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Integral Inequalities for Monotone Functions

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Some integral inequalities for generalized monotone functions of one variable and an integral inequality for monotone functions of several variables are proved. Some applications are presented and discussed.

1. INTRODUCTION

We consider the following recent result of Heinig and Maligranda [10, Theorem 2.1]:

THEOREM 1 [10]. *Let* $-\infty < a < b \leq \infty$ *and let* f *and* g *be positive functions on* (a, b), *where* g *is continuous on* (a, b).

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(a) *Suppose that f is decreasing on* (a, b) *and g is increasing on* (a, b), *where* $g(a+0) = 0$. *Then, for any* $p \in (0, 1]$,

$$\int_a^b f(x)\,dg(x) \le \left(\int_a^b f^p(x)\,d[g^p(x)]\right)^{1/p}. \tag{1.1}$$

If $1 \le p < \infty$, *then the inequality* (1.1) *holds in the reversed direction.*

(b) *Assume that f is increasing on* (a, b) *and g is decreasing on* (a, b) *where* $g(b-0) = 0$. *Then, for any* $p \in (0, 1]$,

$$\int_a^b f(x)\,d[-g(x)] \le \left(\int_a^b f^p(x)\,d[-g^p(x)]\right)^{1/p}. \tag{1.2}$$

If $1 \le p < \infty$, *then the inequality* (1.2) *holds in the reversed direction.*

In this theorem and later on in this paper the terms positive, decreasing, and increasing shall be interpreted as nonnegative, nonincreasing, and nondecreasing, respectively.

Our first observation is that if $b < \infty$, then by making the transformation $t = a + b - x$, we find that the statements in (a) and (b) are in fact equivalent. Next we note that (1.1) can formally be written as

$$\varphi\left(\int_a^b f(x)\,dg(x)\right) \le \int_a^b \varphi'(f(x)g(x))f(x)\,dg(x), \tag{1.1$'$}$$

where $\varphi(u) = u^p$, $0 < p \le 1$. Moreover, we see that (1.1) implies that

$$\left(\int_a^b f^q(x)\,d[g^q(x)]\right)^{1/q} \le \left(\int_a^b f^p(x)\,d[g^p(x)]\right)^{1/p}, \tag{1.1$''$}$$

where $0 < p \le q < \infty$. We also consider the corresponding variants (1.2)′ and (1.2)″ of (1.2). In Section 2 of this paper we will generalize Theorem 1 by proving some direct and reversed inequalities of the type (1.1)′–(1.2)′, where the φ's are concave (or convex) functions (see Theorem 2.2) and also some direct and reversed inequalities of the type (1.1)″–(1.2)″ (see Theorem 2.4). In order to prepare our discussions later on and to be able to compare our results with some other recent results we prove our results in the more general case when decreasing (increasing) functions are replaced by C-decreasing (C-increasing) functions. In Section 3 we prove a multidimensional version of Theorem 1 (see Theorem 3.1). In Section 4 we use our previous results to obtain some new information about Bergh type inequalities (see Corollary 4.1 and Theorem 4.3) and to point out an application (see Corollary 4.6), which unifies and extends some well-known inequalities.

2. SOME EXTENSIONS OF THEOREM 1

In the sequel we consider positive real valued functions f, g defined on an interval (a, b), $-\infty < a < b \le \infty$. We say that f is C-decreasing [C-increasing], $C \ge 1$, if $f(t) \le Cf(s)$ [$f(s) \le Cf(t)$] whenever $s \le t$, $t, s \in (a, b)$.

EXAMPLE 2.1. Let $(a, b) = (0, \infty)$. We note that the increasing function f, defined by $f(x) = 1$, $0 < x \le 1$, and $f(x) = 2$, $1 < x < \infty$, is 2-decreasing. Moreover, the decreasing function $g(x) = 1 + \exp(-x)$ is 2-increasing and the function $h(x) = 1.5 + 0.5\sin x$ is both 2-decreasing and 2-increasing. In all these cases $C = 2$ cannot be replaced by any $C < 2$ (note that both the upper and the lower indices of all these functions are equal to zero, cf. final Remark 4.9(b)).

THEOREM 2.2. *Let* φ: $[0, \infty) \to R$ *be a concave, nonnegative, and differentiable function such that* $\varphi(0) = 0$ *and let* $-\infty \le a < b < \infty$.

