General Upper Bounds on the Minimum Size of Covering Designs

Iliya Bluskov and Katherine Heinrich*

Department of Mathematics and Statistics, Simon Fraser University
Burnaby, British Columbia, Canada, V5A 1S5

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Let $D$ be a finite family of $k$-subsets (called blocks) of a $v$-set $X(v)$. Then $D$ is a $(v, k, t)$ covering design or covering if every $t$-subset of $X(v)$ is contained in at least one block of $D$. The number of blocks is the size of the covering, and the minimum size of the covering is called the covering number. In this paper we find new upper bounds on the covering numbers for several families of parameters.

1. INTRODUCTION

First we discuss some facts and notation that will be used throughout the paper. Let $D = \{B_1, B_2, ..., B_b\}$ be a finite family of $k$-subsets (called blocks) of a $v$-set $X(v) = \{1, 2, ..., v\}$ (with elements called points). Then $D$ is a $(v, k, t)$ covering design or covering if every $t$-subset of $X(v)$ is contained in at least one block of $D$. The number of blocks, $b$, is the size of the covering, and the minimum size of the covering is called the covering number, denoted $C(v, k, t)$. If every $t$-subset of $X(v)$ is contained in exactly one block of $D$, then $D$ is a Steiner system, denoted $S(v, k, t)$. A Steiner system is said to be resolvable if there exists a partition of its set $D$ blocks into subsets called resolution classes each of which in turn partitions the set $X(v)$.

A general lower bound on $C(v, k, t)$ is due to Schönheim [10].

**Theorem 1.1.**

$$C(v, k, t) \geq \left\lfloor \frac{v - 1}{k - 1} \cdot \frac{v - t + 1}{k - t + 1} \right\rfloor.$$ 

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There is an extensive literature on the covering numbers \( C(v, k, t) \). For excellent surveys on the known results we refer to [6, 7, 11]. The survey papers include tables of upper bounds on the size of coverings for small values of \( v \). Recent works on upper bounds on the covering numbers include [3, 5, 6, 8]. In this paper we give several constructions which produce general upper bounds, and, in particular, the best upper bounds in the range of the (most extensive) tables in [6]. Two of the construction (Theorems 2.4 and 2.5) are generalizations of “record-breaking constructions” found in [3].

Let \( X^k(v) \) denote the set of all \( k \)-subsets of \( X(v) \). A \( t \)-\( (v, k_1, k_2, ..., k_n, \lambda) \) design (also called a \( t \)-wise balanced design) is pair \((X(v), D)\), where \( X(v) = \{1, 2, ..., v\} \) is a set of points and \( D \) is a subset of \( X^{k_1}(v) \cup X^{k_2}(v) \cup ... \cup X^{k_n}(v) \) with elements called blocks (of sizes \( k_1, k_2, ..., k_n \)) so that every \( t \)-set of \( X(v) \) is contained in exactly \( \lambda \) blocks. When \( \{k_1, k_2, ..., k_n\} = \{k\} \), we denote the design by \( t \)-\( (v, k, \lambda) \) design. Let \( D \) be a \( t \)-\( (v, k, \lambda) \) design. Given a point \( x \) in \( X(v) \), the blocks obtained on deleting \( x \) from these blocks that contained it, form a \( (t-1)-(v-1, k-1, \lambda) \) design \( D^x \) on \( X \setminus \{x\} \) called the derived design of \( D \) with respect to \( x \).

Let the set \( X \) be the disjoint union of the sets \( X_1 \) and \( X_2 \) of sizes \( n_1 \) and \( n_2 \), respectively. We define an \( [m_1, m_2] \)-set to be an \((m_1 + m_2)\)-subset of \( X \) with \( m_1 \) of its elements in \( X_1 \) and the remaining \( m_2 \) elements in \( X_2 \).

