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A Theorem Relating a Certain Generalized Weyl Fractional Integral with the Laplace Transform and a Class of Whittaker Transforms

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In the present paper the authors prove a theorem which asserts an interesting relationship between the classical Laplace transform, a certain class of Whittaker transforms, and a Weyl fractional integral involving a general class of polynomials with essentially arbitrary coefficients. By specializing the various parameters involved, this general theorem would readily yield several (known or new) results involving simpler integral operators. It is also shown how the relationship asserted by the theorem can be applied to evaluate the generalized Weyl fractional integrals of various special functions. © 1990 Academic Press, Inc.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

Over two decades ago, Srivastava [14] considered an interesting unification of many familiar generalizations of the classical Laplace transform (cf., e.g., [27]; see also [5, Vol. I, Chaps. 4 and 5])

$$\mathscr{L}\{f(t): s\} = \int_0^\infty e^{-st} f(t)\, dt \tag{1.1}$$

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in the form:

$$\mathscr{S}_{q,\kappa,\mu}^{(\rho,\sigma)}\{f(t):s\}=\int_0^\infty e^{-qst/2}(st)^{\sigma-1/2}\,W_{\kappa,\mu}(\rho st)f(t)\,dt, \tag{1.2}$$

where $W_{\kappa,\mu}(z)$ denotes the Whittaker function of the second kind (cf. [26, p. 339, Sect. 16.12 et seq.]; see also [4, p. 264, Sect. 6.9]). Since

$$W_{0,\nu}(2z)=\sqrt{\frac{2z}{\pi}}\,K_\nu(z) \tag{1.3}$$

in terms of the modified Bessel function $K_\nu(z)$, the integral transform (1.2) contains each known generalization of the classical Laplace transform (1.1), involving the Bessel function $K_\nu(z)$ or the Whittaker function $W_{\kappa,\mu}(z)$ in the kernel. Furthermore, since

$$W_{\mu+1/2,\,\pm\mu}(z)=z^{\mu+1/2}e^{-z/2} \qquad \text{and} \qquad K_{\pm 1/2}(z)=\sqrt{\frac{\pi}{2z}}\,e^{-z}, \tag{1.4}$$

each of these generalizations would, in turn, reduce to the classical Laplace transform (1.1) upon suitably specializing the parameters q, κ, μ, ρ, and σ occurring in Srivastava's generalized Whittaker transform (1.2). For several interesting properties and characteristics of Srivastava's transform (1.2), see the subsequent works by (for example) Srivastava and Vyas [21], Srivastava [15], Srivastava and Panda [19], Sinha [13], Munot and Padmanabham [8], Tiwari and Ko [24], Rao [11], Malgonde and Saxena [7], Akhaury [1], and Carmichael and Pathak [2, 3]. Of our concern here is merely the following particular case of (1.2) considered earlier by Varma [25]:

$$\mathscr{V}_{\kappa,\mu}\{f(t):s\}=\mathscr{S}_{1,\kappa,\mu}^{(1,\mu)}\{f(t):s\}, \tag{1.5}$$

which, in view of the first relationship in (1.4), would reduce immediately to the classical Laplace transform (1.1) upon setting $\kappa=(1/2)-\mu$.

The main object of the present paper is to establish an interesting theorem which provides a useful relationship between the classical Laplace transform (1.1), the Varma (or *special* Whittaker) transform (1.5), and the generalized Weyl fractional integral defined by

$$\mathscr{W}_{\mu;\rho,\sigma}^{z;m,n}\{f(t):s\}=\frac{1}{\Gamma(\mu)}\int_s^\infty (t-s)^{\mu-1}\,S_n^m[zt^{-\sigma}(t-s)^\rho]f(t)\,dt, \tag{1.6}$$

where $S_n^m[x]$ denotes a general class of polynomials introduced by Srivastava (cf. [16, p. 1, Eq. (1)]):

$$S_n^m[x]=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}A_{n,j}x^j \qquad (n=0,1,2,\ldots). \tag{1.7}$$

Here, and in what follows, $(\lambda)_\nu = \Gamma(\lambda+\nu)/\Gamma(\lambda)$, m is an arbitrary positive integer, and the coefficients $A_{n,j}$ $(n, j \geqslant 0)$ are arbitrary constants, real or complex (see also Srivastava and Singh [20] and Srivastava and Garg [17]).