(a) *If f is C-decreasing and g is increasing, differentiable and such that* $g(a + 0) = 0$, *then*

$$\varphi\left(C\int_a^b f(x)\,dg(x)\right) \le C\int_a^b \varphi'(f(x)g(x))f(x)\,dg(x). \tag{2.1}$$

(b) *If f is C-increasing and g is increasing, differentiable and such that* $g(a + 0) = 0$, *then*

$$\varphi\left(\frac{1}{C}\int_a^b f(x)\,dg(x)\right) \ge \frac{1}{C}\int_a^b \varphi'(f(x)g(x))f(x)\,dg(x). \tag{2.2}$$

(c) *If f is C-increasing and g is decreasing, differentiable and such that* $g(b - 0) = 0$, *then*

$$\varphi\left(C\int_a^b f(x)d[-g(x)]\right) \le C\int_a^b \varphi'(f(x)g(x))f(x)d[-g(x)]. \tag{2.3}$$

(d) *If f is C-decreasing and g is decreasing, differentiable and such that* $g(b - 0) = 0$, *then*

$$\varphi\left(\frac{1}{C}\int_a^b f(x)d[-g(x)]\right) \ge \frac{1}{C}\int_a^b \varphi'(f(x)g(x))f(x)d[-g(x)]. \tag{2.4}$$

(e) *If the condition "φ is concave" is replaced by "φ is convex," then all the inequalities* (2.1)–(2.4) *hold in the reversed direction.*

Proof. (a) Since f is C-decreasing it yields that

$$f(t)g(t) \le C\int_a^t f(x)\,dg(x). \tag{2.5}$$

Consider now the function

$$F(t) = \varphi\left(C\int_a^t f(x)\,dg(x)\right) - C\int_a^t \varphi'(f(x)g(x))f(x)\,dg(x),$$
$$a \le t \le b.$$

We note that

$$F'(t) = Cf(t)g'(t)\left[\varphi'\left(C\int_a^t f(x)\,dg(x)\right) - \varphi'(f(t)g(t))\right].$$

Therefore, according to (2.5) and the concavity assumption, we find that $F'(t) \le 0$ for $a \le t \le b$. Since, in addition, $F(a) = 0$ we conclude that $F(b) \le 0$ and (2.1) is proved,

(b) The function f is C-increasing and, thus,

$$f(t)g(t) \ge \frac{1}{C}\int_a^t f(x)\,dg(x). \tag{2.6}$$

We differentiate the function

$$G(t) = \varphi\left(\frac{1}{C}\int_a^t f(x)\,dg(x)\right) - \frac{1}{C}\int_a^t \varphi'(f(x)g(x))f(x)\,dg(x),$$
$$a \le t \le b,$$

and find, by (2.6) and the concavity assumption, that

$$G'(t) = \frac{1}{C}f(t)g'(t)\left[\varphi'\left(\frac{1}{C}\int_a^t f(x)\,dg(x)\right) - \varphi'(f(t)g(t))\right] \ge 0,$$
$$a \le t \le b.$$

Moreover, $G(a) = 0$ and we conclude that $G(b) \ge 0$ which means that (2.2) holds.

(c) According to our assumptions we have

$$f(t)g(t) \le C\int_t^b f(x)\,d[-g(x)]. \tag{2.7}$$

The rest of the proof only consists of making obvious modifications of the proof of (a).

(d) In this case our assumptions ensure that

$$f(t)g(t) \geq \frac{1}{C}\int_t^b f(x)d[-g(x)], \tag{2.8}$$

and the proof follows as above.

(e) The proof of the case when φ is convex is obviously similar.

Remark 2.3. For the special case $\varphi(u) = u^p$, $0 < p < 1$, the formulas (2.1) and (2.3) read

$$\left(\int_a^b f(x)\,dg(x)\right)^p \leq C^{1-p}\int_a^b f^p(x)d[g^p(x)], \tag{2.9}$$

respectively,

$$\left(\int_a^b f(x)d[-g(x)]\right)^p \leq C^{1-p}\int_a^b f^p(x)d[-g^p(x)]. \tag{2.10}$$

This shows in particular that Theorem 1 follows from Theorem 2.2(a) and (c) with $C = 1$. In fact, the inequalities (2.9)–(2.10) hold even with the constant C^{1-p} replaced by the smaller constant $C^{p(1-p)}$. More generally, we shall finish this section by proving the following theorem:

THEOREM 2.4. *Assume that* $0 < p < q < \infty$ *and* $-\infty \leq a < b < \infty$.

