Topological entropy of maps on regular curves

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Abstract

In [G.T. Seidler, The topological entropy of homeomorphisms on one-dimensional continua, Proc. Amer. Math. Soc. 108 (1990) 1025–1030], G.T. Seidler proved that the topological entropy of every homeomorphism on a regular curve is zero. Also, in [H. Kato, Topological entropy of monotone maps and confluent maps on regular curves, Topology Proc. 28 (2) (2004) 587–593] the topological entropy of confluent maps on regular curves was investigated. In particular, it was proved that the topological entropy of every monotone map on any regular curve is zero. In this paper, furthermore we investigate the topological entropy of more general maps on regular curves. We evaluate the topological entropy of maps \( f \) on regular curves \( X \) in terms of the growth of the number of components of \( f^{-n}(y) \) (\( y \in X \)).

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1. Introduction

In [9], G.T. Seidler proved that the topological entropy of every homeomorphism on a regular curve is zero. In [3], L.S. Efremova and E.N. Makhrova proved that the topological entropy of every monotone map on a dendrite which satisfies some special condition is zero. In [5], we investigated the topological entropy of confluent maps on regular curves. As a corollary, the topological entropy of every monotone map on any regular curve is zero. In this paper, furthermore we investigate the topological entropy of more general maps on regular curves. In fact, we evaluate the topological entropy of maps \( f \) on regular curves \( X \) in terms of the growth of the number of components of \( f^{-n}(y) \) (\( y \in X \)).

All spaces considered in this paper are assumed to be separable metric spaces. Maps are continuous functions. By a compactum we mean a compact metric space. A continuum is a nonempty connected compactum. For a compactum \( X \), \( \text{Comp}(X) \) denotes the set of all (connected) components of \( X \). For a set \( A \), \( |A| \) denotes the cardinality of the set \( A \).

For each map \( f : X \to X \) of a compactum \( X \) and a natural number \( n \), put

\[
\varphi(f, n) = \sup \{|\text{Comp}(f^{-n}(y))| \ | y \in X\}.
\]
A map \( f : X \to Y \) of compacta is an at most \( k \)-to-1 map if for each \( y \in Y \), \( |f^{-1}(y)| \leq k \).

A map \( f : X \to Y \) of compacta is a finite-to-one map if for each \( y \in Y \) the cardinality of the fiber \( f^{-1}(y) \) is finite, i.e., \( |f^{-1}(y)| < \infty \).

A continuum \( X \) is a regular curve if for each \( x \in X \) and each open neighborhood \( U \) of \( x \) in \( X \), there is an open neighborhood \( V \) of \( x \) in \( U \) such that the boundary set \( \text{Bd}(V) \) of \( V \) is a finite set. Clearly, each regular curve is a Peano curve (=1-dimensional locally connected continuum). Note that no regular curve possesses an uncountable family of mutually disjoint nondegenerate subcontinua of \( X \) and hence for any regular curve \( X \), the set \( \{ C \in \text{Comp}(f^{-n}(y)) \mid y \in X, C \text{ is nondegenerate} \} \) is a countable set.

A continuum \( X \) is a dendrite (=1-dimensional compact AR) if \( X \) is a locally connected continuum which contains no simple closed curve. It is well known that every dendrite is a regular curve. There are many regular curves which are not local dendrites (=1-dimensional compact ANR). Many fractal sets (see [2,4]) are regular curves which are not local dendrites. For example, the Sierpinski’s triangle is a well-known regular curve, but the Menger universal curve is not a regular curve.

2. Topological entropy

For a map \( f : X \to X \) of a compactum \( X \), we define the topological entropy \( h(f) \) of \( f \) as follows (see [10]): Let \( n \) be a natural number and \( \varepsilon > 0 \). A subset \( F \) of \( X \) is an \((n, \varepsilon)\)-spanning set for \( f \) if for each \( x \in X \), there is \( y \in F \) such that

\[
\max \{ d(f^i(x), f^i(y)) \mid 0 \leq i \leq n-1 \} \leq \varepsilon.
\]

Let \( r_n(\varepsilon) \) be the smallest cardinality of all \((n, \varepsilon)\)-spanning sets for \( f \). Put

\[
r(\varepsilon) = \lim_{n \to \infty} \sup \{ 1/n \log r_n(\varepsilon) \}
\]

and

\[
h(f) = \lim_{\varepsilon \to 0} r(\varepsilon).
\]

It is well known that \( h(f) \) is equal to the topological entropy which was defined by Adler, Konheim and McAndrew (see [1]).

