde{\varepsilon} = (\operatorname{Ad} p)\varepsilon$ is any other algebraic Weyl structure (with $p \in P$), then

$$\tilde{\varepsilon}_{\mathbb{V}}(v) = p \cdot \varepsilon_{\mathbb{V}}(p^{-1} \cdot v). \tag{2.4}$$

In particular, if $p \in \exp \mathfrak{p}^{\perp}$, we have $\tilde{\varepsilon}_{\mathbb{V}} = \varepsilon_{\mathbb{V}} \circ (q^{-1} \cdot\ )$. Since the action of $\exp \mathfrak{p}^{\perp}$ on $\mathfrak{w}$ is free and transitive, this dependence implies that if we have a smooth family $\varepsilon = \varepsilon(s)$ of algebraic Weyl structures then $d\varepsilon_{\mathbb{V}}(X)(v) = -\varepsilon_{\mathbb{V}}(\eta(d\varepsilon(X)) \cdot v)$.

Lemma 2.6. *For a smooth map $\varepsilon : S \to \mathfrak{w}$ and a filtered P-module $\mathbb{V}$, the* $\operatorname{End}\mathbb{V}$*-valued* 1*-form $\varepsilon_{\mathbb{V}}^{-1} d\varepsilon_{\mathbb{V}}$ on S is given by the natural action of $-\varepsilon^*\eta$ on $\mathbb{V}$.*

2.2. The tangent module

We end this section by considering the module $\mathfrak{m} := \mathfrak{g}/\mathfrak{p}$, which is the filtered P-module dual to $\mathfrak{p}^{\perp}$. More precisely, the Killing form of $\mathfrak{g}$ gives a nondegenerate pairing between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}^{\perp}$, which will also be denoted by $\mathfrak{m}^*$. This duality depends on the normalization of the Killing form, which we do not specify at present.

We have seen that $\mathfrak{g}$ is a filtered P-module, with $\mathfrak{m}$ and $\mathfrak{m}^*$ as quotient and sub- modules respectively. The associated graded modules $\operatorname{gr}\mathfrak{m}$ and $\operatorname{gr}\mathfrak{m}^*$ are graded nilpotent subalgebras of $\operatorname{gr}\mathfrak{g}$. In particular, as semisimple P-modules, $\operatorname{gr}\mathfrak{g} = \operatorname{gr}\mathfrak{m} \oplus \mathfrak{p}_0 \oplus \operatorname{gr}\mathfrak{m}^*$, although the Lie bracket is not compatible with this direct sum decomposition. An algebraic Weyl structure ε therefore determines a vector space isomorphism $\varepsilon_{\bullet} : \mathfrak{m} \oplus \mathfrak{p}_0 \oplus \mathfrak{m}^* \to \mathfrak{g}$.

Observe that $\mathfrak{f} := \mathfrak{g}_1$ is the lowest geometric weight subspace of $\operatorname{gr}\mathfrak{m}$, and so is a P-submodule of $\mathfrak{m}$; the dual $\mathfrak{f}^*$ is naturally a quotient P-module of $\mathfrak{p}^{\perp} = \mathfrak{m}^*$.

3. Parabolic geometries and Weyl structures

Definition 3.1. A *parabolic geometry* on M is a Cartan geometry $(\mathcal{G}, \theta)$ of type $(\mathfrak{g}, P)$ with $\mathfrak{g}$ semisimple and $\mathfrak{p}$ parabolic, satisfying the conditions of the previous two sections.

We define $\mathcal{G}_0$ to be the principal P_0-bundle $\mathcal{G}/\exp\mathfrak{p}^{\perp}$ and we let π_0 also denote the projection $\mathcal{G} \to \mathcal{G}_0$, so that $\pi_0(yp) = \pi_0(y)\pi_0(p)$ for $y \in \mathcal{G}$ and $p \in P$.

The tangent bundle $TM = \mathcal{G} \times_P \mathfrak{m}$ has a natural filtration induced by the filtration of $\mathfrak{m}$, the smallest nontrivial distribution in the filtration being $\mathfrak{f}_M = \mathcal{G} \times_P \mathfrak{f}$. The cotangent bundle $T^*M = \mathcal{G} \times_P \mathfrak{m}^* = \mathcal{G} \times_P \mathfrak{p}^{\perp}$ is a bundle of nilpotent Lie algebras, the nilradical bundle of $\mathfrak{p}_M = \mathcal{G} \times_P \mathfrak{p}$. The quotient $\mathfrak{p}_M/T^*M$ is a reductive Lie algebra bundle, namely $\mathfrak{p}_{M,0} := \mathcal{G} \times_P \mathfrak{p}_0$. Observe that $\mathfrak{p}_{M,0}$ has a canonical grading section E_0, induced by the grading element ε_0 of $\mathfrak{p}_0$, which is P-invariant.

Definition 3.2. Let $(\mathcal{G}, \theta)$ be a parabolic geometry on M. Then a (geometric) *Weyl structure* E on M is a smooth lift of the grading section E_0 to a section of $\mathfrak{p}_M$.

Thus a Weyl structure amounts to a smooth choice of algebraic Weyl structure at each point. Since algebraic Weyl structures form an affine space, a Weyl structure is a section of an affine bundle, the *bundle of Weyl geometries* $\mathfrak{w}_M = \mathcal{G} \times_P \mathfrak{w}$. In particular, Weyl structures always exist, and form an affine space modelled on the space of 1-forms on M.

Remark 3.3 (*Key observation*). Any construction with algebraic Weyl structures can be carried out with geometric Weyl structures. We can either work with associated bundles or on the principal bundle $\mathcal{G}$, and both points of view are useful.

(i) If $V = \mathcal{G} \times_P \mathbb{V}$ is bundle associated to a filtered P-module $\mathbb{V}$, with graded bundle $\operatorname{gr} V = \mathcal{G} \times_P \operatorname{gr}\mathbb{V} = \mathcal{G}_0 \times_{P_0} \operatorname{gr}\mathbb{V}$, then a Weyl structure E provides an isomorphism $E_V : V \to \operatorname{gr} V$, simply by applying the construction of the previous section pointwise. We also obtain a bundle isomorphism $E_\bullet : TM \oplus \mathfrak{p}_{M,0} \oplus T^*M \to \mathfrak{g}_M$.

(ii) A Weyl structure may equally be regarded as a P-invariant function $\mathcal{E} : \mathcal{G} \to \mathfrak{w}$. For any filtered P-module $\mathbb{V}$, we then have a P-equivariant isomorphism $\mathcal{E}_{\mathbb{V}} : \mathcal{G} \times \mathbb{V} \to \mathcal{G} \times \operatorname{gr}\mathbb{V}$ whose fibre at $y \in \mathcal{G}$ is $\mathcal{E}(y)_{\mathbb{V}}$; the induced isomorphism of associated bundles is E_V. Similarly we get a P-equivariant isomorphism $\mathcal{E}_\bullet : \mathcal{G} \times (\mathfrak{m} \oplus \mathfrak{p}_0 \oplus \mathfrak{m}^*) \to \mathcal{G} \times \mathfrak{g}$ inducing $E_\bullet$.

If we fix an algebraic Weyl structure then any geometric Weyl structure may be written $\mathcal{E} = (\operatorname{Ad} q)\varepsilon$, where $q : \mathcal{G} \to \exp \mathfrak{p}^\perp$ is P-invariant in the sense that $pq(yp)\pi_0(p^{-1}) = q(y)$: here P_0 acts on P via the lift defined by ε. As we discuss in Appendix A, this allows us to relate our approach to Weyl structures to the original approach of Čap and Slovák [12].

4. Ricci-corrected Weyl differentiation

The main difficulty in the study of invariant differential operators on parabolic geometries is that there is no natural covariant derivative on associated bundles: we only have the invariant derivative $\nabla^\theta : \mathrm{C}^\infty(M, V) \to \mathrm{C}^\infty(M, \mathfrak{g}_M^* \otimes V)$ in general. There is no canonical projection $\mathfrak{g}_M^* \to T^*M$; equivalently, the restriction map $\mathfrak{g}^* \to \mathfrak{m}^* = \mathfrak{p}^\perp$ is not P-equivariant.

Weyl structures provide two solutions to this problem, one well known (Weyl connections), the other implicitly known (and closely related to the 'conformal derivation' of Wünch [33]), but not properly formalized (Ricci-corrected Weyl connections). In our theory, both can be defined straightforwardly using Remark 3.3(i).

Definition 4.1. Let $(\mathcal{G} \to M, \theta)$ be a parabolic geometry and E be a Weyl structure on M. Let $V = \mathcal{G} \times_P \mathbb{V}$ be a filtered P-bundle (i.e., associated to a filtered P-module).

(i) The *Ricci-corrected Weyl connection* $D^{(1)} : \mathrm{C}^\infty(M, V) \to \mathrm{C}^\infty(M, T^*M \otimes V)$ is given by $D^{(1)}_X \varphi = \nabla^\theta_{E_\bullet X} \varphi$ for all vector fields X and sections φ of V. In other words $D^{(1)}$ obtained by restricting the invariant derivative to tangent vectors using the isomorphism $E_\bullet : TM \oplus \mathfrak{p}_M \oplus T^*M \to \mathfrak{g}_M$ induced by E.

(ii) The *Weyl connection* $D : \mathrm{C}^\infty(M, V) \to \mathrm{C}^\infty(M, T^*M \otimes V)$ is $D\varphi = E_V^{-1} D^{(1)}(E_V \varphi)$, i.e., the connection on V induced by $D^{(1)}$ on $\operatorname{gr} V$ via the isomorphism $E_V : V \to \operatorname{gr} V$.

