Oscillations of Second-Order Nonlinear Ordinary Differential Equations with Impulses

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1. INTRODUCTION

Several authors including Butler [2], Lakshmikantham et al. [3], Travis [4], and Wong [5], studied oscillations of second-order nonlinear ordinary differential equations. However, relatively less attention has been given to oscillations of the second-order nonlinear ordinary differential equations with impulses. Very recently, Chen and Feng [1] studied second-order nonlinear ordinary differential equations with impulses,

\[ x'' + f(t, x) = 0, \quad t \geq t_0, t \neq k, k = 1,2,\ldots, \]
\[ x(t_k^+) = g_k(x(t_k)), \quad x'(t_k^-) = h_k(x'(t_k)), \quad k = 1,2,\ldots, \]
\[ x(t_0^-) = x_0, \quad x'(t_0^+) = x'_0, \]

and they obtained some interesting results.
In the present paper, we investigate oscillations of the second-order nonlinear ordinary differential equation with impulses,

\[ [r(t)x'(t)]'' + f(t, x) = 0, \quad t \geq t_0, t \neq t_k, k = 1, 2, \ldots, \]

\[ x(t_k^-) = g_k(x(t_k)), \quad x'(t_k^-) = h_k(x'(t_k)), \quad k = 1, 2, \ldots, \quad (1.2) \]

\[ x(t_0^-) = x_0, \quad x'(t_0^-) = x'_0, \]

where \(0 \leq t_0 < t_1 < \cdots < t_k < \cdots\), and \(\lim_{k \to \infty} t_k = +\infty\),

\[ x'(t_k) = \lim_{h \to 0^-} \frac{x(t_k + h) - x(t_k)}{h}, \]

\[ x'(t_k^+) = \lim_{h \to 0^+} \frac{x(t_k + h) - x(t_k^+)}{h}. \]

Clearly, the system (1.2) is more general than the system (1.1).

Throughout this paper, we assume that

(i) \(f(t, x)\) is continuous in \([t_0, \infty) \times (-\infty, +\infty)\), \(xf(t, x) > 0\), \((f(t, x))/((\psi(x)) \geq p(t) (x \neq 0)\), where \(p(t)\) is continuous in \([t_0, \infty)\), \(p(t) \geq 0\), and \(x\psi(x) > 0 (x \neq 0)\), \(\psi'(x) > 0\), and \(r(t)\) is continuous in \([t_0, \infty)\), \(r(t) > 0\).

(ii) \(g_k(x), h_k(x)\) are continuous in \((-\infty, +\infty)\) and there exist positive numbers \(a_k, a_k^+, b_k, b_k^+\) such that

\[ a_k^+ \leq g_k(x)/x \leq a_k, \quad b_k^+ \leq h_k(x)/x \leq b_k. \]

A function \(x: [t_0, t_0 + a] \to \mathbb{R}, t_0 \geq 0, a > 0\), is said to be a solution of (1.2) if

(a) \(x(t_0^-) = x_0, x'(t_0^-) = x'_0\);

(b) \(x(t)\) satisfies \([r(t)x'(t)]'' + f(t, x) = 0\), when \(t \in [t_0, t_0 + a]\), \(t \neq t_k\);

(c) \(x(t_k^+) = g_k(x(t_k)), x'(t_k^+) = h_k(x'(t_k))\), and for any such \(t_k\), both \(x(t)\) and \(x'(t)\) are assumed to be left continuous.

As is customary, a solution of (1.2) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

The main purpose of this paper is to investigate the oscillations of the system (1.2). It is shown that our results generalize and improve those of Chen and Feng [1].
2. MAIN RESULTS

First, we obtain the following lemma.