The main result in this paper is the following.

**Theorem 2.1.** If \( f : X \to X \) is a map of a regular curve \( X \), then

\[
h(f) \leq \lim_{n \to \infty} \sup \{ 1/n \log \varphi(f, n) \}.
\]

**Proof.** We may assume that \( \varphi(f, n) < \infty \) for each \( n \). Let \( \varepsilon > 0 \) and \( n \) a natural number. Since \( X \) is a regular curve, there is a finite open cover \( U = \{ U_1, U_2, \ldots, U_s \} \) of \( X \) such that \( \text{diam } U_i < \varepsilon \) and the boundary set \( \text{Bd}(U_i) \) is a finite set for each \( i \). Put \( A_1 = \text{Cl}(U_1), A_2 = \text{Cl}(U_2) - U_1 \). Inductively we put \( A_{i+1} = \text{Cl}(U_{i+1}) - \bigcup_{k=1}^{i} U_k \). Then we may assume that \( A_i \neq \emptyset \) for each \( i \). Let \( \mathcal{A} = \{ A_i \mid i = 1, 2, \ldots, s \} \). Then \( \mathcal{A} \) is a finite closed cover of \( X \) such that if \( A, A' \in \mathcal{A} \) and \( A \neq A' \), then \( A \cap A' = \text{Bd}(A) \cap \text{Bd}(A') \), \( \text{Bd}(A) \) is a finite set, and diam \( A < \varepsilon \). Put

\[
B = \bigcup \{ \text{Bd}(A) \mid A \in \mathcal{A} \}.
\]

Let \( L = |B| = L_{\mathcal{A}} \). Note that \( L < \infty \).

Suppose that \( 1 \leq i \leq n - 1 \), \( b \in B \) and \( D \in \text{Comp}(f^{-i}(b)) \). If \( D \) contains no element of \( B \), we choose a point \( c = c(b, i, D) \in D \). If \( D \) contains an element of \( B \), we choose a point \( c = c(b, i, D) \in B \cap D \).

Consider the set

\[
F = B \cup \{ c(b, i, D) \mid b \in B, \ 1 \leq i \leq n - 1, \ D \in \text{Comp}(f^{-i}(b)) \}.
\]

Then we see that

\[
|F| \leq L \cdot \sum_{i=0}^{n-1} \varphi(f, i).
\]
We shall show that $F$ is an $(n, \varepsilon)$-spanning set for $f$. Let $x$ be any point of $X$. We may assume $x \in X - F$. First, we will consider the following case (I).

Case (I) $x \in X - \bigcup_{i=0}^{n-1} f^{-i}(B)$: Let $E_j (0 \leq j \leq n - 1)$ be the component of $X - \bigcup_{i=0}^{j} f^{-i}(B)$ containing $f^{n-1-j}(x)$. Note that each regular curve is path connected. First, we consider the component $C_1(= E_{n-1})$ of $X - \bigcup_{i=0}^{n-1} f^{-i}(B)$ containing $x$. By [8, p. 75, (5.7), Boundary Bumping Theorem III],