By definition, $D^{(1)}$ and D agree on bundles associated to semisimple P-modules (when $\exp \mathfrak{p}^\perp$ acts trivially and V and $\operatorname{gr} V$ are canonically isomorphic). In the notation we suppress their dependence on the Weyl structure E. A priori they also depend on the chosen P-module $\mathbb{V}$. This latter dependence is straightforward as they are associated to principal P-connections. To see this, we use the isomorphism $\mathcal{E}_\bullet : \mathcal{G} \times (\mathfrak{m} \oplus \mathfrak{p}_0 \oplus \mathfrak{m}^*) \to \mathcal{G} \times \mathfrak{g}$ defined by the Weyl structure to decompose the Cartan connection $\theta : T\mathcal{G} \to \mathfrak{g}$ as

$$\theta = \mathcal{E}_\bullet \theta_\mathfrak{m} + \theta_\mathfrak{p}, \qquad \theta_\mathfrak{p} = \mathcal{E}_\bullet \theta_{\mathfrak{p}_0} + \theta_{\mathfrak{m}^*}, \tag{4.1}$$

where $\theta_\mathfrak{m} := (\theta \bmod \mathfrak{p}) : T\mathcal{G} \to \mathfrak{m}$ is the solder form, induced by projecting θ onto $\mathfrak{m} = \mathfrak{g}/\mathfrak{p}$ and similarly $\theta_{\mathfrak{p}_0} := (\theta_\mathfrak{p} \bmod \mathfrak{p}^\perp) : T\mathcal{G} \to \mathfrak{p}_0 = \mathfrak{p}/\mathfrak{p}^\perp$. Thus

$$\mathcal{E}_\bullet^{-1} \circ \theta = \theta_\mathfrak{m} + \theta_{\mathfrak{p}_0} + \theta_{\mathfrak{m}^*} \tag{4.2}$$

and we can write the $\mathfrak{p}$-part conceptually as

$$\theta_\mathfrak{p} := (\mathcal{E}_\bullet \theta_{\mathfrak{p}_0} - \mathcal{E}^*\eta) + (\theta_{\mathfrak{m}^*} + \mathcal{E}^*\eta), \tag{4.3}$$

where η is the Maurer–Cartan form on $\mathfrak{w}$. This leads to the following proposition.

Proposition 4.2. *Let $(\mathcal{G} \to M, \theta)$ be a parabolic geometry with Weyl structure E. Then*:

(i) *$\theta_\mathfrak{p}$ is a principal P-connection on $\mathcal{G}$ inducing $D^{(1)}$ on associated bundles*;
(ii) *$\theta_\mathcal{E} = \mathcal{E}_\bullet \theta_{\mathfrak{p}_0} - \mathcal{E}^*\eta$ is a principal P-connection on $\mathcal{G}$ inducing D on associated bundles*;
(iii) *$\rho = \theta_{\mathfrak{m}^*} + \mathcal{E}^*\eta$ is a horizontal, P-invariant $\mathfrak{p}^\perp$-valued* 1*-form on $\mathcal{G}$*; *if r^D is the induced T^*M-valued* 1*-form on M and $\cdot$ is the natural action of $T^*M = \mathfrak{p}^\perp_M$ on V, then*

$$D^{(1)}_X \varphi = D_X \varphi + r^D(X) \cdot \varphi. \tag{4.4}$$

Proof. (i) Clearly $\theta_\mathfrak{p}$ is P-invariant and $\mathfrak{p}$-valued, and so, since θ is a Cartan connection, $\theta_\mathfrak{p}$ is a principal P-connection. Let X be a vector field and φ a section of V, and let $\chi : \mathcal{G} \to \mathfrak{m}$ and $f : \mathcal{G} \to \mathbb{V}$ be the corresponding P-invariant functions. Since the identification of TM with $\mathcal{G} \times_P \mathfrak{m}$ is via the solder form, $\chi = \theta_\mathfrak{m}(\hat{X})$ for any P-invariant lift $\hat{X}$ of X to $\mathcal{G}$.

As a P-invariant function on $\mathcal{G}$, $D^{(1)}_X \varphi$ is then

$$\nabla^\theta_{\mathcal{E}_\bullet \chi} f = \nabla^\theta_{\mathcal{E}_\bullet \theta_\mathfrak{m}(\hat{X})} f = \nabla^\theta_{\theta(\hat{X})} f - \nabla^\theta_{\theta_\mathfrak{p}(\hat{X})} f = df(\hat{X}) + \theta_\mathfrak{p}(\hat{X}) \cdot f,$$

which is precisely the P-invariant function on $\mathcal{G}$ corresponding to the covariant derivative of φ along X induced by $\theta_\mathfrak{p}$.

(ii) We now mirror the construction of D from $D^{(1)}$ on the principal bundle level, using Remark 3.3(ii): if $\mathbb{V}$ is a filtered P-module and $f : \mathcal{G} \to \mathbb{V}$ is P-invariant, corresponding to a section φ of $V = \mathcal{G} \times_P \mathbb{V}$, then $D\varphi$ corresponds to the P-invariant horizontal 1-form

$$\mathcal{E}_\mathbb{V}^{-1}(d + \theta_\mathfrak{p})(\mathcal{E}_\mathbb{V} f) = df + \mathcal{E}_\bullet(\theta_{\mathfrak{p}_0}) \cdot f + \mathcal{E}_\mathbb{V}^{-1}(d\mathcal{E}_\mathbb{V}) f = df + \theta_\mathcal{E} \cdot f$$

with $\theta_\mathcal{E} = \mathcal{E}_\bullet(\theta_{\mathfrak{p}_0}) - \mathcal{E}^*\eta$, by Lemma 2.6, as required.

(iii) This follows immediately because $\theta_\mathfrak{p} = \theta_\mathcal{E} + \rho$. (One can also easily see directly that $\mathcal{E}^*\eta + \theta_{\mathfrak{m}^*}$ is a P-invariant $\mathfrak{p}^\perp$-valued horizontal 1-form on $\mathcal{G}$.) □

In conformal geometry r^D is the *normalized Ricci curvature* (aka. the *Schouten* or *Rho tensor*) of D. This is the origin of the term Ricci-corrected Weyl connection.

We wish to see how the objects we have constructed depend on the choice of Weyl structure. We can either do this on M, or for the corresponding P-invariant objects on $\mathcal{G}$.

Proposition 4.3. *Let E and $\tilde{E} = (\operatorname{Ad} q^{-1})E$ be Weyl structures, with $q : M \to \mathcal{G} \times_P \exp \mathfrak{p}^\perp$ (associated to the adjoint action), and let V be a filtered P-bundle. Then $\tilde{E}_V = E_V \circ (q\cdot)$.*

This is immediate from Eq. (2.4). In practice it suffices to understand infinitesimal variations. Let q_t be a curve of sections of $\mathfrak{p}_M$ with $q_0 = \mathrm{id}$ and $\dot{q}_0 = \gamma$ for a 1-form γ (equivalently a P-invariant function $\mathcal{G} \to \mathfrak{p}^\perp$). Then for any object $F(E)$ depending on E, define $(\partial_\gamma F)(E)$ to be the t-derivative of $F((\operatorname{Ad} q_t^{-1})E)$ at $t = 0$ (so that $\partial_\gamma E = -\gamma$). By the fundamental theorem of calculus, F is independent of the Weyl structure if and only if $(\partial_\gamma F)(E) = 0$ for all Weyl structures E and all 1-forms γ. Proposition 4.3 implies that $\partial_\gamma E_V = E_V \circ (\gamma\cdot)$. It is now straightforward to differentiate the definition of $D^{(1)}$.

Proposition 4.4. *For a Weyl structure E, a 1-form γ, and a vector field X, let $[\gamma, X]^E_{\mathfrak{p}_M} = [\gamma, E_\bullet X] - E_\bullet(\gamma \cdot X)$. Then for any section φ of a filtered P-module V,*

$$\partial_\gamma D^{(1)}_X \varphi = [\gamma, X]^E_{\mathfrak{p}_M} \cdot \varphi. \tag{4.5}$$

Proof. For $X \in TM$, $\partial_\gamma E_\bullet X = E_\bullet(\gamma \cdot X) - [\gamma, E_\bullet X] = -[\gamma, X]^E_{\mathfrak{p}_M}$. This is $\mathfrak{p}_M$-valued (it is the $\mathfrak{p}_M$ component of the Lie bracket $[\gamma, X]$, where the lift of X to $\mathfrak{g}_M$ and the projection to $\mathfrak{p}_M$ are defined using E), and so $\partial_\gamma \nabla^\theta_{E_\bullet X}\varphi = \nabla^\theta_{\partial_\gamma E_\bullet X}\varphi = [\gamma, X]^E_{\mathfrak{p}_M} \cdot \varphi$. $\square$

The Ricci-corrected first derivative $D^{(1)}$ agrees with the Weyl connection on semisimple modules. Ricci corrections start to play a more important role when higher derivatives are considered, because jet modules are not semisimple. The following deceptively simple definition clearly generalizes the previous definition of $D^{(1)}$.

Definition 4.5. Let E be a Weyl structure. Then we define an isomorphism from

$$\hat{J}^k_\theta V \leqslant \bigoplus_{j=0}^{k} (\otimes^j \mathfrak{g}^*_M) \otimes V \quad \text{to} \quad \hat{J}^k_E V := \bigoplus_{j=0}^{k} (\otimes^j T^*M) \otimes V$$

by sending $\phi = (\phi_0, \phi_1, \ldots, \phi_k)$ to $\psi = (\psi_0, \psi_1, \ldots, \psi_k)$ where $\psi_j = (\phi_j \circ E_\bullet)|_{\otimes^j TM}$.

If $\varphi \in C^\infty(M, V)$, then the section of $\hat{J}^k_E V$ corresponding to $\hat{j}^k_\theta \varphi$ is denoted $\hat{j}^k_D \varphi = (\varphi, D^{(1)}\varphi, D^{(2)}\varphi, \ldots, D^{(k)}\varphi)$. We call $\hat{j}^k_D$ the *Weyl jet operator*, and its components $D^{(j)}$ the *Ricci-corrected powers of the Weyl connection*.