For $n=0$ and $A_{0,0}=1$, (1.6) would reduce immediately to the familiar Weyl fractional integral (cf., e.g., [5, Vol. II, Chap. 13]; see also [12]). Moreover, (1.6) with $\rho=\sigma$ corresponds to the one-dimensional case of the generalized Weyl fractional integral considered elsewhere by us [22].

The following result involving the classical Laplace transform will be required in our investigation (cf. [9, p. 24, Entry 3.22]; see also [5, Vol. I, p. 139, Entry 4.3(22) with $b=0$]):

$$\mathscr{L}\{t^{\mu}(t+\zeta)^{\nu} : s\} = \Gamma(\mu+1)\,\zeta^{(\mu+\nu)/2}\, s^{-1-(\mu+\nu)/2} e^{\zeta s/2} \cdot W_{(\nu-\mu)/2,\,(1+\nu+\mu)/2}(\zeta s) \tag{1.8}$$
$$(\operatorname{Re}(s)>0;\ \operatorname{Re}(\mu)>-1;\ |\arg(\zeta)|<\pi).$$

Making use of (1.8) and the definition (1.7), it is fairly straightforward to deduce

LEMMA 1. *Let*

$$\min\{\operatorname{Re}(s), \operatorname{Re}(\mu+\rho j), \operatorname{Re}(\sigma)\}>0$$
$$(j=0, 1, ..., [n/m];\ n=0, 1, 2, ...;\ m=1, 2, 3, ...). \tag{1.9}$$

Suppose also that $|\arg(\zeta)|<\pi$.

Then

$$\mathscr{L}\{t^{\mu-1}(t+\zeta)^{-\lambda}\, S_n^m[zt^{\rho}(t+\zeta)^{-\sigma}] : s\}$$
$$= e^{\zeta s/2} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!}\, \Gamma(\mu+\rho j)\, A_{n,j} z^j \zeta^{\beta(j)-1/2} s^{-\beta(j)-1/2} \cdot W_{\alpha(j),\,\beta(j)}(\zeta s), \tag{1.10}$$

where, for convenience,

$$\alpha(j) = \tfrac{1}{2}\{1-\mu-\lambda-(\rho+\sigma)j\}, \qquad \beta(j) = \tfrac{1}{2}\{\mu-\lambda+(\rho-\sigma)j\}. \tag{1.11}$$

We shall also require two important properties of the classical Laplace transform (1.1). For the sake of ready reference, we recall these properties as Lemma 2 and Lemma 3 below.

LEMMA 2 (cf., e.g., Erdélyi *et al.* [5, Vol. I, p. 129, Entry 4.1(8)]). *Suppose that the Laplace transform of each of the functions* $f^{(k)}(t)$ $(k=0, 1, ..., N)$ *exist. Also let*

$$F(s) = \mathscr{L}\{f(t) : s\}. \tag{1.12}$$

Then

$$\mathscr{L}\{f^{(N)}(t):s\}=s^N F(s)-\sum_{k=0}^{N-1} s^{N-k-1} f^{(k)}(0) \tag{1.13}$$

for every nonnegative integer N, an empty sum being interpreted as zero.

LEMMA 3 (The Parseval–Goldstein Theorem [6, p. 106, Eq. (8)]). *Let $F(s)$ be given by* (1.12), *and suppose that*

$$G(s)=\mathscr{L}\{g(t):s\}. \tag{1.14}$$

Then

$$\int_0^\infty f(t)\,G(t)\,dt=\int_0^\infty F(t)\,g(t)\,dt, \tag{1.15}$$

provided that each integral involved is absolutely convergent.

2. THE MAIN RESULT

We begin by stating our main result contained in the following

THEOREM. *Under the hypotheses of Lemma* 2, *let*

$$f(0)=f'(0)=\cdots=f^{(N-1)}(0)=0, \tag{2.1}$$

where N is a positive integer. Suppose that the hypothesis (1.9) *of Lemma* 1 *holds true, and let $\alpha(j)$ and $\beta(j)$ be defined by Eq.* (1.11).
Then

$$\begin{aligned}\mathscr{W}^{z;\,m,\,n}_{\mu;\,\rho,\,\sigma}&\{t^{N-\lambda}F(t):s\}\\ &=\sum_{j=0}^{[n/m]}(-n)_{mj}\,(\mu)_{\rho j}\,A_{n,j}\frac{z^j}{j!}\\ &\quad\cdot\mathscr{V}_{\alpha(j),\,\beta(j)}\{t^{-2\beta(j)}f^{(N)}(t):s\}\qquad(N=0,1,2,...),\end{aligned} \tag{2.2}$$

provided further that each member of (2.2) *exists.*

Proof. In view of the hypothesis (2.1), Lemma 2 immediately yields

$$s^N F(s)=\mathscr{L}\{f^{(N)}(t):s\}\qquad(N=0,1,2,...), \tag{2.3}$$

where $F(s)$ is given by (1.12).