$$\text{Cl}(C_1) \cap \bigcup_{i=0}^{n-1} f^{-i}(B) \neq \emptyset.$$ 

Choose a point $x_1 \in \text{Cl}(C_1) \cap \bigcup_{i=0}^{n-1} f^{-i}(B)$. Since $\text{Cl}(C_1)$ is also a regular curve, there is an arc $P_1$ from $x = x_0$ to $x_1$ in $\text{Cl}(C_1)$. We may assume that $P_1 - \{x_1\} \subset C_1$. Note that $f^k(P_1 - \{x_1\}) \subset f^k(C_1) \subset E_{n-1-k}$ for each $k = 0, 1, \ldots, n - 1$. If $x_1 \in F$, we put $y = x_1$. If $x_1$ is not in $F$, we choose $1 \leq i_1 \leq n - 1$ such that $x_1 \in f^{-i_1}(B)$ and $x_1$ is not contained in $f^{-i}(B)$ ($0 \leq i < i_1$). Next we choose $D_1 \subset \text{Comp}(f^{-i_1}(B))$ containing $x_1$. If $D_1$ contains no element of $\text{Comp}(f^{-i}(B)) (0 \leq i \leq i_1 - 1)$, we can choose $y = c(b, i_1, D_1) \in D_1 \cap F$. If $D_1$ contains an element of $\text{Comp}(f^{-i}(B)) (0 \leq i \leq i_1 - 1)$, we choose the component $C_2$ of $D_1 - \bigcup_{i=0}^{i_1-1} f^{-i}(B)$ containing $x_1$. Also we choose $x_2 \in \text{Cl}(C_2) \cap \bigcup_{i=0}^{i_1-1} f^{-i}(B)$ and an arc $P_2$ from $x_1$ to $x_2$ in $D_1$ such that $P_2 - \{x_2\} \subset C_2$. If $x_2 \in F$, we put $y = x_2$. If $x_2$ is not in $F$, we choose $1 \leq i_2 < i_1$ such that $x_2 \in f^{-i_2}(B)$ and $x_2$ is not contained in $f^{-i}(B)$ ($0 \leq i < i_2$). We choose $D_2 \subset \text{Comp}(f^{-i_2}(B))$ containing $x_2$. We continue this procedure.

Then for some $1 \leq m \leq n - 1$, we obtain a finite sequences $x = x_0, x_1, \ldots, x_m$ of points of $X$, a finite sequence $P_1, P_2, \ldots, P_m$ of arcs, a finite sequence $D_1, D_2, \ldots, D_m$ of continua and a finite sequence $i_1, i_2, \ldots, i_{m-1}$ of natural numbers such that

1. $0 \leq i_{m-1} < \cdots < i_1 \leq n - 1$,
2. $x_j \in P_j \cap D_j \ (j = 1, 2, \ldots, m),$
3. $x_1, x_2, \ldots, x_{m-1} \not\in X - F$,
4. $P_1$ is an arc from $x_0 = x$ to $x_1$ contained in the component $C_1$ of $X - \bigcup_{i=0}^{n-1} f^{-i}(B)$, and for $m \geq j \geq 2$, $P_j$ is an arc from $x_{j-1}$ to $x_j$ contained in the component $C_j$ of $D_{j-1} - \bigcup_{i=0}^{j-1} f^{-i}(B)$,
5. $P_j - \{x_j\} \subset C_j \ (j = 1, 2, \ldots, m),$
6. $x_j \ (j = 1, 2, \ldots, m - 1)$ is not contained in $f^{-i}(B)$ ($0 \leq i < i_j$),
7. $D_j \subset \text{Comp}(f^{-i}(B)) \ (j = 1, 2, \ldots, m - 1),$ and
8. either (*); $x_m \not\in F$ or (**); $x_m \in X - F$ and there is $i_m (\ < i_{m-1})$ such that $x_m \in D_m \subset \text{Comp}(f^{-i_m}(B))$ and $D_m$ contains no element of $\text{Comp}(f^{-i}(B)) (0 \leq i < i_m)$.