(a) *If f is C-decreasing and g is increasing, differentiable and such that* $g(a+0) = 0$, *then*

$$\left(\int_a^b f^q(x)d[g^q(x)]\right)^{1/q} \leq C^{1-p/q}\left(\int_a^b f^p(x)d[g^p(x)]\right)^{1/p}. \tag{2.11}$$

(b) *If f is C-increasing and g is increasing, differentiable and such that* $g(a+0) = 0$, *then*

$$\left(\int_a^b f^q(x)d[g^q(x)]\right)^{1/q} \geq C^{p/q-1}\left(\int_a^b f^p(x)d[g^p(x)]\right)^{1/p}. \tag{2.12}$$

(c) *If f is C-increasing and g is decreasing, differentiable and such that* $g(b-0) = 0$, *then*

$$\left(\int_a^b f^q(x)d[-g^q(x)]\right)^{1/q} \leq C^{1-p/q}\left(\int_a^b f^p(x)d[-g^p(x)]\right)^{1/p}. \tag{2.13}$$

(d) *If f is C-decreasing and g is decreasing, differentiable and such that $g(b-0)=0$, then*

$$\left(\int_a^b f^q(x)\,d[-g^q(x)]\right)^{1/q} \geq C^{p/q-1}\left(\int_a^b f^p(x)\,d[-g^p(x)]\right)^{1/p}. \quad (2.14)$$

Proof. The idea of the proof is similar to that of Theorem 2.2 but for the reader's convenience we will include the details.

(a) Since f is C-decreasing it yields that f^p is C^p-decreasing. Thus, in particular,

$$f^p(t)g^p(t) \leq C^p \int_a^t f^p(x)\,d[g^p(x)]. \quad (2.15)$$

Consider the function

$$F(t) = C^{q-p}\left(\int_a^t f^p(x)\,d[g^p(x)]\right)^{q/p} - \int_a^t f^q(x)\,d[g^q(x)], \qquad a \leq t \leq b.$$

We note that $F(a)=0$ and, according to (2.15),

$$\begin{aligned} F'(t) &= qg'(t)\left\{C^{q-p}\left[\left(\int_a^t f^p(x)\,d[g^p(x)]\right)^{q/p-1} f^p(t)g^{p-1}(t) \right.\right. \\ &\qquad \left.\left. - f^q(t)g^{q-1}(t)\right]\right\} \\ &\geq qg'(t)\left\{(f^p(t)g^p(t))^{q/p-1} f^p(t)g^{p-1}(t) - f^q(t)g^{q-1}(t)\right\} = 0. \end{aligned}$$

We conclude that $F(b) \geq 0$, i.e., that (2.11) holds.

(b) In this case f^p is C^p-increasing so that, in particular

$$f^p(t)g^p(t) \geq C^{-p}\int_a^t f^p(x)\,d[g^p(x)]. \quad (2.16)$$

Therefore, the proof follows by studying the function

$$F(t) = C^{p-q}\left(\int_a^t f^p(x)\,d[g^p(t)]\right)^{q/p} - \int_a^t f^q(x)\,d[g^q(x)], \qquad a \leq t \leq b,$$

and arguing as in the proof of (a).

(c), (d) In these cases the crucial inequalities corresponding to (2.15)–(2.16) read

$$f^p(t)g^p(t) \le C^p \int_t^b f^p(x)\,d[-g^p(x)] \qquad \text{and}$$

$$f^p(t)g^p(t) \ge C^{-p} \int_t^b f^p(x)\,d[-g^p(x)],$$

respectively. The remaining part of the proof is analogous to the proof of (a) and (b).

Remark 2.5. By applying Theorem 2.2 with $\varphi(u) = u^p$ we also obtain a result as in Theorem 2.4 but with less sharp constants for each $C > 1$. More exactly, in (a) and (c) we obtain the corresponding constants $C^{q/p-1}$ and in (b) and (d), $C^{1-q/p}$. This also means that the results as expected coincide for the case $C = 1$.

Remark 2.6. Theorem 2.4(a) and (c) are genuine generalizations of Theorem 1(a) and (b), respectively. For example, let g be a decreasing function on (0, 2) such that $g(2-0) = 0$ and consider the function f_d, $0 \le d \le 1$, defined by $f_d(x) = x$, $0 \le x < 1$, and $f_d(x) = x - d$, $1 \le x \le 2$. Then the inequality (1.2) cannot directly be applied to any function f_d, $0 < d < 1$, but (2.13) can be applied to all such f_d because f_d is obviously C-increasing with $C = 1/(1-d)$.

3. A MULTIDIMENSIONAL GENERALIZATION

Let $m \in \mathbb{N}$, $\mathbf{a} = (a_1, a_2, \dots, a_m)$, $\mathbf{b} = (b_1, b_2, \dots, b_m)$, and $\mathbf{x} = (x_1, x_2, \dots, x_m)$. We also use the (simplex) notation that if $\mathbf{a} = (a_1, a_2, \dots, a_m)$, then $\mathbf{a}_i = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m)$, $i = 1, 2, \dots, m-1$, and $\mathbf{a}_m = (a_1, a_2, \dots, a_{m\text{-}1})$. The notation $\mathbf{a} < \mathbf{b}$ ($\mathbf{a} \le \mathbf{b}$) means that $a_i < b_i$ ($a_i \le b_i$), $i = 1, 2, \dots, m$, and $(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} | a_i < x_i < b_i,\ i = 1, 2, \dots, m\}$.