It is convenient to represent a covering by a \( b \times k \) array whose rows are the blocks of the covering. Let

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1q} \\
b_{21} & b_{22} & \cdots & b_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
b_{p1} & b_{p2} & \cdots & b_{pq}
\end{pmatrix}
\]

be a set of \( m \) blocks of size \( n \) and a set of \( p \) blocks of size \( q \), respectively. We use the notation \( AB \) to represent the following set of \( mp \) blocks:

\[
\{ \{a_{i1}, a_{i2}, ..., a_{in}, b_{j1}, b_{j2}, ..., b_{jq}\} : i = 1, 2, ..., m; j = 1, 2, ..., p \}.
\]

Stated below are two particular cases of a result obtained by Etzion [4]. We give the proof of the first one as an illustration; note that \( C(6, 4, 3) = 6 \) and \( C(8, 4, 4) = 14 \).

**Theorem 1.2.** There exist four \( (6, 4, 3) \) coverings of size 6 whose union is \( (6, 4, 4) \) covering.

**Proof.** Each array given below is one of the coverings.
Note that \( C(6, 4, 3) = 6 \).

**Theorem 1.3.** There exist six \((8, 4, 3)\) coverings of size 14 whose union is an \((8, 4, 4)\) covering.

Given a vector space \( V = V_n(K) \) of dimension \( n \) over the field \( K \), a code \( C \) is a subset of \( V \). The vectors in the code are called codewords. The *(Hamming)* distance between two codewords \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) is the number of places in which they differ; that is,

\[
d(x, y) = |\{i : 1 \leq i \leq n, x_i \neq y_i\}|.
\]

The *(Hamming)* weight of a vector \( x = (x_1, \ldots, x_n) \) is the number of nonzero coordinates, and is denoted by \( wt(0) \); that is, \( wt(x) = d(x, x) \) where \( 0 \) is the all zero vector. More generally, \( wt(x - y) = d(x, y) \). The minimum distance of a code is

\[
d = \min\{d(x, y) : x \in C, y \in C, x \neq y\}.
\]

Given a code \( C \), and a vector \( v \in V \), the set

\[
v + C = \{v + c : c \in C\}
\]

is called a *translate of the code* \( C \) *by the vector* \( v \). A translate of a code is also a code with the same minimum distance as the original.

### 2. MAIN RESULTS

After studying the paper of Zaitsev et al. [12] we observed that it contains (although not explicitly stated) the following result.

**Theorem 2.1.** There exist a 3-\((4^m, 6, \frac{1}{4}(4^m - 4))\) design \( D \) so that the family of all 4-subsets of \( X(4^m) \) not covered by any block of \( D \) is a Steiner
system $S(4^m, 4, 3)$. This Steiner system can be partitioned into $(2^{2m-1} - 1)$ $S(4^m, 4, 2)$’s.

The decomposition of the $S(4^m, 4, 3)$ into $S(4^m, 4, 2)$’s for every $m \geq 2$, is due to Zaitsev et al. [13]. This result is based on the remarkable proof of Preparata [9] that the binary Hamming code decomposes into translates of the Preparata code. The design $D$ referred to in Theorem 2.1 is obtained from the codewords of weight 6 of the Preparata code. The partition of the Steiner system $S(4^m, 4, 3)$ (formed by the codewords of weight 4 in the Hamming code) into $(2^{2m-1} - 1)$ $S(4^m, 4, 2)$’s is described in [13]. Taking the derived designs of the designs given in Theorem 2.1 we obtain the following.

**Corollary 2.2.** There exists a $2-(4^m - 1, 5, \frac{1}{2}(4^m - 4))$ design $D$. The family of all 3-subsets of $X(4^m - 1)$ that are not covered by any block of $D$ form a resolvable Steiner system $S(4^m - 1, 3, 2)$.

The result of Zaitsev et al. leads to the following general upper bound.