Consequently, for any $x \in X - F$ we obtain a sequence $x = x_0, x_1, \ldots, x_m$ of points of $X$ satisfying the above conditions. First, we will consider the case (**) that $x_m \in X - F$ (see (8)). In this case, since $D_m$ contains no element of $\text{Comp}(f^{-i}(B)) (0 \leq i < i_m)$ (see (8)), we can choose $y \in F \cap D_m$. Let $P_{m+1}$ be an arc from $x_m$ to $y$ in $D_m$. We choose $A_k \in A$ such that $f^k(x) \in \text{Int}(A_k)$ for each $k = 0, 1, 2, \ldots, n - 1$. Since the connected set $P_1 \cup P_2 \cup \ldots \cup P_{m+1}$ does not meet the set $B$, it is contained in $\text{Int}(A_0)$. Also we see that $f^k(P_1 - \{x_1\}) \subset E_{n-1-k}$ ($k = 0, 1, \ldots, n - 1$), for each $j = 2, \ldots, m + 1$ $f^k(P_j - \{x_j\})$ is contained in the component $E_{j-1-k}$ ($k = 0, 1, \ldots, i_{j-1} - 1$), and $f^{i_{j-1}}(P_j) \subset B$. By using these facts, we see that $f^k(P_1 \cup P_2 \cup \ldots \cup P_{m+1}) \subset \text{Cl}(A_k) = A_k$ for each $k = 0, 1, 2, \ldots, n - 1$. Then we see that

$$\max \{d(f^i(x), f^i(y)) \mid 0 \leq i \leq n - 1\} \leq \varepsilon.$$

Secondly, we will consider the case (*) that $x_m = y \in F$ (see (8)). Note that $P_1 \cup P_2 \cup \ldots \cup (P_m - \{y\})$ does not meet the set $B$. In this case, similarly we can prove that $f^k(P_1 \cup P_2 \cup \ldots \cup P_m) \subset \text{Cl}(A_k) = A_k$ for each $k = 0, 1, 2, \ldots, n - 1$.

Next, we will consider the following case (II).

Case (II) $x \in \bigcup_{i=0}^{n-1} f^{-i}(B)$: Choose a natural number $l < n - 1$ such that $x \in f^{-i}(B) - \bigcup_{i=0}^{l-1} f^{-i}(B)$. Take $X' \subset \text{Comp}(f^{-i}(B))$ containing $x$. In this case, by the similar way to the case (I), we choose $y \in F$. If necessary, we may replace $l$ and $X'$ with $n$ and $X$. Also, by the similar way to the case (I), we can prove that

$$\max \{d(f^i(x), f^i(y)) \mid 0 \leq i \leq n - 1\} \leq \varepsilon.$$
Consequently, $F$ is an $(n, \varepsilon)$-spanning set for $f$. For each $n$, choose $i_n$ ($0 \leq i_n \leq n - 1$) such that
\[
\max\left\{\phi(f, i) \mid i = 0, 1, \ldots, n - 1\right\} = \phi(f, i_n).
\]

Then
\[
r(\varepsilon) = \limsup_{n \to \infty} (1/n) \log r_n(\varepsilon)
\leq \limsup_{n \to \infty} (1/n) \log (n \cdot L \cdot \max\left\{\phi(f, i) \mid i = 0, 1, \ldots, n - 1\right\})
= \limsup_{n \to \infty} (1/n) (\log n + \log L + \log \phi(f, i_n))
= \limsup_{n \to \infty} \left(i_n / (n \cdot i_n)\right) \log \phi(f, i_n)
\leq \limsup_{n \to \infty} (1/i_n) \log \phi(f, i_n) = \limsup_{n \to \infty} (1/n) \log \phi(f, n).
\]

Hence
\[
h(f) = \lim_{\varepsilon \to 0} r(\varepsilon) \leq \limsup_{n \to \infty} (1/n) \log \phi(f, n).
\]

This completes the proof. $\square$

For a map $f : X \to X$ of a compactum $X$ and a natural number $n$, put
\[
\Psi(f, n) = \sup \{|\text{Comp}(f^{-n}(C))| \mid C \text{ is a continuum of } X\}.
\]
Note that (1) $\phi(f, n) \leq \Psi(f, n)$ and (2) $\Psi(f, m + n) \leq \Psi(f, m) \cdot \Psi(f, n)$. The condition (2) implies that $\lim_{n \to \infty} (1/n) \log \Psi(f, n)$ exists (see [10, p. 87, Theorem 4.9]).