**Lemma 2.1.** Let \( x(t) \) be a solution of \( (1.2) \). Suppose that there exists some \( t_1 \geq t_0 \) such that \( x(t) > 0 \) \( \text{for} \ t \geq t_1 \). If conditions (i) and (ii) are satisfied and

\[
\int_{t_0}^{t_1} r(t) \frac{ds}{r(s)} + \frac{b_1^*}{a_1} \int_{t_0}^{t_1} r(t) \frac{ds}{r(s)} + \frac{b_1^* b_2^*}{a_1 a_2} \int_{t_2}^{t_0} r(t) \frac{ds}{r(s)} + \ldots
\]

\[
+ \frac{b_1^* b_2^* \cdots b_n^*}{a_1 a_2 \cdots a_n} \int_{t_n}^{t_0} r(t) \frac{ds}{r(s)} + \frac{b_1^* b_2^* \cdots b_n^*}{a_1 a_2 \cdots a_{n+1}} \int_{t_{n+1}}^{t_0} r(t) \frac{ds}{r(s)} + \ldots
\]

\[
= +\infty
\]

holds, then \( x'(t^*_k) \geq 0 \) \( \text{for} \ t \in (t_k, t_{k+1}) \), where \( \lim t_k = T \).

The proof is similar to that of Lemma 1 of Chen and Feng [1], so we omit it.

**Remark 2.1.** When \( r(t) = 1 \), Lemma 2.1 reduces to Lemma 1 of Chen and Feng [2].

Now we give theorems which provide sufficient conditions for the oscillations of \( (1.2) \).

**Theorem 2.1.** Assume that the conditions (i), (ii), and (iii) and Lemma 2.1 hold, and there exist a positive integer \( k_0 \), a nonnegative constant \( c \), and a continuous function \( F(t) \), \( t \in [t_0, \infty) \), such that \( a_k \geq 1 \) for \( k \geq k_0 \), \( \psi'(x) \geq c \geq 0 \), and

\[
\int_{t_0}^{t_1} \left[ p(t) - \frac{cr(t) F^2(t)}{4} \right] \exp \left( \int_{t_0}^{t} c F(s) \, ds \right) \, dt
\]

\[
+ \frac{1}{b_1} \int_{t_1}^{t_2} \left[ p(t) - \frac{cr(t) F^2(t)}{4} \right] \exp \left( \int_{t_0}^{t} c F(s) \, ds \right) \, dt
\]

\[
+ \cdots + \frac{1}{b_1 b_2 \cdots b_n} \int_{t_n}^{t_{n+1}} \left[ p(t) - \frac{cr(t) F^2(t)}{4} \right] \exp \left( \int_{t_0}^{t} c F(s) \, ds \right) \, dt
\]

\[
+ \cdots = +\infty.
\]

Then every solution of \( (1.2) \) is oscillatory.
Proof. Without loss of generality, we can assume $k_0 = 1$. If (1.2) has a nonoscillatory solution $x(t)$, we may also assume $x(t) > 0 \ (t \geq t_0)$. From Lemma 2.1, we can see $x'(t) \geq 0$ for $t \in (t_k, t_{k+1}]$, where $k = 1, 2, \ldots$.

Let

$$u(t) = \frac{r(t)x'(t)}{\psi(x(t))}.$$ 

Then $u(t^*_k) \geq 0 \ (k = 1, 2, \ldots)$, $u(t) \geq 0 \ (t \geq t_0)$. Using condition (i), by (1.2), we obtain, when $t \neq t_k$,

$$u'(t) = -\frac{f(t, x(t))}{\psi(x(t))} - \left[ \frac{r(t)x'(t)}{\psi(x(t))} \right]^2 \frac{\psi'(x(t))}{r(t)}$$

$$\leq -p(t) - [u(t)]^2 \cdot \frac{c}{r(t)}$$

$$= -\left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \cdot \left[ (u(t))^2 \cdot \frac{c}{r(t)} + \frac{cr(t)F^2(t)}{4} \right]$$

$$\leq -\left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] - cu(t)F(t);$$

that is,

$$\left( u(t) \exp \left( \int_{t_0}^{t} cF(s) \, ds \right) \right)' \leq -\left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp \left( \int_{t_0}^{t} cF(s) \, ds \right).$$

(2.3)

Condition (ii) and $a_k^2 \geq 1$, $\psi'(x) \geq c > 0$ yield

$$u(t_k^*) = \frac{r(t_k)x'(t_k)}{\psi(x(t_k))} \leq \frac{b_k r(t_k)x'(t_k)}{\psi(a_k^2 x(t_k))} \leq \frac{b_k r(t_k)x'(t_k)}{\psi(x(t_k))} \leq b_k u(t_k),$$

$$k = 1, 2, \ldots$$

(2.4)
Integrating (2.3) from \( s \) to \( s_1 \), we obtain

\[
\begin{align*}
    u(s_1) & \leq u(s) \exp \left( \int_{s_1}^{s} cF(\tau) \, d\tau \right) \\
    & \quad - \int_{s}^{s_1} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp \left( \int_{s_1}^{s} cF(\tau) \, d\tau \right) \, dt,
\end{align*}
\]