An alternative description of the Weyl jet operator is obtained from the obvious natural isomorphism between $\operatorname{gr} \hat{J}^k_\theta V$ and $\operatorname{gr} \hat{J}^k_E V$. Then $\hat{j}^k_D = E^{-1}_{[b]\hat{J}^k_E V} E_{[b]\hat{J}^k_\theta V} \hat{j}^k_\theta$. It follows that

$$\partial_\gamma \hat{j}^k_D \varphi = E^{-1}_{\hat{J}^k_E V} E_{\hat{J}^k_\theta V} \gamma \cdot \hat{j}^k_\theta \varphi - \gamma \cdot E^{-1}_{\hat{J}^k_E V} E_{\hat{J}^k_\theta V} \hat{j}^k_\theta \varphi = \gamma * \hat{j}^k_D \varphi - \gamma \cdot \hat{j}^k_D \varphi, \tag{4.6}$$

where $*$ is the action of T^*M on $\hat{J}^k_E V$ induced by the isomorphism $E^{-1}_{\hat{J}^k_E V} E_{\hat{J}^k_\theta V}$ with $\hat{J}^k_\theta V$.

The computation of $*$ is a straightforward exercise in algebra: it suffices to describe the jet $\tilde{\phi} = \gamma \cdot \phi$ for $\phi \in \hat{J}^k_0 \mathbb{V}$, $\gamma \in \mathfrak{p}^\perp$, in terms of the elements $\tilde{\psi}$ and ψ of $\hat{J}^k_\varepsilon \mathbb{V} := \bigoplus_{j=0}^k (\otimes^j \mathfrak{m}^*) \otimes \mathbb{V}$ corresponding to $\tilde{\phi}$ and ϕ using an algebraic Weyl structure ε. This $\mathfrak{p}$-module structure on $\hat{J}^k_\varepsilon \mathbb{V}$ is computed in [12,30]: let us sketch briefly the computation. We write $\psi = (\psi_0, \psi_1, \dots, \psi_k)$ with $\psi_j \in (\otimes^j \mathfrak{m}^*) \otimes \mathbb{V}$ and similarly for $\tilde{\psi}$.

First, note that the existence of natural projections $\hat{J}_0^k \mathbb{V} \to \hat{J}_0^\ell \mathbb{V}$ (for $k \geqslant \ell$) implies that the ψ_j component contributes only to $\tilde{\psi}_{j+s}$ for $s \geqslant 0$, independently of $k \geqslant j+s$. We may therefore write $\tilde{\psi}_\ell = (\gamma * \psi)_\ell = \sum_{j+s=\ell} \gamma *_s \psi_j$ for any $\ell \leqslant k$, where $\gamma *_s \psi_j$ denotes the contribution from ψ_j. Clearly $\gamma *_0 \psi_j = \gamma \cdot \psi_j$ is the ordinary action of $\mathfrak{p}^\perp \leqslant \mathfrak{p}$.

Secondly, observe that the natural inclusions $\hat{J}_0^{k+1} \mathbb{V} \to J_0^1 \hat{J}_0^k \mathbb{V}$ mean that we can compute $\gamma *_s \psi_j$ inductively using the definition of $J_0^1 \mathbb{V}$. Recall that this consists of the pairs (ϕ_0, ϕ_1) in $\mathbb{V} \oplus (\mathfrak{g}^* \otimes \mathbb{V})$ with $\phi_1(\xi) + \xi \cdot \phi_0 = 0$ for $\xi \in \mathfrak{p}$. The identification with $\mathbb{V} \oplus (\mathfrak{m}^* \otimes \mathbb{V})$ is obtained by restricting ϕ_1 to $\mathfrak{m}$, using ε. The induced $\mathfrak{p}$-module structure on $\mathbb{V} \oplus (\mathfrak{m}^* \otimes \mathbb{V})$ is given by $\xi \cdot (\phi_0, \phi_1|_\mathfrak{m}) = (\xi \cdot \phi_0, (\xi \cdot \phi_1)|_\mathfrak{m})$, and one easily computes that $(\xi \cdot \phi_1)|_\mathfrak{m} = \xi \cdot (\phi_1|_\mathfrak{m}) + [\xi, \cdot]_\mathfrak{p}^\varepsilon \cdot \phi_0$, where ε is used to split the natural maps $\mathfrak{p} \to \mathfrak{g} \to \mathfrak{m}$. Therefore

$$\gamma * (\psi_0, \psi_1) = (\gamma \cdot \psi_0, \gamma \cdot \psi_1 + \gamma *_1 \psi_0) \quad \text{with} \quad \gamma *_1 \psi_0 = [\gamma, \cdot]_\mathfrak{p}^\varepsilon \cdot \psi_0.$$

(Applying this pointwise on M, we rederive Eq. (4.5).) Iterating this action we have

$$\gamma *_s \psi_0 = \big[[\dots[\gamma, \cdot]_\mathfrak{p}^\varepsilon, \dots]_\mathfrak{p}^\varepsilon, \cdot\big]_\mathfrak{p}^\varepsilon \cdot \psi_0.$$

The formula for $\gamma *_s \psi_j$ is also computed inductively, by considering the action of γ on $J_0^1 \hat{J}_0^{j+s-1} \mathbb{V}$: apply this action to $(0, \dots, 0, \phi_j, 0, \dots, 0) \in \hat{J}_0^{j+s-1} \mathbb{V}$ to obtain $\gamma *_s \psi_j$ as the $\mathfrak{m}^* \otimes (\otimes^{j+s-1} \mathfrak{m}^*) \otimes \mathbb{V}$ component with respect to ε.

Passing from the algebra to associated bundles, we obtain the following result.

Proposition 4.6. *The action of $\gamma \in T^*M$ on $\psi \in \hat{J}_E^k V$ induced by the identification with $\hat{J}_\theta^k V$ is given by*

$$(\gamma * \psi)_\ell = \sum_{j+s=\ell} \gamma *_s \psi_j$$

where, if we suppose that $\psi_j = A_1 \otimes \dots \otimes A_j \otimes v$ for $A_i \in T^*M$ and $v \in V$, we have

$$\begin{aligned}
\gamma *_0 \psi_j &= \gamma \cdot \psi_j, \\
\gamma *_1 \psi_j &= \sum_{0 \leqslant i \leqslant j} A_1 \otimes A_2 \otimes \dots \otimes A_i \otimes [\gamma, \cdot]_{\mathfrak{p}_M}^E \cdot (A_{i+1} \otimes \dots \otimes A_j \otimes v), \\
\gamma *_s \psi_j &= \sum_{\substack{0 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_s \leqslant j \\ a_1, a_2, \dots, a_s}} A_1 \otimes \dots \otimes A_{i_1} \otimes e^{a_1} \otimes A_{i_1+1} \otimes \dots \otimes A_{i_2} \otimes e^{a_2} \otimes A_{i_2+1} \otimes \cdots \\
&\quad \otimes A_{i_{s-1}} \otimes e^{a_{s-1}} \otimes A_{i_{s-1}+1} \otimes \dots \otimes A_{i_s} \otimes e^{a_s} \otimes \\
&\quad \big[[\dots[\gamma, e_{a_1}]_{\mathfrak{p}_M}^E, e_{a_2}]_{\mathfrak{p}_M}^E, \dots e_{a_{s-1}}]_{\mathfrak{p}_M}^E, e_{a_s}\big]_{\mathfrak{p}_M}^E \cdot (A_{i_s+1} \otimes \dots \otimes A_j \otimes v).
\end{aligned}$$

In these formulae e_a is a frame of TM with dual frame e^a.

This action not only gives an explicit formula for $\partial_\gamma \hat{j}_D$: it also provides an explicit inductive formula for the Ricci-corrected powers $D^{(k)}$ of the Weyl connection D. Indeed, since the order $k+1$ part of $j_\theta^1 \hat{j}_\theta^k \varphi$ is the same as that of $\hat{j}_\theta^{k+1} \varphi$, we obtain:

$$\iota_X D^{(k+1)} \varphi = D_X D^{(k)} \varphi + \mathrm{proj}_{(\otimes^k T^*M) \otimes V} \big(r^D(X) * \hat{j}_D^k \varphi\big)$$

$$= D_X D^{(k)}\varphi + \sum_{j+s=k} r^D(X) *_s D^{(j)}\varphi. \tag{4.7}$$

Explicit formulae for invariant differential operators will follow by computing projections of $D^{(k)}$ using this inductive expression and some representation theory.

5. Strongly invariant operators

5.1. Jet module homomorphisms

Our goal is to explain how Ricci-corrected Weyl differentiation leads to explicit formulae for a class of invariant differential operators. These are the *strongly invariant* operators of [15,16,20], defined as follows. Let $\mathbb{V}$ and $\mathbb{W}$ be P-modules and let $\Phi : \hat{J}^k_0\mathbb{V} \to \mathbb{W}$ be a P-homomorphism. Then Φ induces a bundle map $F : \mathcal{G} \times_P \hat{J}^k_0\mathbb{V} \to \mathcal{G} \times_P \mathbb{W}$, and hence an invariant differential operator $F \circ \hat{j}^k_\theta$ from V to W.

In practice, such P-homomorphisms are constructed by lifting a P_0-homomorphism $\operatorname{gr}\Phi : \operatorname{gr}\hat{J}^k_0\mathbb{V} \to \operatorname{gr}\mathbb{W}$ using an algebraic Weyl structure ε to give $\Phi_\varepsilon = \varepsilon_{\mathbb{W}}^{-1} \circ \operatorname{gr}\Phi \circ \varepsilon_{\mathbb{V}}$. Since $\Phi_{(\mathrm{Ad}\, q)\varepsilon} v = q \cdot \Phi_\varepsilon(q^{-1} \cdot v)$, it follows that $\Phi = \Phi_\varepsilon$ is a P-homomorphism if and only if it is independent of the algebraic Weyl structure ε.

If we mirror this construction on associated bundles, for any Weyl structure E, a bundle map $\operatorname{gr} F : \operatorname{gr}\hat{J}^k V \to \operatorname{gr} W$ (associated to $\operatorname{gr}\Phi : \operatorname{gr}\hat{J}^k_0\mathbb{V} \to \operatorname{gr}\mathbb{W}$) induces a differential operator $E_W^{-1} \circ \operatorname{gr} F \circ E_{\hat{J}^k V} \circ \hat{j}^k_\theta$ from V to W, which will be invariant if it is independent of the choice of Weyl structure E. An obvious sufficient condition is that $E_W^{-1} \circ \operatorname{gr} F \circ E_{\hat{J}^k V}$ is independent of the choice of Weyl structure and these are the strongly invariant operators. (The condition is not necessary because $\hat{j}^k_\theta\varphi$ will satisfy some Bianchi identities not satisfied by general sections of $\hat{J}^k V$.)