**Theorem 2.3.** Let $0 \leq d \leq 2^{2m-2} - 3$.

$$C(3(2^{2m-1}) - 2d, 6, 4) \leq \left( \frac{4^m}{3} \right) \left( \frac{1}{15} (4^{m-1} - 1) + \left( \frac{4^{m-2} - d}{4} \right) \right)$$

$$+ 2^{2m-1} C(2^{2m-1} - 2d, 4, 3) + C(2^{2m-1} - 2d, 6, 4).$$

**Proof.** Partition $X(3(2^{2m-1}) - 2d)$ into two sets, $X_1 = \{1, 2, \ldots, 4^m\}$ and $X_2 = \{4^m + 1, 4^m + 2, \ldots, 3(2^{2m-1}) - 2d\}$. Let $D$ be the $3-(4^m, 6, \frac{1}{2}(4^m - 4))$ design on $X_1$ from Theorem 2.1, and $D^*$ be the corresponding $S(4^m, 4, 3)$. Let $A_1, A_2, \ldots, A_{2^{2m-1}-1}$ be the partition of $D^*$ into $(2^{2m-1} - 1)$ $S(4^m, 4, 2)$’s. Let $B_1, B_2, \ldots, B_{2^{2m-1}-1}$ be the 1-factors of a 1-factorization of the complete graph $K_{2^{2m-1}-2d}$ on $X_2$. Let $B_{2^{2m-1}-1} = B_{2^{2m-1}-2d} = \ldots = B_{2^{2m-1}-1} = B_1$. Let $E$ be a 1-factor of the complete graph $K_w$ on $X_1$ and $F$ a $(2^{2m-1} - 2d, 4, 3)$ covering of size $C(2^{2m-1} - 2d, 4, 3)$ on $X_2$. Let $H$ be a $(2^{2m-1} - 2d, 6, 4)$ covering of size $C(2^{2m-1} - 2d, 6, 4)$ on $X_2$. We claim that the blocks of

$$D, A_iB_i, \quad i = 1, 2, \ldots, 2^{2m-1} - 1$$

form a $(3(2^{2m-1}) - 2d, 6, 4)$ covering.
All of the \([4,0]\)-sets are covered because the blocks of \(D\) and \(A_iB_i\) contain as subblocks the blocks of the \(4(4^m, \{6,4\}, 1)\) design formed by the union of the design \(D\) and the Steiner system \(S(4^m, 4, 3)\).

The \([3,1]\)-sets and \([2,2]\)-sets are covered by the blocks of \(A_iB_i\), \(i = 1, 2, \ldots, 2^{2m-1}-1\).

The \([1,3]\)-sets are covered by the blocks of \(EF\).

The \([0,4]\)-sets are covered by the blocks of \(H\).

Finally, it is easy to check that the number of blocks of this covering is exactly the right hand side of the inequality of the theorem, which completes the proof.

A slightly better bound can be obtained under the condition given in the next theorem.

**Theorem 2.4.** Let \(0 \leq d \leq 2^{2m-2} - 2\). If there exist \(2^{2m-1} (2^{2m-1} - 2d, 4, 3)\) coverings each of size \(C(2^{2m-1} - 2d, 4, 3)\) whose union is a \((2^{2m-1} - 2d, 4, 4)\) covering, then

\[
C(3(2^{2m-1}) - 2d, 6, 4) \leq \left( \frac{4^m}{3} \right) \left[ \frac{1}{15} (4^{m-1} - 1) + \left( \frac{4^{m-2} - d}{4} \right) \right] + 2^{2m-1} \left[ \frac{1}{2} (4^{m-1} - d) \left( \frac{1}{3} (2^{2m-1} - 2d - 1)(4^{m-1} - d - 1) \right) \right].
\]

**Proof.** We basically follow the proof of the preceding theorem. The difference is in the covering of the \([1,3]\) and \([0,4]\)-sets. Let \(F_i, i = 1, 2, \ldots, 2^{2m-1}, \) be the \((2^{2m-1} - 2d, 4, 3)\) coverings on \(X\) whose union is a \((2^{2m-1} - 2d, 4, 4)\) covering. Instead of using the blocks of \(EF\) and \(H\) we use the blocks of \(EF_i, i = 1, 2, \ldots, 2^{2m-1}\), to cover the \([1,3]\) and \([0,4]\)-sets. The proof is completed by using the known result

\[
C(2^{2m-1} - 2d, 4, 3) = \left[ \frac{2^{2m-1} - 2d}{4} \left( \frac{2^{2m-1} - 2d - 1}{3} \right) \right] + \left[ \frac{1}{2} (4^{m-1} - d) \left( \frac{1}{3} (2^{2m-1} - 2d - 1)(4^{m-1} - d - 1) \right) \right] [7].
\]