By appealing appropriately to Lemma 1, we also have

$$\begin{aligned}
&\mathscr{L}\{t^{-\lambda}(t-\tau)^{\mu-1}\, S_n^m[zt^{-\sigma}(t-\tau)^\rho]\, H(t-\tau) : s\} \\
&= e^{-\tau s/2} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!}\, \Gamma(\mu+\rho j)\, A_{n,j} z^j \tau^{\beta(j)-1/2} s^{-\beta(j)-1/2} \\
&\quad \cdot W_{\alpha(j),\beta(j)}(\tau s) \qquad (\tau \geqslant 0),
\end{aligned} \tag{2.4}$$

where $H(t)$ denotes the Heaviside unit function, and $\alpha(j)$ and $\beta(j)$ are given by Eq. (1.11).

Now make use of the Laplace transform pairs (2.3) and (2.4) in Lemma 3, and we obtain

$$\begin{aligned}
&\int_0^\infty t^{N-\lambda}(t-\tau)^{\mu-1}\, S_n^m[zt^{-\sigma}(t-\tau)^\rho]\, H(t-\tau)\, F(t)\, dt \\
&= \int_0^\infty e^{-\tau t/2} f^{(N)}(t) \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!}\, \Gamma(\mu+\rho j)\, A_{n,j} z^j \\
&\quad \cdot \tau^{\beta(j)-1/2} t^{-\beta(j)-1/2} W_{\alpha(j),\beta(j)}(\tau t)\, dt,
\end{aligned} \tag{2.5}$$

or, equivalently,

$$\begin{aligned}
&\frac{1}{\Gamma(\mu)} \int_\tau^\infty t^{N-\lambda}(t-\tau)^{\mu-1}\, S_n^m[zt^{-\sigma}(t-\tau)^\rho]\, F(t)\, dt \\
&= \sum_{j=0}^{[n/m]} (-n)_{mj}\, (\mu)_{\rho j}\, A_{n,j} \frac{z^j}{j!} \\
&\quad \cdot \int_0^\infty e^{-\tau t/2} (\tau t)^{\beta(j)-1/2}\, t^{-2\beta(j)} f^{(N)}(t)\, W_{\alpha(j),\beta(j)}(\tau t)\, dt \\
&\quad (N = 0, 1, 2, \ldots),
\end{aligned} \tag{2.6}$$

provided that the integrals involved converge absolutely.

The assertion (2.2) follows when we interpret this last result (2.6) by means of the definitions (1.5) and (1.6), and the proof of the theorem is thus completed.

3. Applications and Illustrative Examples

The relationship (2.2) asserted by the theorem can be suitably applied not only to deduce several (known or new) results connecting simpler integral operators, but also to evaluate the generalized Weyl fractional integrals of various special functions. First of all, setting $\lambda = \sigma = 0$ in the theorem, and applying the first reduction formula in (1.4), we get

COROLLARY 1. *Under the hypothesis* (2.1) *of the theorem, let* $F(s)$ *be defined by Eq.* (1.12).
Then

$$\mathscr{W}_{\mu;\rho,0}^{z;m,n}\{t^N F(t):s\} = \sum_{j=0}^{[n/m]} (-n)_{mj}\,(\mu)_{\rho j} A_{n,j} \frac{z^j}{j!} \cdot \mathscr{L}\{t^{-\mu-\rho j} f^{(N)}(t):s\} \qquad (N=0,1,2,...), \quad (3.1)$$

provided that each member of (3.1) *exists.*

For $N=0$, the assertion (2.2) immediately yields

COROLLARY 2. *If* $F(s)$ *is defined by Eq.* (1.12), *and* $\alpha(j)$ *and* $\beta(j)$ *are given by Eq.* (1.11), *then*

$$\mathscr{W}_{\mu;\rho,\sigma}^{z;m,n}\{t^{-\lambda}F(t):s\} = \sum_{j=0}^{[n/m]} (-n)_{mj}\,(\mu)_{\rho j}\, A_{n,j} \frac{z^j}{j!} \cdot \mathscr{V}_{\alpha(j),\beta(j)}\{t^{-2\beta(j)} f(t):s\}, \quad (3.2)$$

provided that each member of (3.2) *exists.*

Finally, upon setting

$$n=0 \qquad \text{and} \qquad A_{0,0}=1$$

in the theorem, we obtain the following known relationship between the familiar Weyl fractional integral and the Varma (or *special* Whittaker) transform (1.5):