The functions f considered in this section are assumed to be measurable and positive on $(\mathbf{a}, \mathbf{b})$. Moreover, f is increasing (decreasing) if $f(\mathbf{x}) \le f(\mathbf{y})$ ($f(\mathbf{y}) \le f(\mathbf{x})$) for $\mathbf{a} < \mathbf{x} \le \mathbf{y} < \mathbf{b}$. We also consider $\mathbf{g} = \{g_i\}$, $g_i = g_i(x_i)$, $i = 1, 2, \dots, m$, and use the notations

$$\int_{\mathbf{a}}^{\mathbf{b}} \cdots \mathbf{dg}(\mathbf{x}) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} \cdots\, dg_1(x_1)\,dg_2(x_2) \cdots dg_m(x_m)$$

and

$$\int_{\mathbf{a}}^{\mathbf{b}} \cdots \mathbf{d}[-\mathbf{g}(\mathbf{x})]$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} \cdots d[-g_1(x_1)]\, d[-g_2(x_2)] \cdots d[-g_m(x_m)].$$

Moreover, we say that $\mathbf{g}$ is increasing (decreasing, positive, or differentiable) if g_i, $i = 1, 2, \dots, m$, are increasing (decreasing, positive, or differentiable, respectively). Our main result in this section reads:

THEOREM 3.1. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, f: $(\mathbf{a}, \mathbf{b}) \to R$ *is positive*, $\mathbf{g} = \{g_i\}$, g_i: $(a_i, b_i) \to R$ *and* $\mathbf{g}$ *is positive and differentiable. Moreover, let* φ: $[0, \infty) \to R$ *be a two times differentiable function such that* $\varphi(0) = 0$ *and* $\lim_{t \to 0} t\varphi'(t) = 0$.

(a) *Suppose that* f *is decreasing and* $\mathbf{g}$ *is increasing, where* $g_i(a_i + 0) = 0$, $i = 1, 2, \dots, m$. *If* φ *is concave, then*

$$\varphi\left(\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{g}(x) \right) \le \int_{\mathbf{a}}^{\mathbf{b}} \varphi'\left(f(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i) \right) f(\mathbf{x})\, \mathbf{dg}(\mathbf{x}). \tag{3.1}$$

If φ *is convex, then* (3.1) *holds in the reversed direction.*

(b) *Assume that* f *is increasing and* $\mathbf{g}$ *is decreasing, where* $g_i(b_i - 0) = 0$, $i = 1, 2, \dots, m$. *If* φ *is concave, then*

$$\varphi\left(\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, \mathbf{d}[-\mathbf{g}(\mathbf{x})] \right) \le \int_{\mathbf{a}}^{\mathbf{b}} \varphi'\left(f(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i) \right) f(\mathbf{x})\, \mathbf{d}[-\mathbf{g}(\mathbf{x})]. \tag{3.2}$$

If φ *is convex, then* (3.2) *holds in the reversed direction.*

Remark 3.2. Let $\varphi(u) = u^p$, $0 < p < \infty$. In this case we have equality in (3.1) for every $f = C_0 \prod_1^m \chi(a_i, e_i)$, where C_0 is a positive constant, χ denotes the characteristic function, and $a_i \le e_i \le b_i$, $i = 1, 2, \dots, m$. Moreover, we have equality in (3.2) for every $f = C_0 \prod_1^m \chi(e_i, b_i)$. This means that the inequalities in Theorem 3.1 cannot be improved in general. Finally we note that Theorem 3.1 coincides with Theorem 1 when $\varphi(u) = u^p$ and $m = 1$.

For the proof of Theorem 3.1 we need the following lemma of independent interest:

LEMMA 3.3. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, f: $(\mathbf{a}, \mathbf{b}) \to R$ *be positive*, $\mathbf{g} = \{g_i\}$, g_i: $(a_i, b_i) \to R$, *and* $\mathbf{g}$ *be positive and differentiable. Moreover, let* ϕ: $[0, \infty) \to R$ *be a positive differentiable function such that* $\lim_{t \to 0} t\phi(t) = 0$.