**Corollary 2.2.** Let $X$ be a regular curve. If $f : X \to X$ is a map, then
\[
h(f) \leq \lim_{n \to \infty} (1/n) \log \Psi(f, n).
\]
In particular, if there is a natural number $k$ such that for any subcontinuum $C$ of $X$
\[
|\text{Comp}(f^{-1}(C))| \leq k,
\]
then $h(f) \leq \log k$.

**Proof.** Note that $\Psi(f, n) \leq k^n$ for each $n = 0, 1, \ldots$. Then
\[
h(f) \leq \lim_{n \to \infty} (1/n) \log \Psi(f, n) \leq \log k.
\]

**Corollary 2.3.** Let $X$ be a regular curve. If $f : X \to X$ is an at most $k$-to-1 map, then $h(f) \leq \log k$.

In particular, we have

**Corollary 2.4.** Let $G$ be a connected graph (=1-dimensional finite polyhedron). If $f : G \to G$ is a map, then
\[
h(f) \leq \lim_{n \to \infty} (1/n) \log \phi(f, n).
\]

**Proof.** Since $G$ is a graph, there is a natural number $l$ such that for any continuum $C$ of $G$, $|\text{Bd}(C)| < l$. Let $C$ be a continuum of $G$. If $D \in \text{Comp}(f^{-n}(C))$, then $D \cap f^{-n}(\text{Bd}(C)) \neq \emptyset$. Take a function $h : \text{Comp}(f^{-n}(C)) \to \text{Comp}(f^{-n}(\text{Bd}(C)))$ such that $D \cap h(D) \neq \emptyset$ for $D \in \text{Comp}(f^{-n}(C))$. Note that $h$ is injective. Hence $(1/l)\psi(f, n) \leq \psi(f, n) \leq \Psi(f, n)$. Then
\[
\log \Psi(f, n) - \log l \leq \log \phi(f, n) \leq \log \Psi(f, n).
\]
Since \( \lim_{n \to \infty} (1/n) \log \Psi(f, n) \) exists, we see that
\[
\lim_{n \to \infty} (1/n) \log \varphi(f, n) = \lim_{n \to \infty} (1/n) \log \Psi(f, n).
\]

A point \( p \) of a locally connected continuum \( X \) is a local separating point of \( X \) if there is a connected neighborhood \( C \) of \( p \) such that \( C - \{p\} \) is not connected. For example, the set of all local separating points of the Sierpinski’s triangle is a countable set. For a map \( f : X \to X \) of a regular curve \( X \), put
\[
\varphi_{ls}(f, n) = \sup \{|\text{Comp}(f^{-n}(y))| \mid y \text{ is a local separating point of } X\}.
\]

Then \( \varphi_{ls}(f, n) \leq \varphi(f, n) \).

**Theorem 2.5.** If \( f : X \to X \) is a map of a regular curve \( X \), then
\[
h(f) \leq \limsup_{n \to \infty} (1/n) \log \varphi_{ls}(f, n).
\]

**Proof.** We see that in the proof of Theorem 2.1, the finite closed cover \( A \) satisfies the further property: Each point of \( \text{Bd}(A) \) (\( A \in \mathcal{A} \)) is a local separating point of \( X \).

**Remark 1.** The followings are well known:

1. If \( f : I = [0, 1] \to I \) is a (finite) piecewise monotone map, then \( h(f) = \lim_{n \to \infty} (1/n) \log l(f^n) \), where \( l(f^n) \) denotes the lap number of \( f^n \) (see [7]).
2. If \( f : G \to G \) is a map of a graph \( G \) and \( h(f) > 0 \), then there exist sequences \( (k_n)_{n=1}^{\infty} \) and \( (s_n)_{n=1}^{\infty} \) of positive integers such that for each \( n \) the map \( f^{k_n} \) has an \( s_n \)-horseshoe and \( \limsup_{n \to \infty} \log s_n / k_n = h(f) \) (see [6]).

**Remark 2.** There is a homeomorphism \( f : \mu^1 \to \mu^1 \) of the Menger universal curve \( \mu^1 \) such that \( h(f) = \infty \). Note that the Menger universal curve is a locally connected curve which is not a regular curve.

**References**