COROLLARY 3 (cf. Pathan [10, p. 885, Theorem I]). *Under the hypothesis* (2.1) *of the theorem, let* $F(s)$ *be defined by Eq.* (1.12).
Then

$$\mathscr{W}_{\mu}\{t^{N-\lambda}F(t):s\} = \mathscr{V}_{(1-\nu-\lambda)/2,\,(\nu-\lambda)/2}\{t^{\lambda-\mu} f^{(N)}(t):s\} \qquad (N=0,1,2,...), \quad (3.3)$$

provided that each member of (3.3) *exists.*

The relationship (2.2), as well as its special cases (3.1), (3.2), and (3.3), can be used (for example) to evaluate the Weyl fractional integrals of various special functions by computing the corresponding simpler integral transforms involved. We illustrate this aspect of applicability of our results by considering the following examples.

EXAMPLE 1. Let

$$f(t)=t^{\nu}e^{-\zeta t} \qquad (t \geqslant 0; \mathrm{Re}(\nu)>0). \tag{3.4}$$

Then

$$F(s)=\mathscr{L}\{t^{\nu}e^{-\zeta t}:s\}=\frac{\Gamma(\nu+1)}{(s+\zeta)^{\nu+1}} \qquad (\mathrm{Re}(\nu)>-1; \mathrm{Re}(s+\zeta)>0). \tag{3.5}$$

Furthermore, by the Leibniz rule for derivatives, we have

$$f^{(N)}(t)=e^{-\zeta t}\sum_{k=0}^{N}\binom{N}{k}\frac{\Gamma(\nu+1)}{\Gamma(\nu-k+1)}(-\zeta)^{N-k}\,t^{\nu-k}, \tag{3.6}$$

which shows that the hypothesis (2.1) of the theorem is satisfied when

$$\mathrm{Re}(\nu)>N-1 \qquad (N=1,2,3,\ldots). \tag{3.7}$$

Substituting these values of $f(t)$ and $F(s)$ in the theorem, and evaluating the resulting Varma (or *special* Whittaker) transform on the right-hand side of (2.2) by means of a well-known integral formula (cf. [5, Vol. I, p. 216, Entry 4.22(16)]; see also [9, p. 189, Entry 17.105]), we obtain

$$\begin{aligned}
&\mathscr{W}_{\mu;\rho,\sigma}^{z;m,n}\{t^{N-\lambda}(t+\zeta)^{-\nu-1}:s\}\\
&\quad=\sum_{j=0}^{[n/m]}\sum_{k=0}^{N}\binom{N}{k}(-n)_{mj}(\mu)_{\rho j}(-\zeta)^{N-k}\\
&\qquad\cdot A_{n,j}\,s^{\mu-\nu-\lambda+(\rho-\sigma)j+k-1}\,\frac{\Gamma\{\lambda-\mu+\nu-(\rho-\sigma)j-k+1\}}{\Gamma(\nu+\lambda+\sigma j-k+1)}\frac{z^j}{j!}\\
&\qquad\cdot {}_2F_1\left[\begin{matrix}\nu-k+1,\ \lambda-\mu+\nu-(\rho-\sigma)j-k+1;\\ \nu+\lambda+\sigma j-k+1;\end{matrix}\ -\frac{\zeta}{s}\right],
\end{aligned} \tag{3.8}$$

provided that

$$\begin{aligned}
&\mathrm{Re}(s+\zeta)>0; \mathrm{Re}\{\lambda-\mu+\nu-(\rho-\sigma)j-k+1\}>0;\\
&\mathrm{Re}(\nu-k+1)>0\ (j=0,1,\ldots,[n/m]; k=0,1,\ldots,N;\\
&n,N=0,1,2,\ldots; m=1,2,3,\ldots).
\end{aligned} \tag{3.9}$$