(a) *Suppose that f is decreasing and* $\mathbf{g}$ *is increasing, where* $g_i(a_i + 0) = 0$, $i = 1, 2, \ldots, m$. *If* ϕ *is decreasing, then*

$$\phi\left(\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{g}(\mathbf{x})\right)\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{g}(\mathbf{x}) \leq \int_{\mathbf{a}}^{\mathbf{b}} \phi\left(f(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x})\, \mathbf{dg}(\mathbf{x}). \tag{3.3}$$

If ϕ *is increasing, then* (3.3) *holds in the reversed direction.*

(b) *Suppose that f is increasing and* $\mathbf{g}$ *is decreasing, where* $g_i(b_i - 0) = 0$, $i = 1, 2, \ldots, m$. *If* ϕ *is decreasing, then*

$$\phi\left(\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\right)\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})] \leq \int_{\mathbf{a}}^{\mathbf{b}} \phi\left(f(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]. \tag{3.4}$$

If ϕ *is increasing, then* (3.4) *holds in the reversed direction.*

Proof. (a) Let ϕ be decreasing. We will make an induction proof in the dimension m and consider first the case $m = 1$.

Let $t \in (a, b)$ and consider the function

$$h(t) = \phi\left(\int_a^t f(x)\, dg(x)\right)\int_a^t f(x)\, dg(x) - \int_a^t \phi(f(x)g(x)) f(x)\, dg(x).$$

The assumptions on ϕ imply that $h(a +) = 0$. Moreover, since $g'(t) \geq 0$, ϕ is decreasing, and $f(t)g(t) \leq \int_a^t f(x)\, dg(x)$, we find that

$$h'(t) = f(t)g'(t)\left\{\phi'\left(\int_a^t f(x) dg(x)\right)\int_a^t f(x) dg(x) + \phi\left(\int_a^t f(x) dg(x)\right) - \phi(f(t)g(t))\right\} \leq 0.$$

This means in particular that $h(b) \leq 0$ so that (3.3) holds for $m = 1$. Next we make the induction assumption that (a) holds for dimension $m - 1$.

Now we consider the function

$$h(\mathbf{t}) = \phi\left(\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\, \mathbf{dg}(\mathbf{x})\right)\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\, \mathbf{dg}(\mathbf{x}) - \int_{\mathbf{a}}^{\mathbf{t}} \phi\left(f(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x})\, \mathbf{dg}(\mathbf{x}).$$

As before we see that $h(\mathbf{a}+)=0$. We differentiate and find that

$$\begin{aligned}\frac{\partial h}{\partial t_1} = g_1'(t_1)\Bigg\{\phi'\left(\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x})\right) \\ \times \int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x}) \\ +\phi\left(\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x})\right)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1) \\ -\int_{\mathbf{a}_1}^{\mathbf{t}_1}\phi\left(f(t_1,\mathbf{x}_1)g_1(t_1)\prod_{i=2}^{m} g_i(x_i)\right)f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\Bigg\}. \quad (3.5)\end{aligned}$$

Moreover, since f is decreasing,

$$g_1(t_1)\int_{a_1}^{t_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1) \le \int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x}), \quad (3.6)$$

and, according to our induction assumption,

$$\begin{aligned}\phi\left(g(t_1)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\right)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1) \\ \le \int_{\mathbf{a}_1}^{\mathbf{t}_1}\phi\left(f(t_1,\mathbf{x}_1)g_1(t_1)\prod_{i=2}^{m} g_i(x_i)\right)f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1). \quad (3.7)\end{aligned}$$

Now by using the fact that ϕ is decreasing together with (3.5)–(3.7) we find that $\partial h/\partial t_1 \le 0$. In the same way we find that $\partial h/\partial t_i \le 0$, $i = 2,\dots,m$, and we conclude that $h(\mathbf{b}) \le 0$ which means that (3.1) holds for the dimension m and the proof is complete.

The proof of the case when ϕ is increasing follows by making some obvious modifications of the proof above.

(b) The proof is completely analogous to the proof of (a) so we do not include all details. We only note that in this case we consider the function

$$\begin{aligned}h(\mathbf{t}) = \phi\left(\int_{\mathbf{t}}^{\mathbf{b}} f(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\right)\int_{\mathbf{t}}^{\mathbf{b}} f(\mathbf{x})\,\mathbf{d}[-g(x)] \\ -\int_{\mathbf{t}}^{\mathbf{b}}\phi\left(f(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right)f(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})].\end{aligned}$$

Then our assumptions imply that the formulas (3.5)–(3.7) hold with integrals over the intervals $[\mathbf{t},\mathbf{b}]$ or $[\mathbf{t}_1,\mathbf{b}_1]$ instead of $[\mathbf{a},\mathbf{t}]$ or $[\mathbf{a}_1,\mathbf{t}_1]$, respectively.