The method we shall adopt for constructing strongly invariant operators is to construct a P-homomorphism $\Phi : \hat{J}^k_\varepsilon\mathbb{V} \to \mathbb{W}$, inducing a bundle map $F : \hat{J}^k_E V \to W$ and hence, for any Weyl structure E, a differential operator $F \circ \hat{j}^k_D$ from V to W. Since $\hat{J}^k_\varepsilon\mathbb{V}$ has the same associated graded module as $\hat{J}^k_0\mathbb{V}$, it is straightforward to obtain conditions that Φ induces a P-homomorphism $\hat{J}^k_0\mathbb{V} \to \mathbb{W}$, and hence a strongly invariant operator. The expression $F \circ \hat{j}^k_D$ then gives an explicit formula for this operator in terms of a Weyl structure.

The P-homomorphisms we construct here all factor through the projections

$$\hat{J}^k_\varepsilon\mathbb{V} \to \otimes^k\mathfrak{m}^* \otimes \mathbb{V} \to \otimes^k\mathfrak{f}^* \otimes \mathbb{V},$$

where the second projection is induced by the restriction map $\mathfrak{m}^* \to \mathfrak{f}^*$. Our first task is to apply these projections to the jet module action $*$. Then we apply further projections $(\otimes^k\mathfrak{f}^*) \otimes \mathbb{V} \to \mathbb{W}$ to obtain the P-homomorphisms we seek. In geometric terms, operators obtained from homomorphisms factoring through these projections are given by applying a bundle map $(\otimes^k\mathfrak{f}^*_M) \otimes V \to W$ to the restriction of $D^{(k)}$ to $\otimes^k\mathfrak{f}_M$.

5.2. Restricting the jet module action

We first restrict $\gamma * \psi$ to $\mathfrak{f}$. For notational simplicity we give the result algebraically: the formulae on associated bundles easily follow.

Proposition 5.1. *Suppose $\mathbb{V}$ is a semisimple P-module. Let $\pi_{\mathfrak{f}}$ denote the restriction maps $\otimes^j \mathfrak{m}^* \to \otimes^j \mathfrak{f}^*$ and let e_a, e^a be dual bases for $\mathfrak{f}$, $\mathfrak{f}^*$. Then $(\gamma * \psi)_\ell = \sum_{j+s=\ell} \gamma *_s \psi_j$, where for each j we have, supposing $\psi_j = A_1 \otimes \cdots \otimes A_j \otimes v$ as before,*

$$\begin{aligned}
\pi_{\mathfrak{f}}(\gamma *_0 \psi_j) &= 0,\\
\pi_{\mathfrak{f}}(\gamma *_1 \psi_j) &= \sum_{0\leqslant i\leqslant j} \pi_{\mathfrak{f}}(A_1 \otimes A_2 \otimes \cdots \otimes A_i) \otimes \Psi\big(\gamma \otimes \pi_{\mathfrak{f}}(A_{i+1} \otimes \cdots \otimes A_j \otimes v)\big),\\
\pi_{\mathfrak{f}}(\gamma *_s \psi_j) &= \sum_{\substack{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_s\leqslant j\\ a_1,a_2,\dots,a_{s-1}}} \pi_{\mathfrak{f}}(A_1 \otimes \cdots \otimes A_{i_1}) \otimes e^{a_1} \otimes \pi_{\mathfrak{f}}(A_{i_1+1} \otimes \cdots \otimes A_{i_2}) \otimes e^{a_2} \otimes \cdots\\
&\quad \otimes e^{a_{s-1}} \otimes \pi_{\mathfrak{f}}(A_{i_{s-1}+1} \otimes \cdots \otimes A_{i_s}) \otimes\\
&\quad \Psi\big(\big[\dots[[\gamma, e_{a_1}]^\varepsilon_{\mathfrak{p}}, e_{a_2}]^\varepsilon_{\mathfrak{p}}, \dots e_{a_{s-1}}\big]^\varepsilon_{\mathfrak{f}^*} \otimes \pi_{\mathfrak{f}}(A_{i_s+1} \otimes \cdots \otimes A_j \otimes v)\big).
\end{aligned}$$

Here, for any semisimple P-module $\tilde{\mathbb{V}}$, we define $\Psi : \mathfrak{f}^* \otimes \tilde{\mathbb{V}} \to \mathfrak{f}^* \otimes \tilde{\mathbb{V}}$ by

$$\Psi(A \otimes \tilde{v})(\chi) = [A, \chi] \cdot \tilde{v} \tag{5.1}$$

for $\chi \in \mathfrak{f}$, which is well defined since $\tilde{\mathbb{V}}$ is semisimple. In the above formulae for $*_1$ and $*_s$, we have $\tilde{\mathbb{V}} = (\otimes^{j-i}\mathfrak{f}^*) \otimes \mathbb{V}$ and $(\otimes^{j-i_s}\mathfrak{f}^*) \otimes \mathbb{V}$ respectively.

Proof. This is immediate from (the algebraic version of) Proposition 4.6, and the equality

$$\begin{aligned}
&\pi_{\mathfrak{f}}\big(\big[[\dots[[\gamma, e_{a_1}]^\varepsilon_{\mathfrak{p}}, e_{a_2}]^\varepsilon_{\mathfrak{p}}, \dots e_{a_{s-1}}]^\varepsilon_{\mathfrak{p}}, e_{a_s}\big]^\varepsilon_{\mathfrak{p}} \cdot (A_{i_s+1} \otimes \cdots \otimes A_j \otimes v)\big)\\
&\quad = \big[[\dots[[\gamma, e_{a_1}]^\varepsilon_{\mathfrak{p}}, e_{a_2}]^\varepsilon_{\mathfrak{p}}, \dots e_{a_{s-1}}]^\varepsilon_{\mathfrak{f}^*}, e_{a_s}\big]^\varepsilon_{\mathfrak{p}_0} \cdot \big(\pi_{\mathfrak{f}}(A_{i_s+1} \otimes \cdots \otimes A_j) \otimes v\big),
\end{aligned}$$

which holds because the projection of the Lie bracket $\mathfrak{p}^\perp \otimes \mathfrak{p}^\perp \to \mathfrak{p}^\perp \to \mathfrak{f}^*$ vanishes and the module $(\otimes^{j-i_s}\mathfrak{f}^*) \otimes \mathbb{V}$ is semisimple. □

The formulae of this proposition may not seem simpler than the full formulae of Proposition 4.6, but they are easier to handle, since Ψ is an operator on semisimple modules.

5.3. Special types of projections

To progress further, we need to understand the map

$$\Psi : \mathfrak{f}^* \otimes \mathbb{V} \to \mathfrak{f}^* \otimes \mathbb{V}, \qquad \Psi(A \otimes v) = \sum_a e^a \otimes \big([A, e_a] \cdot v\big),$$

where $\mathbb{V}$ is a semisimple P-module and e_a and e^a are dual bases of $\mathfrak{f}$ and $\mathfrak{f}^*$ respectively. Since Ψ is a P-homomorphism, it acts by a scalar on every irreducible component of $\mathfrak{f}^* \otimes \mathbb{V}$ and these scalars can be computed explicitly using Casimirs. Note now that all weights of the $\mathfrak{p}_0$-module $\mathfrak{m}^*$ have multiplicity one (they are just positive roots of $\mathfrak{g}$). Hence results from [5,27,31] show that all irreducible components of $\mathfrak{m}^* \otimes \mathbb{V}$ are multiplicity free.

We can write $\mathfrak{f}^* \otimes \mathbb{V}$ as a sum of well-defined irreducible P_0-components $\mathbb{V}_b$. Consequently, $(\otimes^2\mathfrak{f}^*) \otimes \mathbb{V} = \mathfrak{f}^* \otimes (\bigoplus_{b_1} \mathbb{V}_{b_1})$ can be again written as a sum of invariant subspaces labelled by a couple (b_1, b_2) of indices indicating that it is the isotypic component with label b_2 inside $\mathfrak{f}^* \otimes \mathbb{V}_{b_1}$. Inductively, we get a well defined decomposition of $(\otimes^k \mathfrak{f}^*) \otimes \mathbb{V}$ into P_0-invariant subspaces labelled by paths $b = (b_1, \dots, b_k)$ of indices showing their positions in consecutive decompositions. Note that the full isotypic component of $(\otimes^k \mathfrak{f}^*) \otimes \mathbb{V}$ with label b_k is the direct sum of all subspaces $\mathbb{V}_b$ labelled by paths b ending with b_k.

We now suppose that our component $\mathbb{V}_b$ is in the *symmetric* tensor product $S^k \mathfrak{f}^* \otimes \mathbb{V}$.