The k-series involved in (3.8) can be expressed in a closed form by appealing to the special case $p=q=1$ of the hypergeometric identity:

$$\begin{aligned}
&\sum_{k=0}^{N}\binom{N}{k}\frac{\prod_{j=1}^{p}\Gamma(a_j-k)}{\prod_{j=1}^{q}\Gamma(b_j-k)}\,{}_{p+1}F_q\left[\begin{matrix}c-k,\ a_1-k,\ldots,a_p-k;\\ b_1-k,\ldots,b_q-k;\end{matrix}\ z\right]z^{N-k}\\
&\quad=\frac{\prod_{j=1}^{p}\Gamma(a_j-N)}{\prod_{j=1}^{q}\Gamma(b_j-N)}\,{}_{p+1}F_q\left[\begin{matrix}c,\ a_1-N,\ldots,a_p-N;\\ b_1-N,\ldots,b_q-N;\end{matrix}\ z\right]
\end{aligned} \tag{3.10}$$

or, equivalently,

$$\sum_{k=0}^{N} \binom{N}{k} \frac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} {}_{p+1}F_q \left[\begin{matrix} c+k, a_1+k, \ldots, a_p+k; \\ b_1+k, \ldots, b_q+k; \end{matrix} z \right] z^k$$

$$= {}_{p+1}F_q \left[\begin{matrix} c+N, a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z \right], \tag{3.11}$$

which can indeed be deduced from the Leibniz rule in view of the derivative formulas

$$\frac{d^N}{dz^N} \left\{ z^{c+N-1} {}_{p+1}F_q \left[\begin{matrix} c, a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z \right] \right\}$$

$$= (c)_N z^{c-1} {}_{p+1}F_q \left[\begin{matrix} c+N, a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z \right] \tag{3.12}$$

and

$$\frac{d^N}{dz^N} \left\{ {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z \right] \right\} = \frac{\prod_{j=1}^{p} (a_j)_N}{\prod_{j=1}^{q} (b_j)_N} {}_pF_q \left[\begin{matrix} a_1+N, \ldots, a_p+N; \\ b_1+N, \ldots, b_q+N; \end{matrix} z \right]. \tag{3.13}$$

We thus find from (3.8) that

$$\mathscr{W}_{\mu;\rho,\sigma}^{z;m,n} \{ t^{N-\lambda} (t+\zeta)^{-\nu-1} : s \}$$

$$= \sum_{j=0}^{[n/m]} (-n)_{mj} (\mu)_{\rho j} A_{n,j} \frac{z^j}{j!}$$

$$\cdot s^{\mu-\nu-\lambda+(\rho-\sigma)j+N-1} \frac{\Gamma\{\lambda-\mu+\nu-(\rho-\sigma)j-N+1\}}{\Gamma(\nu+\lambda+\sigma j-N+1)}$$

$$\cdot {}_2F_1 \left[\begin{matrix} \nu+1, \lambda-\mu+\nu-(\rho-\sigma)j-N+1; \\ \nu+\lambda+\sigma j-N+1; \end{matrix} -\frac{\zeta}{s} \right], \tag{3.14}$$

which holds true under the constraints listed in (3.9).

EXAMPLE 2. In terms of the modified Bessel function $I_\nu(z)$, we put

$$f(t) = I_\nu(\zeta t) \qquad (t \geqslant 0; \mathrm{Re}(\nu) > 0), \tag{3.15}$$

so that [5, Vol. I, p. 195, Entry 4.16(1)]

$$F(s) = \frac{\zeta^\nu (s+\sqrt{s^2-\zeta^2})^{-\nu}}{\sqrt{s^2-\zeta^2}} \qquad (\mathrm{Re}(\nu) > -1; \mathrm{Re}(s) > |\mathrm{Re}(\zeta)|). \tag{3.16}$$

It also follows easily from the Leibniz rule that

$$f^{(N)}(t) = \left(\frac{1}{2}\zeta\right)^N \sum_{k=0}^{N} \binom{N}{k} I_{\nu-N+2k}(\zeta t). \tag{3.17}$$

Thus the hypothesis (2.1) is satisfied when the constraint (3.7) holds true.