For example, if \(m = 2\) and \(d = 1\), Theorem 2.3 gives \(C(22, 6, 4) \leq 581\) while Theorem 2.4 gives \(C(22, 6, 4) \leq 580\) which is the best known bound.
(both use Theorem 1.2). For more results on \((v, 4, 3)\) coverings whose union is an \((v, 4, 4)\) covering we refer to [4]. Letting \(m = 2\) and \(d = 0, 2\) in Theorem 2.4 yields the best known bounds \(C(24, 6, 4) \leq 784\) (via Theorem 1.3) and \(C(20, 6, 4) \leq 400\).

Corollary 2.2 leads to the following upper bound.

**Theorem 2.5.** Let \(0 \leq d \leq 2^{2m-2} - 3\). Then

\[
C(3(2^{2m-1}) - 2d - 1, 5, 3) \\
\leq \left(\frac{4^m - 1}{2}\right) \left[\frac{1}{15}(2^{2m-1} - 2) + \frac{1}{3}(4^{m-1} - d)\right] + C(2^{2m-1} - 2d, 5, 3),
\]

**Proof.** Partition \(X(3(2^{2m-1}) - 2d - 1)\) into the two sets \(X_1 = \{1, 2, ..., 4^m - 1\}\) and \(X_2 = \{4^m, 4^m + 1, ..., 3(2^{2m-1}) - 2d - 1\}\). Let \(D\) be the 2-\((4^m - 1, 5, \frac{1}{2}(4^m - 4))\) design on \(X_1\) from Corollary 2.2. Let \(A_1, A_2, ..., A_{2^{2m-1}-1}\) be the resolution classes of the Steiner system \(S(4^m - 1, 3, 2)\). Let \(B_1, B_2, ..., B_{2^{2m-1}-1} = 2d\) be the 1-factors of a 1-factorization of the complete graph \(K_{2^{2m-1}-1} = 2d\) on \(X_2\) and \(B_{2^{2m-1}-1} = 2d + 1 = \cdots = B_{2^{2m-1}} = B_1\). Let \(C\) be a \((2^{2m-1} - 2d, 5, 3)\) covering of minimum size on \(X_2\). We claim that the blocks of

\[
D \\
A_iB_i, \quad i = 1, 2, ..., 2^{2m-1} - 1 \\
C
\]

form a \((3(2^{2m-1}) - 2d - 1, 5, 3)\) covering.

Since the blocks of \(D\) and \(A_iB_i\) contain as subblocks the blocks of the 3-\((4^m - 1, \{5, 3\}, 1)\) design formed by the union of the design \(D\) and the Steiner system \(S(4^m - 1, 3, 2)\), all of the \([3, 0]\)-sets are covered.

The \([2, 1]\)-sets and \([1, 2]\)-sets are covered by the blocks of \(A_iB_i\), \(i = 1, 2, ..., 2^{2m-1} - 1\).

The \([0, 3]\)-sets are covered by the blocks of \(C\).

Again, the number of blocks of the constructed covering is exactly the right hand of the desired inequality, which completes the proof.

For example, the values \(m = 2, d = 0, 1\) yield the best known bounds \(C(23, 5, 3) \leq 190\) and \(C(21, 5, 3) \leq 151\) (using the known covering numbers \(C(8, 5, 3) = 8\) and \(C(6, 5, 3) = 4\) [7]).

We now formulate a theorem that can be used to find good upper bounds on the size of covering designs provided appropriate \(t\)-wise balanced designs exist.
Theorem 2.6. If there exists a $t$-$\{v, \{k_1, k_2\}, 1\}$ design, where $k_1 < k_2$ with $n_i$ blocks of size $k_i$, $i = 1, 2$, then

$C(v, k_1, t) \leq n_1 + n_2 C(k_2, k_1, t)$.

Proof. Substitute each block $B$ of size $k_2$ in the $t$-wise balanced design with a $(k_2, k_1, t)$ covering on the points contained in $B$. □

The next result follows from a discussion in [1] based on a construction of Wilson.