(2.5)

where \( t_0 < s < s_1 < t_1 \). Let \( s \to t_0^+ \) and \( s_1 \to t_1^+ \). It follows from (2.4) and (2.5) that

\[
\begin{align*}
    u(t_1^+) & \leq b_1 u(t_1) \leq b_1 \left[ u(t_0^+) \exp \left( \int_{t_0}^{t_1} cF(\tau) \, d\tau \right) \\
    & \quad - \int_{t_0}^{t_1} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp \left( \int_{t_1}^{t} cF(\tau) \, d\tau \right) \, dt \right] \exp \int_{t_0}^{t_1} cF(\tau) \, d\tau.
\end{align*}
\]

(2.6)

Similarly, the following inequality holds

\[
\begin{align*}
    u(t_2^+) & \leq b_2 u(t_2) \\
    & \leq b_2 \left[ u(t_0^+) \exp \left( \int_{t_0}^{t_2} cF(\tau) \, d\tau \right) \\
    & \quad - \int_{t_0}^{t_2} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp \left( \int_{t_2}^{t} cF(\tau) \, d\tau \right) \, dt \right] \exp \int_{t_0}^{t_2} cF(\tau) \, d\tau.
\end{align*}
\]

(2.7)
By the principle of mathematical induction, for any natural number $n$, we obtain

$$u(t_n^+) \leq b_1 b_2 \cdots b_n u(t_0^+) \exp\left\{ \int_{t_0}^{t_n} c F(\tau) \, d\tau \right\}$$

$$- b_1 b_2 \cdots b_n \int_{t_0}^{t_1} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_0}^{t_1} c F(\tau) \, d\tau \right\} \, dt$$

$$- b_2 b_3 \cdots b_n \int_{t_1}^{t_2} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_1}^{t_2} c F(\tau) \, d\tau \right\} \, dt$$

$$- \cdots - b_{n-1} b_n \int_{t_n-1}^{t_n} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_{n-1}}^{t_n} c F(\tau) \, d\tau \right\} \, dt$$

$$- b_n \int_{t_{n-2}}^{t_n} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_{n-2}}^{t_n} c F(\tau) \, d\tau \right\} \, dt$$

$$= b_1 b_2 \cdots b_n \exp\left\{ \int_{t_0}^{t_n} c F(\tau) \, d\tau \right\}$$

$$\times \left\{ u(t_0^+) - \int_{t_0}^{t_1} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_0}^{t_1} c F(\tau) \, d\tau \right\} \, dt \right.$$

$$- \frac{1}{b_1} \int_{t_1}^{t_2} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_1}^{t_2} c F(\tau) \, d\tau \right\} \, dt \cdots$$

$$- \frac{1}{b_1 b_2} \cdots \frac{1}{b_{n-2} b_{n-1}} \int_{t_{n-2}}^{t_{n-1}} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_{n-1}}^{t_{n-2}} c F(\tau) \, d\tau \right\} \, dt$$

$$\times \exp\left\{ \int_{t_0}^{t_n} c F(\tau) \, d\tau \right\} \, dt$$

$$- \frac{1}{b_1 b_2} \cdots \frac{1}{b_{n-1} b_n} \int_{t_{n-1}}^{t_n} \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \exp\left\{ \int_{t_{n-1}}^{t_n} c F(\tau) \, d\tau \right\} \, dt$$

$$\times \exp\left\{ \int_{t_0}^{t_n} c F(\tau) \, d\tau \right\} \, dt \right\}. \quad (2.8)$$

Equations (2.2), (2.8) and $u(t_k^+) \geq 0 \ (k = 1, 2, \ldots)$ lead to a contradiction. Hence, every solution of (1.2) is oscillatory. The proof of Theorem 2.1 is complete.
Remark 2.2. When \( r(t) = 1 \) and \( c = 0 \), i.e., \( \psi'(x) \geq 0 \), Theorem 2.1 reduces to Theorem 1 of Cheng and Feng [1]. So Theorem 2.1 generalizes and improves Theorem 1 of [1] for Eq. (1.2).