Upon evaluating the Laplace transform resulting on the right-hand side of (3.1) by means of a known formula [5, Vol. I, p. 196, Entry 4.16(8)], Corollary 1 yields

$$\begin{aligned}
&\mathscr{W}_{\mu;\,\rho,\,0}^{z;\,m,\,n}\{t^N(t^2-\zeta^2)^{-1/2}\ (t+\sqrt{t^2-\zeta^2})^{-\nu} : s\} \\
&= 2^{-N}\zeta^{N-\nu} \sum_{j=0}^{[n/m]} \sum_{k=0}^{N} \binom{N}{k} (-n)_{mj}\, (\mu)_{\rho j}\, A_{n,j} \frac{z^j}{j!} \\
&\quad \cdot \Gamma(\nu-\mu-\rho j+2k-N+1)(s^2-\zeta^2)^{(\mu+\rho j-1)/2}\, P_{-\mu-\rho j}^{N-\nu-2k}\left(\frac{s}{\sqrt{s^2-\zeta^2}}\right),
\end{aligned} \tag{3.18}$$

in terms of the associated Legendre function, provided that

$$\begin{aligned}
&\mathrm{Re}(s) > |\mathrm{Re}(\zeta)|;\ \mathrm{Re}(\nu-\mu-\rho j+2k-N+1) > 0; \\
&\mathrm{Re}(\nu) > N-1\ (j=0, 1, ..., [n/m];\ k=0, 1, ..., N; \\
&n, N=0, 1, 2, ...;\ m=1, 2, 3, ...).
\end{aligned} \tag{3.19}$$

EXAMPLE 3. As an interesting generalization of Example 1, we take

$$f(t) = t^\nu\, {}_pF_q\left[\begin{matrix} a_1, ..., a_p; \\ b_1, ..., b_q; \end{matrix} -\zeta t\right] \qquad (t \geqslant 0;\ \mathrm{Re}(\nu) > 0), \tag{3.20}$$

which reduces immediately to (3.4) when $p=q=0$. It is easily seen that

$$f^{(N)}(t) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-N+1)} t^{\nu-N}\, {}_{p+1}F_{q+1}\left[\begin{matrix} \nu+1, a_1, ..., a_p; \\ \nu-N+1, b_1, ..., b_q; \end{matrix} -\zeta t\right], \tag{3.21}$$

where, for the validity of the hypothesis (2.1), the constraint (3.7) is assumed to hold true.

We also have [5, Vol. I, p. 219, Entry 4.23(17)]

$$\begin{aligned}
&F(s) = \frac{\Gamma(\nu+1)}{s^{\nu+1}}\, {}_{p+1}F_q\left[\begin{matrix} \nu+1, a_1, ..., a_p; \\ b_1, ..., b_q; \end{matrix} -\frac{\zeta}{s}\right] \\
&(\mathrm{Re}(\nu) > -1;\ p \leqslant q;\ \mathrm{Re}(s) > 0 \text{ if } p < q; \\
&\mathrm{Re}(s+\zeta) > 0 \text{ if } p = q).
\end{aligned} \tag{3.22}$$

Upon making use of the known integral formula [5, Vol. I, p. 216, Entry 4.22(16)] once again, the assertion (2.2) of the theorem yields the following generalization of (3.14):

$$\mathscr{W}^{z;\,m,\,n}_{\mu;\,\rho,\,\sigma}\left\{t^{N-\nu-\lambda-1}\,{}_{p+1}F_q\left[\begin{matrix}\nu+1, a_1, ..., a_p; \\ b_1, ..., b_q;\end{matrix}\ -\frac{\zeta}{t}\right] : s\right\}$$
$$= \sum_{j=0}^{[n/m]} (-n)_{mj}\,(\mu)_{\rho j}\,A_{n,j}\,\frac{z^j}{j!}\,s^{\mu-\nu-\lambda+(\rho-\sigma)j+N-1}$$
$$\cdot\frac{\Gamma\{\lambda-\mu+\nu-(\rho-\sigma)j-N+1\}}{\Gamma(\nu+\lambda+\sigma j-N+1)}$$
$$\cdot\,{}_{p+2}F_{q+1}\left[\begin{matrix}\nu+1, \lambda-\mu+\nu-(\rho-\sigma)j-N+1, a_1, ..., a_p; \\ \nu+\lambda+\sigma j-N+1, b_1, ..., b_q;\end{matrix}\ -\frac{\zeta}{s}\right], \tag{3.23}$$

provided that

$$\begin{gathered} p\leqslant q;\ \mathrm{Re}(s)>0 \text{ if } p<q;\ \mathrm{Re}(s+\zeta)>0 \text{ if } p=q;\ \mathrm{Re}(\nu-N+1)>0; \\ \mathrm{Re}\{\lambda-\mu+\nu-(\rho-\sigma)j-N+1\}>0\ (j=0, 1, ..., [n/m]; \\ n, N, p, q=0, 1, 2, ...;\ m=1, 2, 3, ...). \end{gathered} \tag{3.24}$$