Proof of Theorem 3.1. (a) Assume that φ is concave. We consider the function

$$h(\mathbf{t}) = \varphi\left(\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x})\right) - \int_{\mathbf{a}}^{\mathbf{t}} \varphi'\left(f(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x})\,\mathbf{dg}(\mathbf{x}),$$

and differentiate with respect to the first variable, i.e.,

$$\frac{\partial h}{\partial t_1} = g_1'(t_1)\left\{\varphi'\left(\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x})\right)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\right.$$
$$\left. - \int_{\mathbf{a}_1}^{\mathbf{t}_1}\varphi'\left(f(t_1,\mathbf{x}_1)g_1(t_1)\prod_{i=2}^{m} g_i(x_i)\right) f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\right\}.$$

Moreover, since f is decreasing and φ is concave,

$$\varphi'\left(\int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x})\right) \leq \varphi'\left(g_1(t_1)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\right),$$

and, according to Lemma 3.3(a),

$$\varphi'\left(g_1(t_1)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)\right)\int_{\mathbf{a}_1}^{\mathbf{t}_1} f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1)$$
$$\leq \int_{\mathbf{a}_1}^{\mathbf{t}_1}\varphi'\left(f(t_1,\mathbf{x}_1)g_1(t_1)\prod_{i=2}^{m} g_i(x_i)\right) f(t_1,\mathbf{x}_1)\,\mathbf{dg}_1(\mathbf{x}_1).$$

Therefore we find that $\partial h/\partial t_1 \leq 0$. Analogously we find that $\partial h/\partial t_i \leq 0$, $i = 2,\ldots,m$, and we conclude that $h(\mathbf{b}) \leq 0$ which means that (3.1) holds.

The proof in the case when φ is convex only consists of making obvious modifications of the proof above.

(b) The proof is completely analogous. We only need to study the function

$$h(\mathbf{t}) = \varphi\left(\int_{\mathbf{t}}^{\mathbf{b}} f(\mathbf{x})\,\mathbf{dg}(\mathbf{x})\right) - \int_{\mathbf{t}}^{\mathbf{b}} \varphi'\left(f(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x})\,\mathbf{dg}(\mathbf{x}),$$

argue as in the proof of (a), and use Lemma 3.3(b) instead of Lemma 3.3(a).

4. FURTHER RESULTS AND APPLICATIONS

Partly guided by some results and motivations recently presented in [13] we will here study the following classes of generalized monotone functions f: For $C_1, C_2 \geq 1$, $-\infty < \alpha_1 \leq \alpha_2 \leq \infty$, we say that $f \in Q^{\alpha_1}(C_1)$ if

$f(x)x^{-\alpha_1}$ is C_1-increasing and $f \in Q_{\alpha_2}(C_2)$ if $f(x)x^{-\alpha_2}$ is C_2-increasing. Moreover, let $Q^{\alpha_1}_{\alpha_2}(C_1, C_2) = Q^{\alpha_1}(C_1) \cap Q_{\alpha_2}(C_2)$.

In particular, by using our Theorem 2.4 we obtain a new proof of the following recent result [13, Theorem 2.1]:

COROLLARY 4.1 [13]. *Let* $0 < p \le q < \infty$.

(a) *If* $f \in Q^{\alpha_1}(C)$, $\alpha > \alpha_1$, *then, for any* $x > 0$,

$$\left(\int_x^\infty (t^{-\alpha} f(t))^q \frac{dt}{t}\right)^{1/q} \le p^{1/p} q^{-1/q} (\alpha - \alpha_1)^{1/p - 1/q} C^{1-p/q} \left(\int_x^\infty (t^{-\alpha} f(t))^p \frac{dt}{t}\right)^{1/p}. \quad (4.1)$$

(b) *If* $f \in Q_{\alpha_2}(C)$, $\alpha_2 > \alpha$, *then, for any* $x \ge 0$,

$$\left(\int_0^x (t^{-\alpha} f(t))^q \frac{dt}{t}\right)^{1/q} \le p^{1/p} q^{-1/q} (\alpha_2 - \alpha)^{1/p - 1/q} C^{1-p/q} \left(\int_0^x (t^{-\alpha} f(t))^p \frac{dt}{t}\right)^{1/p}. \quad (4.2)$$

Proof.

(a) We note that (4.1) is equivalent to the inequality

$$\left(\int_x^\infty (f(t)t^{-\alpha_1})^q d[-t^{q(\alpha_1 - \alpha)}]\right)^{1/q} \le C^{1-p/q} \left(\int_x^\infty (f(t)t^{-\alpha_1})^p d[-t^{p(\alpha_1 - \alpha)}]\right)^{1/p},$$

and this is just a consequence of our Theorem 2.4(c).

(b) We see that (4.2) is equivalent to the inequality

$$\left(\int_0^x (f(t)t^{-\alpha_2})^q d(t^{q(\alpha_2 - \alpha)})\right)^{1/q} \le C^{1-p/q} \left(\int_0^x (f(t)t^{-\alpha_2})^p d(t^{p(\alpha_2 - \alpha)})\right)^{1/p},$$

and this is a special case of our Theorem 2.4(a).