Proposition 5.2. *Let π be a projection of $\hat{J}^k_\varepsilon \mathbb{V}$ to an invariant subspace in $(S^k \mathfrak{f}^* \otimes \mathbb{V}) \cap \mathbb{V}_b$ for some $\mathbb{V}_b$ in the decomposition described above. Then for any element ψ of $\hat{J}^k_\varepsilon \mathbb{V}$, the only contribution to $\pi(\gamma * \psi)$ is from ψ_{k-1} and if $\psi = A_1 \otimes \dots \otimes A_{k-1} \otimes v$ we have*

$$\pi(\gamma * \psi) = \sum_{0 \leqslant i \leqslant k-1} \pi\bigl(A_1 \otimes A_2 \otimes \dots \otimes A_i \otimes \Psi\bigl(\pi_{\mathfrak{f}}(\gamma) \otimes \pi_{\mathfrak{f}}(A_{i+1} \otimes \dots \otimes A_{k-1} \otimes v)\bigr)\bigr).$$

Proof. Note first that $\pi(\gamma * \psi) = \sum_{j+s=k} \pi(\gamma *_s \psi_j)$ and that the $s = 0$ term vanishes (since $\mathbb{V}$ is semisimple). Hence it remains to show that the terms with $s \geqslant 2$ are zero. To do this is suffices to show that terms in which Ψ is applied to a Lie bracket are killed by the projection. Consider therefore an expression of the form

$$\sum_a e^a \otimes \Psi\bigl([\gamma, e_a]^\varepsilon_{\mathfrak{f}^*} \otimes v\bigr)$$

and suppose we apply a projection π to $(\otimes^2 \mathfrak{f}^*) \otimes \mathbb{V}$ which factors through $S^2 \mathfrak{f}^* \otimes \mathbb{V}$ and is of the form $\pi_2 \circ \mathrm{id} \otimes \pi_1$ where $\pi_1 : \mathfrak{f}^* \otimes \mathbb{V} \to \mathbb{V}_1$ is a projection onto an isotypic component. Since Ψ acts by a scalar on such a component, the result is a multiple of π applied to

$$\sum_a e^a \otimes [\gamma, e_a]^\varepsilon_{\mathfrak{f}^*} \otimes v.$$

The projection π factorizes through the symmetric product, so it suffices to note that

$$\left(\sum_a e^a \otimes [\gamma, e_a]^\varepsilon_{\mathfrak{f}^*}\right)(\chi_1, \chi_2) = [\gamma, \chi_1]^\varepsilon_{\mathfrak{f}^*}(\chi_2) = \gamma\bigl([\varepsilon_\bullet \chi_1, \varepsilon_\bullet \chi_2] \bmod \mathfrak{p}\bigr),$$

which is clearly antisymmetric in $\chi_1, \chi_2 \in \mathfrak{f}$. The terms appearing in the action for $s \geqslant 2$ are all of this form, with γ and v replaced by iterated brackets and suitable tensor products. $\square$

5.4. Casimir computations

In this section we compute the eigenvalues of Ψ and hence prove that Ψ acts by a scalar on each isotypic component. This is the first point at which we need detailed information from representation theory, so we set up the necessary notation [22].

Let $\mathfrak{g}$ be a semisimple Lie algebra with complexification $\mathfrak{g}^{\mathbb{C}}$. We choose a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$, a set Δ^+ of positive roots, and its subset $S = \{\alpha_1, \dots, \alpha_r\}$ of simple roots. Using the Killing form $(\cdot\,,\cdot)$, with any normalization, the fundamental weights $\omega_1, \dots, \omega_r$ are defined by $(\alpha_i^\vee, \omega_j) = \delta_{ij}$, where $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$.

The dominant Weyl chamber $\mathcal{C}$ is given by linear combinations of fundamental weights with non-negative coefficients. Finite dimensional complex irreducible representations of $\mathfrak{g}^{\mathbb{C}}$ (as well as of $\mathfrak{g}$) are characterized by their highest weights $\lambda \in \mathcal{C}$, which lie in the weight lattice $\{\sum \lambda_i \omega_i \colon \lambda_i \in \mathbb{Z}\}$. The corresponding representation will be denoted by $\mathbb{V}_\lambda$.

A reductive algebra is a direct sum of a commutative and a semisimple algebra (either of which can be trivial). Its irreducible (complex) representations are tensor products of irreducible representations of the summands, where irreducible representations of a commutative Lie algebra $\mathfrak{a}$ are one dimensional, characterized by an element of $\mathrm{Hom}(\mathfrak{a}, \mathbb{C})$.

Remark 5.3. For simplicity, we focus on complex representations of the Lie algebras in question. In practice, we may well be more interested in real representations. For this, it is sufficient to use the following description: a real or quaternionic structure on a complex $\mathfrak{g}$-module $\mathbb{V}$ is a conjugate-linear $\mathfrak{g}$-map $J : \mathbb{V} \to \mathbb{V}$ with $J^2 = 1$ or $J^2 = -1$ respectively. An irreducible real representation of a Lie algebra can be identified either with an irreducible complex representations, or with such a representation endowed with a real or quaternionic structure. We note also that if $\mathbb{V}$ is a complex $\mathfrak{g}$-module and $\mathbb{U}$ is a real $\mathfrak{g}$-module with complexification $\mathbb{U}^c$, then $\mathbb{U} \otimes \mathbb{V}$ and $\mathbb{U}^c \otimes_{\mathbb{C}} \mathbb{V}$ are equivalent as complex modules.

We now specialize to the situation where $\mathfrak{g}$ is a semisimple Lie algebra with parabolic subalgebra $\mathfrak{p}$ and Levi factor $\mathfrak{p}_0$. The set S of simple roots for $\mathfrak{g}^{\mathbb{C}}$ can be chosen in such a way that all positive root spaces are contained in $\mathfrak{p}^{\mathbb{C}}$. This fixes an algebraic Weyl structure, and the positive root spaces lying in $\mathfrak{p}_0^{\mathbb{C}}$ correspond to roots in the span of the subset S_0 of 'uncrossed' simple roots—we write $S = S_\times \cup S_0$ for the decomposition into crossed and uncrossed simple roots. In this situation, we shall say that a weight λ (integral for $\mathfrak{g}$) is dominant for $\mathfrak{p}$, if its restriction to $\mathfrak{h}_{ss}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{p}_{0,ss}^{\mathbb{C}}$ is dominant for $\mathfrak{p}_{0,ss}^{\mathbb{C}}$. Such a weight specifies uniquely an irreducible P-module.

Let us denote by δ the half sum of positive roots for $\mathfrak{g}^{\mathbb{C}}$ and by δ_0 the half sum of those positive roots for $\mathfrak{g}^{\mathbb{C}}$ for which the corresponding root space belongs to $\mathfrak{g}_0^{\mathbb{C}}$.

Proposition 5.4. *Let $\mathbb{V}_\lambda$ be an irreducible representation of $\mathfrak{p}_0$ with highest weight $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$. Let $\mathfrak{f}^* \otimes \mathbb{V}_\lambda = \bigoplus_{\mu \in A} \mathbb{V}_\mu$ be the decomposition of the tensor product into the sum of isotypic components with highest weight μ and let π_μ be the projection to $\mathbb{V}_\mu$. Let δ denote the half sum of positive roots for the Lie algebra $\mathfrak{g}$. Then*

$$\Psi = \sum c_\mu \pi_\mu,$$

with

$$c_\mu = \frac{1}{2}\big(|\mu + \delta|^2 - |\lambda + \delta|^2\big)$$

and $|\alpha|^2 = (\alpha, \alpha)$. (Note that Ψ depends upon the normalization of the Killing form, since we used it to identify $\mathfrak{p}^\perp$ with $(\mathfrak{g}/\mathfrak{p})^*$, and hence the bracket of $\mathfrak{f}^*$ with $\mathfrak{f}$ depends on $(\cdot\,,\cdot)$.)

Proof. We first give a formula for Ψ in terms of the Casimir operator C of $\mathfrak{p}_0$, as in [21,31]. Let E^i and E_i be bases for $\mathfrak{p}_0$ which are dual with respect to $(\cdot\,,\cdot)$. Then

$$\Psi(\gamma \otimes v) = \sum_a e^a \otimes [\gamma, e_a] \cdot v = \sum_i [E^i, \gamma] \otimes E_i \cdot v = \sum_i [E_i, \gamma] \otimes E^i \cdot v$$

and so

$$2\Psi(\gamma \otimes v) = \sum_i E^i \cdot E_i \cdot (\gamma \otimes v) - \gamma \otimes \left(\sum_i E^i \cdot E_i \cdot v\right) - \left(\sum_i E^i \cdot E_i \cdot \gamma\right) \otimes v$$

$$= C(\gamma \otimes v) - \gamma \otimes C(v) - C(\gamma) \otimes v.$$

It is well known that on an irreducible representation with highest weight λ, C acts by the scalar $(\lambda, \lambda + 2\delta_0)$: with the definition of δ_0 above, this holds even though $\mathfrak{p}_0$ is reductive rather than semisimple—see [31].

Let us write $S_\times = \{\beta_i\colon i = 1, \dots, r_\times \leqslant r\}$ for the crossed simple roots of $\mathfrak{g}$. We know that $-\beta_i$ are precisely the highest weights of the irreducible components $\mathfrak{f}_i^*$ of the P_0-module $\mathfrak{f}^*$, so that the irreducible components of $\mathfrak{f}_i^* \otimes \mathbb{V}_\lambda$ have highest weights of the form $\lambda - \beta_i + \gamma$, where γ is an integral linear combination of simple roots for $\mathfrak{p}_0^{\mathbb{C}}$. Hence for any isotypic component $\mathbb{V}_\mu$ there is an i so that $\mathbb{V}_\mu$ is an invariant subspace of $\mathfrak{f}_i^* \otimes \mathbb{V}_\lambda$.

Since the action of the Casimir depends only on the highest weight, we deduce, following [31], that Ψ acts on the entire isotypic component $\mathbb{V}_\mu$ by the scalar

$$c = \frac{1}{2}\big[(\mu, \mu + 2\delta_0) - (\lambda, \lambda + 2\delta_0) - (-\beta_i, -\beta_i + 2\delta_0)\big].$$

It remains to identify c with the constant c_μ above. For any simple root β, we have

$$2(\delta, \beta) = \big(\delta, 2\beta/|\beta|^2\big)|\beta|^2 = \sum_{j=1}^{n} (\omega_j, \beta^\vee)|\beta|^2 = |\beta|^2.$$

Hence $(\mu, \mu + 2\delta_0) - (\lambda, \lambda + 2\delta_0) - (-\beta_i, -\beta_i + 2\delta_0)$ is given by

$$\begin{aligned}&(\mu, \mu + 2\delta) - (\lambda, \lambda + 2\delta) - 2(\delta - \delta_0, \mu - \lambda) - 2(\beta_i, \delta - \delta_0)\\ &\quad = |\mu + \delta|^2 - |\lambda + \delta|^2 + 2(\delta - \delta_0, -\beta_i - \mu + \lambda).\end{aligned}$$

We know that $\mu - \lambda$ is a weight of $\mathfrak{f}_i^*$, hence $-\beta_i - (\mu - \lambda) = \sum_{\alpha \in S_0} n_\alpha \alpha$. But for all $\alpha \in S_0$, we have $(\alpha, \delta - \delta_0) = 0$, so the last term vanishes. □

6. Explicit constructions of invariant operators

In this section, we are going to construct a large class of invariant differential operators. Most of them belong to the class of standard regular operators, but a certain subclass are standard singular operators. The class of operators constructed here does not cover all standard regular operators, but we shall see in examples that it covers many of them.