Theorem 2.7. There exists a $5$-$\{2^n, \{6, 8\}, 1\}$ design with $2^n$ blocks of size 8, and

$\frac{2^{n+2}}{45} \prod_{i=0}^{3} (2^{n-i} - 1)$ blocks of size 6, for every $n \geq 4$.

Now we can prove the following bound.

Theorem 2.8.

$C(2^n, 6, 5) \leq \frac{7 \cdot 2^{2n-1} - 7 \cdot 2^{n+2} + 45 \cdot 2^{n-1}}{315} \prod_{i=0}^{2} (2^{n-i} - 1)$,

for every $n \geq 4$.

Proof. Apply Theorem 2.6 to the design from Theorem 2.7. The proof is completed by using the known covering number $C(8, 6, 5) = 12$ [7]. □

For example, $n = 4$ yields the best known bound $C(16, 6, 5) \leq 808$ (Etzion et al. [5] have established this bound by a different construction). The case $n = 5$ gives $C(32, 6, 5) \leq 35216$, an improvement of more than a thousand blocks over the old bound from [6]. By removing a point from the blocks that contain it, we get $C(31, 5, 4) \leq \frac{5}{315}(35216) = 6603$ (the average number of blocks a point lies in), which is also an improvement (the old bound in [6] was 6852.) A natural question is how good is the general bound. In what follows, we answer this question by proving that asymptotically, the bound found in Theorem 2.8, equals the covering number. Let $s(n)$ denote the right hand side of the inequality of Theorem 2.8. We have the following.

Theorem 2.9.

$$\lim_{n \to \infty} \frac{s(n)}{C(2^n, 6, 5)} = 1.$$
Proof. By counting,

\[ C(2^n, 6, 5) \geq \prod_{i=0}^{4} \frac{2^n - i}{6 - i} = \frac{2^{2n} - 3 \cdot 2^n}{90} \prod_{i=0}^{2} (2^n - i - 1). \]

Then

\[ 1 \leq \frac{s(n)}{C(2^n, 6, 5)} \leq \frac{7 \cdot 2^{2^n-1} - 7 \cdot 2^{n+2} + 45 \cdot 2^{n-1}}{315} \leq \frac{9}{2^{2n} - 3 \cdot 2^n} = \frac{2^n - (11/7)}{2^n - 5}. \]

The result follows from the fact that the limit of the rightmost expression is 1.

Another application of Theorem 2.6 is the following.

Corollary 2.10. If there exists a resolvable 2-(nk, k, 1) design, then

\[ C(nk + p, k + 1, 2) \leq np + C(p, k + 1, 2), \]

where \( p = (nk - 1)/(k - 1) \) is the number of parallel classes.

Proof. It is well-known that we can adjoin a point to each block of any parallel class, and then form an extra block (of size the number of parallel classes) from the new points to obtain a 2-(nk + p, \( \{k, p\} \), 1) design with np blocks of size k (the blocks of the resolvable design) and one block of size p. The result now follows by Theorem 2.6.

As an application, consider a resolvable Steiner system \( S(q^3 + 1, q + 1, 2) \), where \( q \) is a prime power \( [2, p. 408] \), with \( q^2 \) resolution classes and \( q^2 - q + 1 \) blocks in each class. By Corollary 2.10 we get

\[ C(q^3 + 1 + q^2, q + 2, 2) \leq q^2(q^2 - q + 1) + C(q^2, q + 2, 2). \]

For example, if \( q = 2 \), we obtain \( C(13, 4, 2) \leq 12 + C(4, 4, 2) = 13 \), which is, in fact, the covering number (there exists a Steiner system \( S(13, 4, 2) \)). For \( q = 3 \), using the known covering number \( C(9, 5, 2) = 5 \) \( [7] \), we get \( C(37, 5, 2) \leq 63 + C(9, 5, 2) = 68 \), which gives the best known upper bound (the covering number \( C(37, 5, 2) \) is unknown; the Schönheim theorem gives \( C(37, 5, 2) \geq 67 \)).

REFERENCES