In its special case when $p=q=0$, this last result (3.23) reduces at once to (3.14). With a view to giving a similar generalization of Example 2, we note that

$$I_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left[\begin{matrix}\overline{\quad\quad}; \\ \nu+1;\end{matrix}\ \tfrac{1}{4}z^2\right], \tag{3.25}$$

which naturally motivates the choice

$$f(t)=t^\nu\,{}_pF_q\left[\begin{matrix}a_1, ..., a_p; \\ b_1, ..., b_q;\end{matrix}\ \tfrac{1}{4}\zeta^2t^2\right] \qquad (t\geqslant 0;\ \mathrm{Re}(\nu)>0). \tag{3.26}$$

Then it is readily observed that

$$f^{(N)}(t)=\frac{\Gamma(\nu+1)}{\Gamma(\nu-N+1)}$$
$$\cdot\,t^{\nu-N}\,{}_{p+2}F_{q+2}\left[\begin{matrix}\Delta(2;\nu+1), a_1, ..., a_p; \\ \Delta(2;\nu-N+1), b_1, ..., b_q;\end{matrix}\ \tfrac{1}{4}\zeta^2t^2\right], \tag{3.27}$$

where, for convenience, $\Delta(m;\lambda)$ abbreviates the array of m parameters

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, ..., \frac{\lambda+m-1}{m} \qquad (m=1, 2, 3, ...),$$

it being assumed that the constraint (3.7) holds true in order to validate the hypothesis (2.1) of the theorem.

We also have [5, Vol. I, p. 219, Entry 4.23(18)]

$$F(s)=\frac{\Gamma(\nu+1)}{s^{\nu+1}}\,{}_{p+2}F_q\left[\begin{matrix}\Delta(2;\nu+1),a_1,\dots,a_p;\frac{\zeta^2}{s^2}\\ b_1,\dots,b_q;\end{matrix}\right]$$

$$(\mathrm{Re}(\nu)>-1;\,p\leqslant q-1;\,\mathrm{Re}(s)>0 \text{ if } p<q-1; \tag{3.28}$$

$$\mathrm{Re}(s)>|\mathrm{Re}(\zeta)| \text{ if } p=q-1).$$

Upon substituting these values for $f(t)$ and $F(s)$ in the assertion (2.2) of the theorem, and evaluating the resulting Varma (or *special* Whittaker) transform as above, we finally have

$$\begin{aligned}
&\mathscr{W}^{z;m,n}_{\mu;\rho,\sigma}\left\{t^{N-\nu-\lambda-1}\,{}_{p+2}F_q\left[\begin{matrix}\Delta(2;\nu+1),a_1,\dots,a_p;\frac{\zeta^2}{t^2}\\ b_1,\dots,b_q;\end{matrix}\right]:s\right\}\\
&=\sum_{j=0}^{[n/m]}(-n)_{mj}\,(\mu)_{\rho j}A_{n,j}\frac{z^j}{j!}\,s^{\mu-\nu-\lambda+(\rho-\sigma)j+N-1}\\
&\cdot\frac{\Gamma\{\lambda-\mu+\nu-(\rho-\sigma)j-N+1\}}{\Gamma(\nu+\lambda+\sigma j-N+1)}\\
&\cdot{}_{p+4}F_{q+2}\left[\begin{matrix}\Delta(2;\nu+1),\Delta\{2;\lambda-\mu+\nu-(\rho-\sigma)j-N+1\},a_1,\dots,a_p;\frac{\zeta^2}{s^2}\\ \Delta(2;\nu+\lambda+\sigma j-N+1),b_1,\dots,b_q;\end{matrix}\right],
\end{aligned}\tag{3.29}$$

provided that

$$\begin{aligned}
&p\leqslant q-1;\,\mathrm{Re}(s)>0 \text{ if } p<q-1;\,\mathrm{Re}(s)>|\mathrm{Re}(\zeta)| \text{ if } p=q-1;\\
&\mathrm{Re}(\nu-N+1)>0;\,\mathrm{Re}\{\lambda-\mu+\nu-(\rho-\sigma)j-N+1\}>0\ (j=0,1,\dots,[n/m];\\
&n,N,p=0,1,2,\dots;\,m,q=1,2,3,\dots).
\end{aligned}\tag{3.30}$$

In view of the relationship (3.25), a special case of our result (3.29) when $p=q-1=0$ would correspond to (3.18) if we further set $\sigma=0$.