Various special cases of the main results of this section can be found in [6,15,16,21,31].

Theorem 6.1. *Let $\alpha \in (\mathfrak{h}^{\mathbb{C}})^*$ be a positive root with $\mathfrak{g}_\alpha \subset \mathfrak{f}^*$. In the case that $\mathfrak{g}$ has roots of different lengths, we shall suppose that α is a long root. Let λ, μ be two integral dominant weights of $\mathfrak{p}_0$ with the property*

$$\mu + \delta = \sigma_\alpha(\lambda + \delta) = \lambda + \delta - (\lambda + \delta, \alpha^\vee)\alpha.$$

Interchanging λ and μ if necessary, suppose that $k := -(\lambda + \delta, \alpha^\vee)$ is positive.

(i) *There is a unique irreducible component $\mathbb{V}_\mu$ with highest weight μ in $(\otimes^k \mathfrak{m}^*) \otimes \mathbb{V}_\lambda$. Furthermore, $\mathbb{V}_\mu$ belongs to $S^k\mathfrak{f}^* \otimes \mathbb{V}_\lambda$ and is of the form $\mathbb{V}_b$, where $\mathbb{V}_{b_j} = \mathbb{V}_{\lambda + j\alpha}$ for $b = (b_1, \dots, b_k)$.*

(ii) *If $\pi : \hat{J}^k_\varepsilon \mathbb{V}_\lambda \to \mathbb{V}_\mu$ is the corresponding projection, then π induces a P-homomorphism $J^k_0 \mathbb{V}_\lambda \to \mathbb{V}_\mu$ and hence a strongly invariant differential operator $\pi \circ D^{(k)}$ of order k from sections of V_λ to sections of V_μ.*

Proof. (i) To prove uniqueness, let us note first that all weights of $\mathfrak{m}^*$ are positive roots. Weights of $\otimes^k \mathfrak{m}^*$ are hence sums of k positive roots. The highest weight of any irreducible components of $(\otimes^k \mathfrak{m}^*) \otimes \mathbb{V}_\lambda$ is of a form $\lambda + \beta$, where β is a weight of $\otimes^k \mathfrak{m}^*$. The unicity claim is therefore true by the triangle inequality, α being a long root.

We now show that there is such a component $\mathbb{V}_\mu$. By assumption, both λ and $\mu = \lambda + k\alpha$ are P-dominant, and $j\alpha$ is an extremal weight of $\boxtimes^j \mathfrak{f}_i^*$ for all $j = 1, \ldots, k$. The so-called Parthasarathy–Ranga-Rao–Varadarajan conjecture (proved in [28]) states that if λ, ν are highest weights of two irreducible (complex) $\mathfrak{p}_0$-modules $\mathbb{V}_\lambda, \mathbb{V}_\nu$, and if α is an extremal weight of $\mathbb{V}_\nu$, then an irreducible component $\mathbb{V}_{\lambda+\alpha}$ with extremal weight $\lambda + \alpha$ will appear with multiplicity at least one in $\mathbb{V}_\lambda \otimes_{\mathbb{C}} \mathbb{V}_\nu$. (In concrete cases, more elementary arguments are available.) It follows that there is an irreducible component of $(\boxtimes^j \mathfrak{f}_i^*) \otimes \mathbb{V}_\lambda$ having $\lambda + j\alpha$ as its highest weight. Hence necessarily $\mathbb{V}_\mu \subset (\boxtimes^k \mathfrak{f}_i^*) \otimes \mathbb{V}_\lambda \subset S^k \mathfrak{f}_i^* \otimes \mathbb{V}_\lambda$. Furthermore, the same holds with λ replaced by $\lambda + j'\alpha$ for all $j' = 1, \ldots, k-1$. Hence, by uniqueness, the projection factors through $(\boxtimes^j \mathfrak{f}_i^*) \otimes \mathbb{V}_\lambda$.

(ii) Let us prove now that π induces a P-homomorphism. The action of $\mathfrak{m}^*$ on $\mathbb{V}_\mu$ is trivial, hence we must show that $\pi(\gamma * \psi)$ vanishes for any $\psi \in \hat{J}^k_\varepsilon \mathbb{V}_\lambda$ and any $\gamma \in \mathfrak{m}^*$.

Since π is the projection to an irreducible piece of $S^k \mathfrak{f}^* \otimes \mathbb{V}_\lambda$ lying in a component of the form $\mathbb{V}_b$, the action is given by Proposition 5.2. Using the Casimir computation of Proposition 5.4, the projection of the action by γ on an element $\psi \in \hat{J}^k \mathbb{V}$ is given by

$$\pi(\gamma * \psi) = c\pi(\gamma \otimes \psi_{k-1})$$

with

$$2c = \sum_{j=1}^{k} \bigl(|\lambda + j\alpha + \delta|^2 - |\lambda + (j-1)\alpha + \delta|^2\bigr) = |\mu + \delta|^2 - |\lambda + \delta|^2,$$

which is zero because $\mu + \delta = \sigma_\alpha(\lambda + \delta)$ and σ_α is an isometry. □

We turn now the formulae for these operators in terms of a Weyl structure. Explicit formulae for the coefficients of various curvature terms for standard operators were first found in the conformal case [3, 24], and later extended to the $|1|$-graded case [2,15]. A very surprising fact was that the formulae were quite universal and did not depend on the specific parabolic structure or on the highest weights of the representations involved. The general structure of coefficients described in [15] was quite complicated. Here we notice that the organization of the terms produced by the Ricci-corrected derivative leads to much simpler coefficients. At the same time, the formulae are extended from the $|1|$-graded case to the broad class of standard operators in all parabolic geometries with no extra complications: the form of the operator depends only on its order.

Theorem 6.2. *Let a positive integer k and a long root α with $\mathfrak{g}_\alpha \subset \mathfrak{f}^*$ be given. Define differential operators $\mathcal{D}_{k,j}$ (of order $j = 0, \ldots, k$), acting on sections of any associated bundle, by the recurrence relation*

$$\iota_X \mathcal{D}_{k,j+1} = D_X \circ \mathcal{D}_{k,j} + j(k-j)\Gamma(X) \otimes \mathcal{D}_{k,j-1} \tag{6.1}$$

with $\mathcal{D}_{k,0} = \mathrm{id}$, $\mathcal{D}_{k,1} = D$. Here D be the covariant derivative given by the choice of the Weyl structure and $\Gamma = -\frac{1}{2}|\alpha|^2 r^D$. Let $\mathcal{D}_k = \mathcal{D}_{k,k}$. Then any invariant operator of order k constructed in Theorem 6.1, mapping sections of V_λ to sections of V_μ, $\mu = \lambda + k\alpha$ is given by $\pi \circ \mathcal{D}_k$ where π is the projection onto V_μ.

Proof. We know that the invariant operator is given by $\pi \circ D^{(k)}$ and we have given a recurrence formula for $D^{(k)}$ in Section 4. Hence we only have to compute the projection of the action of r^D on $S^j\mathfrak{f}^* \otimes \mathbb{V}_\lambda$, which is straightforward using the results of Sections 5.2–5.4 we find that the action is the tensor product with cr^D, where $2c = |\lambda + j\alpha + \delta|^2 - |\lambda + \delta|^2$.

Now, since $k = -(\lambda + \delta, \alpha^\vee)$, we have

$$2c = |\lambda + j\alpha + \delta|^2 - |\lambda + \delta|^2 = (2\lambda + j\alpha + 2\delta, j\alpha) = |\alpha|^2 j(-k + j).$$

Substituting this into the projection of the recurrence formula for $D^{(j+1)}$ gives (6.1). □

Hence the universal nature of the explicit formulae for invariant operators arises from the fact that $\mathcal{D}_k$ only depends upon k. It is straightforward to compute $\mathcal{D}_k$ for small k. Omitting the tensor product sign when tensoring with $\Gamma^j = \Gamma \otimes \cdots \otimes \Gamma$, we have

$$\begin{aligned}
\mathcal{D}_1 s &= Ds,\\
\mathcal{D}_2 s &= D^2 s + \Gamma s,\\
\mathcal{D}_3 s &= D^3 s + 2D(\Gamma s) + 2\Gamma Ds,\\
\mathcal{D}_4 s &= D^4 s + 3D^2(\Gamma s) + 4D(\Gamma Ds) + 3\Gamma D^2 s + 9\Gamma^2 s,\\
\mathcal{D}_5 s &= D^5 s + 4D^3(\Gamma s) + 6D^2(\Gamma Ds) + 6D(\Gamma D^2 s) + 4\Gamma D^3 s\\
&\quad + 24D(\Gamma^2 s) + 16\Gamma D(\Gamma s) + 24\Gamma^2 Ds,\\
\mathcal{D}_6 s &= D^6 s + 5D^4(\Gamma s) + 8D^3(\Gamma Ds) + 9D^2(\Gamma D^2 s) + 8D(\Gamma D^3 s) + 5\Gamma D^4 s\\
&\quad + 45D^2(\Gamma^2 s) + 40D(\Gamma D(\Gamma s)) + 25\Gamma D^2(\Gamma s)\\
&\quad + 64D(\Gamma^2 Ds) + 40\Gamma D(\Gamma Ds) + 45\Gamma^2 D^2 s + 225\Gamma^3 s,\\
\mathcal{D}_7 s &= D^7 s + 6D^5(\Gamma s) + 10D^4(\Gamma Ds) + 12D^3(\Gamma D^2 s)\\
&\quad + 12D^2(\Gamma D^3 s) + 10D(\Gamma D^4 s) + 6\Gamma D^5 s\\
&\quad + 72D^3(\Gamma^2 s) + 72D^2\big(\Gamma D(\Gamma s)\big) + 120D^2(\Gamma^2 Ds) + 60D\big(\Gamma D^2(\Gamma s)\big)\\
&\quad + 100D\big(\Gamma D(\Gamma Ds)\big) + 36\Gamma D^3(\Gamma s)\\
&\quad + 60\Gamma D^2(\Gamma Ds) + 120D(\Gamma^2 D^2 s) + 72\Gamma D(\Gamma D^2 s) + 72\Gamma^2 D^3 s\\
&\quad + 720D(\Gamma^3 s) + 432\Gamma D(\Gamma^2 s) + 432\Gamma^2 D(\Gamma s) + 720\Gamma^3 Ds,\\
\mathcal{D}_8 s &= D^8 s + 7D^6(\Gamma s) + 12D^5(\Gamma Ds) + 15D^4(\Gamma D^2 s)\\
&\quad + 16D^3(\Gamma D^3 s) + 15D^2(\Gamma D^4 s) + 12D(\Gamma D^5 s) + 7\Gamma D^6 s\\
&\quad + 105D^4(\Gamma^2 s) + 112D^3\big(\Gamma D(\Gamma s)\big) + 192D^3(\Gamma^2 Ds) + 105D^2\big(\Gamma D^2(\Gamma s)\big)\\
&\quad + 180D^2\big(\Gamma D(\Gamma Ds)\big) + 84D\big(\Gamma D^3(\Gamma s)\big) + 225D^2(\Gamma^2 D^2 s)\\
&\quad + 144D\big(\Gamma D^2(\Gamma s)\big) + 49\Gamma D^4(\Gamma s) + 84\Gamma D^3(\Gamma Ds) + 180D\big(\Gamma D(\Gamma D^2 s)\big)
\end{aligned}$$