**Theorem 2.2.** Assume that the conditions (i), (ii), and (iii) of Lemma 2.1 hold and there exist a nonnegative constant \( c \) and a continuous function \( F(t) \), \( t \in [t_0, \infty) \), such that \( \psi(a, b) \geq \psi(a)\psi(b) \) for any \( ab > 0 \), \( \psi(x) \geq c \), and

\[
\int_{t_0}^{t_1} G(t) \, dt + \frac{\psi(a^1_k)}{b_1} \int_{t_1}^{t_2} G(t) \, dt + \frac{\psi(a^2_k)\psi(a^3_k)}{b_1b_2} \int_{t_2}^{t_3} G(t) \, dt + \cdots \\
+ \frac{\psi(a^1_k)\psi(a^2_k)\cdots\psi(a^n_k)}{b_1b_2\cdots b_n} \int_{t_{n-1}}^{t_n} G(t) \, dt + \cdots + \cdots = +\infty, \tag{2.9}
\]

where

\[
G(t) = \left[ p(t) - \frac{cr(t)F^2(t)}{4} \right] \cdot \exp \left( \int_t^{t_0} cF(s) \, ds \right).
\]

Then every solution of (1.2) is oscillatory.

**Proof.** If (1.2) has a nonoscillatory solution \( x(t) \), without loss of generality, we can assume that \( x(t) > 0 \) \( (t \geq t_0) \). By Lemma 2.1, \( x'(t) \geq 0 \) \( (t \geq t_0) \). Let

\[
u(t) = \frac{r(t)x'(t)}{\psi(x(+)^+)}.
\]

Then \( u(t) \geq 0 \) \( (t \geq t_0) \), \( u(t_k^+) \geq 0 \) \( (k = 1, 2, \ldots) \). Relation (1.2) yields (2.3). It is easy to see that

\[
u(t_k^+) \leq \frac{r(t)x'(t_k^+)}{\psi(x(t_k^+))} \leq \frac{r(t)b_kx'(t_k)}{\psi(a_k^+\psi(t_k))} \leq \frac{r(t)x'(t_k^+)}{\psi(a_k^+\psi(t_k))} \leq \frac{b_k}{\phi(a_k^+)} u(t_k).
\]

(2.10)
Similar to the proof of (2.8) of Theorem 2.1 by induction, we get, for any natural number $n$,

\[ u(t_n^+) \leq (b_1 b_2 \cdots b_n) \psi(a_1^n) \psi(a_2^n) \cdots \psi(a_n^n)^{-1} \exp \left( \int_{t_n}^{t_0} c F(s) \, ds \right) \]

\[
\times \left( u(t_n^+) - \int_{t_0}^{t_n} G(t) \, dt - \frac{\psi(a_1^n) \psi(a_2^n) \cdots \psi(a_{n-2}^n)}{b_1 b_2 \cdots b_{n-2}} \int_{t_{n-2}}^{t_n} G(t) \, dt \right. \\
- \frac{\psi(a_1^n) \cdots \psi(a_{n-1}^n)}{b_1 \cdots b_{n-1}} \int_{t_{n-1}}^{t_n} G(t) \, dt \right), \tag{2.11}
\]

Relations (2.9), (2.11) and $u(t_n^+) \geq 0 (k = 1, 2, \ldots)$ lead to a contradiction. Hence, every solution of (1.2) is oscillatory. The proof of Theorem 2.2 is complete.

Remark 2.3. If $r(t) \equiv 1$ and $c = 0$, Theorem 2.2 reduces the Theorem 2 of Wong [5].

Let \( \int_{\pm \frac{\varphi(0)}{c}}^{\pm \infty} \frac{du}{\psi(u)} < +\infty \) denote

\[
\int_{\pm \frac{\varphi(0)}{c}}^{\infty} \frac{du}{\psi(u)} < +\infty \quad \text{and} \quad \int_{-\infty}^{\pm \frac{\varphi(0)}{c}} \frac{du}{\psi(u)} < +\infty, \tag{2.12}
\]

The following theorems provide the oscillations of Eq. (1.2) when (2.12) holds. The proof is very similar to those of Theorems 2 and 3 of Cheng and Feng [1] and hence, we omit them.