Numerous other examples leading to the generalized Weyl fractional integrals of the various classes of multivariable hypergeometric functions (cf. [18, 23]) can be given in a manner illustrated above.

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REFERENCES

1. S. K. AKHAURY, Some theorems for a distributional generalized Laplace transform, *Jñānābha* **16** (1986), 129–144.
2. R. D. CARMICHAEL AND R. S. PATHAK, Abelian theorems for Whittaker transforms, *Internat. J. Math. Math. Sci.* **10** (1987), 417–431.
3. R. D. CARMICHAEL AND R. S. PATHAK, Asymptotic behaviour of the H-transform in the complex domain, *Math. Proc. Cambridge Philos. Soc.* **102** (1987), 533–552.
4. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, "Higher Transcendental Functions," Vol. I, McGraw–Hill, New York/Toronto/London, 1953.
5. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, "Tables of Integral Transforms," Vols. I and II, McGraw–Hill, New York/Toronto/London, 1954.
6. S. GOLDSTEIN, Operational representations of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function, *Proc. London Math. Soc.* (2) **34** (1932), 103–125.
7. S. P. MALGONDE AND R. K. SAXENA, Some Abelian theorems for the distributional H-transformation, *Indian J. Pure Appl. Math.* **15** (1984), 365–370.
8. P. C. MUNOT AND P. A. PADMANABHAM, On a generalized integral transform. II, *Indian J. Pure Appl. Math.* **12** (1981), 1235–1239.
9. F. OBERHETTINGER AND L. BADII, "Tables of Laplace Transforms," Springer-Verlag, New York/Heidelberg/Berlin, 1973.
10. M. A. PATHAN, A theorem on Weyl (fractional) integral, *Proc. Nat. Acad. Sci. India Sect. A* **36** (1966), 885–888.
11. G. L. N. RAO, A unification of generalizations of the Laplace transform and generalized functions, *Acta Math. Hungar.* **41** (1983), 119–126.
12. S. G. SAMKO, A. A. KILBAS, AND O. I. MARICHEV, "Integrals and Derivatives of Fractional Order and Some of Their Applications" (in Russian), Nauka i Tekhnika, Minsk, 1987.
13. S. K. SINHA, A complex inversion formula and a Tauberian theorem for the generalized Whittaker transform, *Jñānābha* **11** (1981), 31–39.
14. H. M. SRIVASTAVA, Certain properties of a generalized Whittaker transform, *Mathematica (Cluj)* **10 (33)** (1968), 385–390.
15. H. M. SRIVASTAVA, On a generalized integral transform. II, *Math. Z.* **121** (1971), 263–272.
16. H. M. SRIVASTAVA, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972), 1–6.
17. H. M. SRIVASTAVA AND M. GARG, Some integrals involving a general class of polynomials and the multivariable H-function, *Rev. Roumaine Phys.* **32** (1987), 685–692.
18. H. M. SRIVASTAVA AND P. W. KARLSSON, "Multiple Gaussian Hypergeometric Series," Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York/Chichester/Brisbane/Toronto, 1985.
19. H. M. SRIVASTAVA AND R. PANDA, Certain multidimensional integral transformations. II, *Nederl. Akad. Wetensch. Indag. Math.* **40** (1978), 132–144.

20. H. M. Srivastava and N. P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) **32** (1983), 157–187.
21. H. M. Srivastava and O. D. Vyas, A theorem relating generalized Hankel and Whittaker transforms, *Nederl. Akad. Wetensch. Indag. Math.* **31** (1969), 140–144.
22. H. M. Srivastava, S. P. Goyal, and R. M. Jain, Fractional integral operators involving a general class of polynomials, *J. Math. Anal. Appl.* **148** (190), 87–100.
23. H. M. Srivastava, K. C. Gupta, and S. P. Goyal, "The H-Functions of One and Two Variables with Applications," South Asian Publishers, New Delhi/Madras, 1982.
24. A. K. Tiwari and A. Ko, Certain properties of a distributional generalized Whittaker transform, *Indian J. Pure Appl. Math.* **13** (1982), 348–361.
25. R. S. Varma, On a generalization of Laplace integral, *Proc. Nat. Acad. Sci. India Sect. A* **20** (1951), 209–216.
26. E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," 4th ed., Cambridge Univ. Press, Cambridge/London/New York, 1927.
27. D. V. Widder, "The Laplace Transform," Princeton Univ. Press, Princeton, NJ, 1941.