$$
\begin{aligned}
&+105\Gamma D^2(\Gamma D^2 s)+192D(\Gamma^2 D^3 s)+112\Gamma D(\Gamma D^3 s)+105\Gamma^2 D^4 s\\
&+1575D^2(\Gamma^3 s)+1260D\big(\Gamma D(\Gamma^2 s)\big)+1344D\big(\Gamma^2 D(\Gamma s)\big)+735\Gamma D^2(\Gamma^2 s)\\
&+2304D(\Gamma^3 Ds)+784\Gamma D\big(\Gamma D(\Gamma s)\big)\\
&+735\Gamma^2 D^2(\Gamma s)+1344\Gamma D(\Gamma^2 Ds)+1260\Gamma^2 D(\Gamma Ds)+1575\Gamma^3 D^2 s\\
&+11025\Gamma^4 s.
\end{aligned}
$$

The combinatorics of the coefficients are simpler than in [15] and the numbers are generally smaller; the formulae there are obtained from those here by expanding the derivatives of Γ using the product rule. Notice that the coefficients depend only on the position of the Γ's, and the coefficients of the nonlinear terms in Γ are easily computed as products of the coefficients of the linear terms, as is clear from the inductive definition of each $\mathcal{D}_k$.

We are still free to choose the normalization of the Killing form $(\cdot\,,\cdot)$: since $-\frac{1}{2}|\alpha|^2$ is independent of the long root α, we could arrange that this is 1 and $\Gamma = r^D$. This is the normalization that gives the formulae stated in the introduction for the conformal case.

7. Scope of the construction

We now show that the class of operators constructed in the previous section includes many standard invariant operators, at least for the 'large' parabolic subgroups occuring in interesting examples. We shall also show that in conformal geometry, the operators we construct include (at least in the conformally flat case) those coming from AdS/CFT correspondence for partially massless fields in string theory.

7.1. Lagrangian contact structure

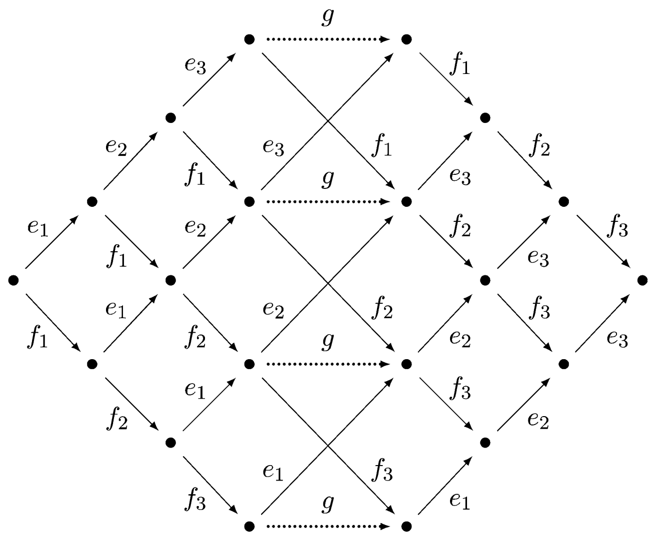
Let us consider the case of a Lagrangian contact structure (see [9,30,32]). This is the real split case of the complex parabolic algebra corresponding to the Dynkin diagram

$$\overset{\alpha_1}{\times}\!\!-\!\!\overset{\alpha_2}{\bullet}\!-\cdots-\!\overset{\alpha_n}{\bullet}\!\!-\!\!\overset{\alpha_{n+1}}{\times},\quad n\geqslant 1. \tag{7.1}$$

The Lie group is $G = PSL(n+2,\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+2,\mathbb{R})$, $\operatorname{gr}\mathfrak{g}$ being equipped with the $|2|$-grading given by block matrices of size $1, n, 1$. The $\mathfrak{g}_1$ part decomposes further into a direct sum $\mathfrak{g}_1 = \mathfrak{g}_1^L \oplus \mathfrak{g}_1^R$ of two irreducibles; the $\mathfrak{g}_2$ part is one-dimensional.

In geometric terms, we have a contact structure on a real manifold of dimension $2n+1$ with a direct sum decomposition of the contact distribution into two Lagrangian subbundles.

To be explicit, we consider the case $n = 3$, when the two irreducible components of $\mathfrak{g}_{-1}$ are three dimensional. Let us denote roots corresponding to both by e_1, e_2, e_3 and f_1, f_2, f_3 respectively, ordered so that e_1, f_1 are the highest weights for $\mathfrak{g}_1$, considered as a $\mathfrak{p}_0$-module, and e_3, f_3 are the lowest ones. Let g be the root corresponding to $\mathfrak{g}_{-2}$.



The (labelled) Hasse diagram for standard operators is shown above. The labels on the arrows indicate the ‘directions’ α for the corresponding operators. We have constructed all operators indicated by full arrows, hence only the horizontal arrows are missing.

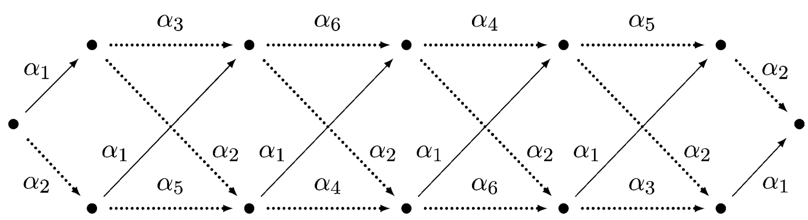
A similar diagram applies in CR geometry, this being another real form of Lagrangian contact geometry, except that some representations and operators become conjugate.

7.2. G_2-case

For a more exotic example, let us consider the split real case of the G_2 complex algebra. In this case, the root system has 12 elements. If we denote simple roots by α_1 (the longer one) and α_2 (the shorter one), then the set of positive roots is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ with $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = \alpha_1 + 2\alpha_2$, $\alpha_5 = \alpha_1 + 3\alpha_2$, $\alpha_6 = 2\alpha_1 + 3\alpha_2$. Let us consider the case that the parabolic is a Borel (maximal solvable) subgroup of G_2. Then the associated graded algebra $\operatorname{gr}\mathfrak{g}$ is $|5|$-graded with

$$\mathfrak{g}_1 = \mathbb{R} \cdot \{\alpha_1\} \oplus \mathbb{R} \cdot \{\alpha_2\}, \qquad \mathfrak{g}_2 = \mathbb{R} \cdot \{\alpha_3\}, \qquad \mathfrak{g}_3 = \mathbb{R} \cdot \{\alpha_4\},$$
$$\mathfrak{g}_4 = \mathbb{R} \cdot \{\alpha_5\}, \qquad \mathfrak{g}_5 = \mathbb{R} \cdot \{\alpha_6\}.$$

The (labelled) Hasse diagram then has the form



The operators constructed in Section 6, indicated by full arrows, are now not so numerous.

There are two other parabolic subgroups of G_2 up to isomorphism, one inducing a $|3|$-grading, the other a $|2|$-grading of the Lie algebra of G_2. For the $|3|$-grading, all roots in $\mathfrak{g}_1$ are short, and so no operators are constructed. For the $|2|$-grading, we have:

$$\mathfrak{g}_1 = \mathbb{R} \cdot \{\alpha_1\} \oplus \mathbb{R} \cdot \{\alpha_3\} \oplus \mathbb{R} \cdot \{\alpha_4\} \oplus \mathbb{R} \cdot \{\alpha_5\}, \qquad \mathfrak{g}_2 = \mathbb{R} \cdot \{\alpha_6\}.$$

The Hasse graph in this case is

$$\bullet \xrightarrow{\alpha_1} \bullet \overset{\alpha_3}{\dashrightarrow} \bullet \overset{\alpha_6}{\dashrightarrow} \bullet \overset{\alpha_4}{\dashrightarrow} \bullet \xrightarrow{\alpha_5} \bullet$$

with approximately half of the operators constructed in Section 6.

7.3. *The conformal case*

In the even-dimensional case, *all* operators in the BGG sequence are obtained by the construction in Section 6, although there are nonstandard operators which are not. In the odd dimensional case there is one arrow in the Hasse diagram which has a special character, as was already noted in [24]. In dimension $2n-1$, $\mathfrak{g} = \mathfrak{so}(p,q,\mathbb{R})$ with $p+q = 2n+1$ and a $|1|$-grading. We denote positive roots with root spaces included in $\mathfrak{g}_1$ by $\pm\alpha_1, \dots, \pm\alpha_n, \alpha_{n+1}$, where α_{n+1} is the short simple root. Suppose that the roots $\alpha_1, \dots, \alpha_n$ are ordered, i.e., that α_1 is the highest among them and α_n is the smallest. Then the (labelled) Hasse diagram has the form

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \quad \cdots \quad \bullet \xrightarrow{\alpha_n} \bullet \overset{\alpha_{n+1}}{\dashrightarrow} \bullet \xrightarrow{-\alpha_n} \bullet \quad \cdots \quad \bullet \xrightarrow{-\alpha_2} \bullet \xrightarrow{-\alpha_1} \bullet$$

The middle operator is labelled by the only short root, hence the construction of Section 6 does not apply; however, the other operators are all constructed.