Our next aim is to prove a corresponding inequality for functions satisfying *two* generalized monotone conditions, but first we state the following technical lemma:

LEMMA 4.2. *Let* $0 < p \le q < \infty$, *let* $I_1\colon (a,b) \to (0, I)$ *be an increasing homeomorphism, and* $I_2(x) = I - I_1(x)$. *Moreover let* $J_1\colon (a,b) \to (0,J)$ *and* $J_2(x) = J - J_1(x)$. *Assume that there are constants* A *and* B *such that* $(I_1(x))^{1/q} \le A(J_1(x))^{1/p}$ *and* $(I_2(x))^{1/q} \le B(J_2(x))^{1/p}$ *for every* $x \in (a,b)$. *Then*

$$I^{1/q} \le C_{p,q} J^{1/p}, \qquad \text{where } C_{p,q} = AB/\left(A^{qp/(q-p)} + B^{qp/(q-p)}\right)^{(q-p)/qp}.$$

Proof. First we choose $x \in (a,b)$ such that $I_1(x) = \beta I$, where $\beta = B^{qp/(q-p)}/(A^{qp/(q-p)} + B^{qp/(q-p)})$. Then we have

$$\begin{aligned} I^{p/q} &= \beta I^{p/q} + (1-\beta) I^{p/q} \\ &= \beta^{1-p/q}\left(I_1(x)\right)^{p/q} + (1-\beta)^{1-p/q}\left(I_2(x)\right)^{p/q} \\ &\le \beta^{1-p/q} A^p J_1(x) + (1-\beta)^{1-p/q} B^p J_2(x). \end{aligned}$$

Moreover, our choice of β gives that $\beta^{1-p/q}A^p = (1-\beta)^{1-p/q}B^p = C_{p,q}^p$, which means that $I^{p/q} \le C_{p,q}^p J$ and the proof is complete.

THEOREM 4.3. *Let* $0 < p \le q < \infty$. *If* $f \in Q_{\alpha_2}^{\alpha_1}(C_1, C_2)$ *and* $\alpha_1 < \alpha < \alpha_2$, *then*

$$\left(\int_0^\infty \left(t^{-\alpha} f(t)\right)^q \frac{dt}{t}\right)^{1/q} \le p^{1/p} q^{-1/q} \left[\frac{(\alpha - \alpha_1)(\alpha_2 - \alpha) C_1^p C_2^p}{(\alpha - \alpha_1) C_1^p + (\alpha_2 - \alpha) C_2^p}\right]^{1/p - 1/q} \times \left(\int_0^\infty \left(t^{-\alpha} f(t)\right)^p \frac{dt}{t}\right)^{1/p}. \tag{4.3}$$

Remark 4.4. For the case $C_1 = C_2$, $\alpha_1 = 0$, and $\alpha_2 = 1$, Theorem 4.3 coincides with the original inequality of Bergh [5, p. 84] (for his proof see [2]). This inequality implies at once sharp embeddings between real interpolation spaces (because the Peetre K-functional belongs to $Q_1^0 = Q_1^0(1,1)$). For the case $C = 1$ another proof was presented in [4] (cf. also [3]). For the case $C_1 = C_2 = C$ see also [13, Theorem 3.3].

Proof. Let $a = 0$, $b = \infty$, and consider

$$I_1(x) = \int_0^x \left(t^{-\alpha} f(t)\right)^q \frac{dt}{t}, \qquad J_1(x) = \int_0^x \left(t^{-\alpha} f(t)\right)^p \frac{dt}{t},$$

$$I = I_1(\infty), \qquad \text{and} \qquad J = J_1(\infty).$$

According to Corollary 4.1 we have

$$(I_1(x))^{1/q} \le A(J_1(x))^{1/p},$$
$$\text{where } A = p^{1/p}q^{-1/q}(\alpha_2 - \alpha)^{1/p-1/q}C_2^{1-p/q},$$

and

$$(I_2(x))^{1/q} \le B(J_2(x))^{1/p},$$
$$\text{where } B = p^{1/p}q^{-1/q}(\alpha - \alpha_1)^{1/p-1/q}C_1^{1-p/q}.$$

Thus, by Lemma 4.2, $I^{1/q} \le C_{p,q}J^{1/p}$, where

$$C_{p,q} = p^{1/p}q^{-1/q}\left(\frac{(\alpha - \alpha_1)(\alpha_2 - \alpha)C_1^pC_2^p}{(\alpha - \alpha_1)C_1^p + (\alpha_2 - \alpha)C_2^p}\right)^{1/p-1/q},$$

and the proof is complete.