Theorem 2.3. Assume that conditions (i), (ii), and (iii) and Lemma 2.1 hold, and there exists a positive integer $k_0$ such that $a_k^+ \geq 1$ for $k \geq k_0$. If

\[
\int_{\pm \frac{\varphi(0)}{c}}^{\pm \infty} \frac{du}{\psi(u)} < +\infty \tag{2.13}
\]

hold for some $\varepsilon > 0$ and

\[
\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \left[ \int_{t_k}^{t_{k+1}} p(t) \, dt + \frac{1}{b_{k+1} b_{k+1}} \int_{t_k}^{t_{k+1}} p(t) \, dt + \frac{1}{b_{k+1} b_{k+2} b_{k+2}} \int_{t_k}^{t_{k+1}} p(t) \, dt + \cdots \\
+ \frac{1}{b_{k+1} b_{k+2} \cdots b_{k+n+1}} \int_{t_k}^{t_{k+n+1}} p(t) \, dt + \cdots \right] \, ds = +\infty, \tag{2.14}
\]

then every solution of (1.2) is oscillatory.
Theorem 2.4. Assume that conditions (i), (ii), and (iii) and Lemma 2.1 hold, and there exists a positive $k_0$ such that $a_k^s \geq 1$ for $k \geq k_0$. Suppose that $\psi(ab) \geq \psi(a)\psi(b)$ for any $ab > 0$, (2.13) and (2.14) hold, and

$$\int_{\pm \infty}^{\infty} \frac{du}{\psi(u)} < +\infty \quad \text{for some } \epsilon > 0,$$

(2.15)

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \left[ \int_{t_k}^{t_{k+1}} p(t) \, dt \right. $$

$$+ \frac{\psi(a_{k+1}^s)}{b_{k+1}} \int_{t_k}^{t_{k+1}} p(t) \, dt $$

$$+ \left. \cdots + \frac{\psi(a_{k+n}^s)}{b_{k+n}} \int_{t_{k+n}}^{t_{k+n+1}} p(t) \, dt \right] ds $$

$$= +\infty.$$  

(2.16)

Then every solution of (1.2) is oscillatory.

Remark 2.4. When $r(t) = 1$, Theorems 2.3 and 2.4 reduce to Theorems 3 and 4 of Chen and Feng [1].

3. Examples

Example 3.1. Consider

$$x'' + \frac{1}{t^2} x = 0, \quad t \geq \frac{1}{2}, \ t \neq k, \ k = 1, 2, \ldots,$$

$$x(k^+) = \frac{k + 1}{k} x(k), \quad x'(k^+) = x'(k), \ k = 1, 2, \ldots,$$

(2.17)

$$x\left(\frac{1}{2}\right) = x_0, \quad x'\left(\frac{1}{2}\right) = x'_0.$$

It is not difficult to see that conditions (i), (ii), and (iii) are satisfied. Taking $F(t) = \frac{1}{t}$, by Theorem 2.1, it follows that every solution of (2.17) is oscillatory. But Theorem 1 of Wong [5] does not apply to system (2.17).
EXAMPLE 3.2. Consider
\[
\left(\frac{1}{t^2}x'\right)' + \frac{5}{4t^4}x = 0, \quad t \geq \frac{1}{2}, t \neq k, k = 1, 2, \ldots,
\]
\[
x(k^+) = \frac{k + 1}{k}x(k), \quad x'(k^+) = x'(k), \quad k = 1, 2, \ldots, \quad (2.18)
\]
\[
x\left(\frac{1}{2}\right) = x_0, \quad x'\left(\frac{1}{2}\right) = x'_0.
\]
Taking \(F(t) = t^{-3/4}, \quad c = 1\), we can show that the conditions of Theorem 2.1 are satisfied. Hence every solution of (2.18) is oscillatory. But the ordinary differential equation \((1/t^2)x' + (5/4t^4)x = 0\) has a nonnegative solution \(x = \sqrt{t}\). This example shows that impulses play an important role in the oscillations of the solution equation.

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