7.4. *AdS/CFT for partially massless fields*

The operators constructed in the conformal case include some operators closely related to the AdS/CFT correspondence for partially massless fields.

We recall that, besides massive or strictly massless fields, partially massless fields of higher spin were studied on vacuum Einstein manifolds with a cosmological constant [7]. In Dolan, Nappi and Witten [17], the following situation was considered. Let M be an Einstein manifold asymptotic to anti-de Sitter space of dimension 4 and let X be its boundary with the induced conformal structure [26]. Then the AdS/CFT correspondence [1] yields a correspondence between a partially massless field ϕ on M and a field L on X, satisfying a certain conformally invariant equation (called a 'partial conservation law' in [17]).

In the case that the field L is a symmetric traceless tensor field $L^{i_1,\dots,i_s}$, the equation is, to leading order,

$$\nabla_{i_1} \cdots \nabla_{i_{s-n}} L^{i_1 \dots i_s} + \cdots = 0.$$

The source and the target for the equation are easily identified and the operator is the last operator in the BGG sequence (described in Section 7.3 in the odd dimensional case). Hence Section 6 provides an explicit formula for a conformally invariant operator of this form.

A detailed study of the case $s=2$, $n=0$ in [17] leads to the equation on $\mathbb{R}^d$

$$\nabla_i \nabla_j L^{ij} + \frac{1}{d-2} R_{ij} L^{ij} = 0,$$

which agrees with the formula of Section 6 since the tracefree parts of $\frac{1}{n-2} R_{ij}$ and Γ agree.

The higher order equations with curvature corrections in Section 6 are natural candidates for the corresponding higher order equations (the cases with $s-n>2$) for the field L. As an example, let us consider the case $s=3, n=0$, where we get

$$\nabla_i\nabla_j\nabla_k L^{ijk} + \frac{2}{d-2}\big[\nabla_i(R_{jk}L^{ijk}) + R_{jk}\nabla_i L^{ijk}\big] = 0 \tag{7.2}$$

for a traceless symmetric tensor field with three indices on $\mathbb{R}^d$. (Of course, after expanding the covariant derivatives using the Leibniz rule, we obtain the formulae already present in [15,24].) In the conformally flat situation, this must be the equation arising from the AdS/CFT correspondence, because representation theory shows that conformally invariant operators are unique up to a multiple in this case. In general, there may be conformally invariant curvature corrections in the lower order terms.

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We would like to thank Fran Burstall and Jan Slovak for helpful discussions concerning Weyl structures.

Appendix A. Weyl structures as reductions

In this appendix we relate our approach to Weyl structures and the original approach of Čap and Slovák [12], who define a Weyl structure to be a P_0-equivariant section σ of $\pi_0 : \mathcal{G} \to \mathcal{G}_0$. For this definition to make sense, an algebraic Weyl structure ε must be fixed so that $\pi_0 : P \to P_0$ is split.

By means of the equation $yq(y) = \sigma(\pi_0(y))$ such a section σ is equivalent to a P-invariant trivialization $q : \mathcal{G} \to \exp \mathfrak{p}^\perp$ of the principal $\exp \mathfrak{p}^\perp$-bundle $\pi_0 : \mathcal{G} \to \mathcal{G}_0$, where the P-invariance means that $pq(yp)\pi_0(p)^{-1} = q(y)$ for all $p \in P$, $y \in \mathcal{G}$.

On the other hand, using the algebraic Weyl structure ε, any geometric Weyl structure on M is given by $\mathcal{E} = (\mathrm{Ad}\, q)\varepsilon$ for a unique P-invariant $q : \mathcal{G} \to \exp \mathfrak{p}^\perp$. To summarize:

Proposition A.1. *Let $(\mathcal{G} \to M, \theta)$ be a parabolic geometry and fix an algebraic Weyl structure ε. Then there is a natural bijection between Weyl structures E on M and P_0-equivariant sections σ of $\pi_0 : \mathcal{G} \to \mathcal{G}_0$.*

In our development, we used the Weyl structure $\mathcal{E}$ to give a P-invariant direct sum decomposition $\mathcal{E}_\bullet : \mathcal{G} \times \mathfrak{m} \oplus \mathfrak{p}_0 \oplus \mathfrak{m}^* \to \mathcal{G} \times \mathfrak{g}$ and hence write

$$\mathcal{E}_\bullet^{-1} \circ \theta = \theta_{\mathfrak{m}} + \theta_{\mathfrak{p}_0} + \theta_{\mathfrak{m}^*} = \theta_{\mathfrak{m}} + \mathcal{E}_\bullet^{-1} \circ \theta_{\mathfrak{p}} = \theta_{\mathfrak{m}} + \mathcal{E}_\bullet^{-1} \circ \theta_{\mathcal{E}} + \rho, \tag{A.1}$$

where $\theta_{\mathfrak{p}}$ and $\theta_{\mathcal{E}}$ are principal P-connections on $\mathcal{G}$ and ρ is a P-invariant $\mathfrak{p}^\perp$-valued horizontal 1-form inducing the normalized Ricci curvature r^D. On the other hand, an algebraic Weyl structure ε gives a fixed direct sum decomposition $\varepsilon_* : \mathfrak{m} \oplus \mathfrak{p}_0 \oplus \mathfrak{m}^* \to \mathfrak{g}$, related to $\mathcal{E}_\bullet$ by conjugating with the action of q, where $\mathcal{E} = (\mathrm{Ad}\, q)\varepsilon$.

Since the fixed decomposition is only P_0-invariant, Čap and Slovák define the *Weyl form* to be the pull back $\tau = \sigma^*\theta$ of θ to $\mathcal{G}_0$, then decompose τ into a solder form, a principal P_0-connection and a P_0-invariant $\mathfrak{p}^\perp$-valued 1-form. In our approach (A.1) is a P-invariant lift of this decomposition to $\mathcal{G}$.

Appendix B. Dependence of D and r^D on the Weyl structure

In Eq. (4.5), we obtained the (infinitesimal) dependence of the Ricci-corrected Weyl connection $D^{(1)}$ on the Weyl structure. We now do the same for the Weyl connection D and the normalized Ricci curvature r^D (we do not need these results in the body of the paper, but include them for general interest).

Proposition B.1. *For $\gamma \in C^\infty(M, \mathfrak{p}_M^\perp)$ and $X \in TM$, $\partial_\gamma r^D(X) = -D_X\gamma + [\gamma, X]^E_{\mathfrak{p}_M^\perp}$, where X is lifted to $\mathfrak{g}_M$ and the Lie bracket is projected onto $\mathfrak{p}_M^\perp$ using E.*

Proof. r^D is the $\mathfrak{p}_M^\perp$-valued 1-form on M induced by $\rho = \mathcal{E}^*\eta + \theta_{\mathfrak{m}^*}$, where $\mathcal{E}^*\eta_y = \eta_{\mathcal{E}(y)} d\mathcal{E}_y$ and $\theta_{\mathfrak{m}^*}$ is shorthand for the $\mathfrak{m}^*$ component $(\mathcal{E}_\bullet^{-1}\theta)_{\mathfrak{m}^*}$. Viewing γ as a P-invariant $\mathfrak{p}^\perp$-valued function on $\mathcal{G}$ with $\partial_\gamma\mathcal{E} = -\gamma$, we easily compute that $\partial_\gamma(\mathcal{E}^*\eta) = -d\gamma - [\gamma, \mathcal{E}^*\eta]$ and $\partial_\gamma\theta_{\mathfrak{m}^*} = (\mathcal{E}_\bullet^{-1}[\gamma, \theta])_{\mathfrak{m}^*} - [\gamma, \theta_{\mathfrak{m}^*}] = [\gamma, \mathcal{E}_\bullet\theta_{\mathfrak{p}_0}] + (\mathcal{E}_\bullet^{-1}[\gamma, \mathcal{E}_\bullet\theta_{\mathfrak{m}}])_{\mathfrak{m}^*}$. Hence

$$\partial_\gamma\rho = -\big(d\gamma - [\mathcal{E}^*\eta, \gamma] + [\mathcal{E}_\bullet\theta_{\mathfrak{p}_0}, \gamma]\big) + [\gamma, \theta_{\mathfrak{m}}]^{\mathcal{E}}_{\mathfrak{p}^\perp}$$

as required, since $\theta_{\mathcal{E}} = -\mathcal{E}^*\eta + \mathcal{E}_\bullet\theta_{\mathfrak{p}_0}$ is the principal connection inducing D. □

Proposition B.2. *Let V be a filtered P-bundle and φ a section of V. Then for $\gamma \in C^\infty(M, \mathfrak{p}_M^\perp)$ and $X \in TM$, $\partial_\gamma D_X\varphi = ([\gamma, X]^E_{\mathfrak{p}_{M,0}} + D_X\gamma)\cdot\varphi$.*

Proof. $D\varphi = E_V^{-1}D^{(1)}(E_V\varphi)$ and so $\partial_\gamma D_X\varphi = E_V^{-1}([\gamma, X]^E_{\mathfrak{p}_M}\cdot(E_V\varphi)) + D_X(\gamma\cdot\varphi) - \gamma\cdot D_X\varphi$. As $\operatorname{gr} V$ is semisimple, the result follows. □

Since $D^{(1)}_X\varphi = D_X\varphi + r^D(X)\cdot\varphi$, these computations are not independent: we check

$$\partial_\gamma(D_X\varphi + r^D(X)\cdot\varphi) = [\gamma, X]^E_{\mathfrak{p}_{M,0}}\cdot\varphi + [\gamma, X]^E_{\mathfrak{p}_M^\perp}\cdot\varphi = [\gamma, X]^E_{\mathfrak{p}_M}\cdot\varphi$$

in accordance with Eq. (4.5). Unlike $D^{(1)}$, the Weyl connection D does not depend algebraically on the Weyl structure. This, of course, was the whole reason for introducing Ricci corrections in the first place.

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