Remark 4.5. If $\alpha_1 \to -\infty$ in (4.3) and $f(t) = 0$ on $[x, \infty)$, then we obtain (4.2) with $C = C_2$. If $\alpha_2 \to \infty$ in (4.3) and $f(t) = 0$ on $[0, x]$, then we obtain (4.1) with $C = C_1$. Thus the inequalities in Corollary 4.1 may be regarded as limiting cases of (4.3). On the other hand our proof above shows that (4.3) can be regarded as an intermediate inequality between (4.1) and (4.2).

Next we state the following obvious consequence of our Theorem 2.4 applied with $C = 1$ and $\varphi(t) = t^p$ (and $f(x)$ and $g(x)$ replaced by $f^p(x)$ respectively $g^p(x)$):

COROLLARY 4.6. *Let* $0 < p < \infty$ *and consider*

$$A_p = \left(\int_a^b f^p(x)\, dg^p(x)\right)^{1/p} \quad \text{and} \quad B_p = \left(\int_a^b f^p(x) d[-g^p(x)]\right)^{1/p}.$$

(a) *If f is decreasing and g is increasing, then* A_p *is decreasing.*
(b) *If f is increasing and g is increasing, then* A_p *is increasing.*
(c) *If f is increasing and g is decreasing, then* B_p *is decreasing.*
(d) *If f is decreasing and g is decreasing, then* B_p *is increasing.*

Remark 4.7. By applying Corollary 4.6 to concrete functions g we obtain inequalities which in some cases already are well-known. For example, according to Corollary 4.6(a) applied with $g(x) = x - a$, it yields that

$$\left(\int_a^b f^q(x)(x-a)^{q-1}dx\right)^{p/q} \le pq^{-p/q}\int_a^b f^p(x)(x-a)^{p-1}dx, \quad (4.4)$$
$$0 < p < q < \infty,$$

for every decreasing function f. The inequality is sharp and equality occurs for every function of the type $f(x) = C_0\,\chi[a,t]$, $a \le t \le b$ (χ denotes the characteristic function and C_0 any positive constant). Moreover, according to Corollary 4.6(b), (4.4) holds in the reversed direction if f is increasing. For the case $q = 1$ the inequality (4.4) was probably first discovered by Lorentz [11, p. 39] and various other proofs of this case can be found in the literature (see, e.g., [3, Lemma 2.1; 6; 9, p. 100; 10, Remark 2.2; 12, Lemma 2.2; 15, Theorem 3.11i]). Further, by using Corollary 4.6(c) with $g(x) = b - x$ we find that

$$q\int_a^b f^q(x)(b-x)^{q-1}\,dx \le \left(p\int_a^b f^p(x)(b-x)^{p-1}\,dx\right)^{q/p},$$
$$0 < p < q < \infty, \quad (4.5)$$

for every increasing function f. Also (4.5) is sharp and equality occurs for $f(x) = C\chi[t,b]$, $a \le t \le b$. Moreover, Corollary 4.6(d) implies that (4.5) holds in the reversed direction if f is decreasing. For the case $p = 1$ the inequality (4.5) was also proved in [10, 16] (cf. also [7; 8, Theorem 5]).

EXAMPLE 4.8. By applying Corollary 4.6(a) with $a = 0$, $b = \infty$, and $g(t) = t^{1/p}$ we obtain the well-known embedding $L_{p,q_1} \subset L_{p,q_0} 0 < p < \infty$, $0 < q_1 \le q_0 < \infty$, between Lorentz spaces with the sharp imbedding constant $q_1^{1/q_1} q_0^{-1/q_0} p^{1/q_0 - 1/q_1}$ (equality occurs when $f^*(t) = C_0\,\chi[0,b]$). A corresponding inequality in higher dimensions m reads: Let f be a positive and decreasing function on $(\mathbf{0},\boldsymbol{\infty})$. If $a_i > 0$, $i = 1, 2, \ldots, m$, and $0 < q_1 \le q_0 < \infty$, then

$$\left(\int_{\mathbf{0}}^{\boldsymbol{\infty}} \left(f(\mathbf{x})\prod_1^m x_i^{\alpha_i}\right)^{q_0} \prod_1^m \frac{dx_i}{x_i}\right)^{1/q_0}$$
$$\le \left(\frac{q_1}{q_0}\right)^{1/q_1} q_0^{m/q_1 - m/q_0}\left(\prod_1^m \alpha_i\right)^{1/q_1 - 1/q_0}$$
$$\times\left(\int_{\mathbf{0}}^{\boldsymbol{\infty}} \left(f(\mathbf{x})\prod_1^m x_i^{\alpha_i}\right)^{q_1} \prod_1^m \frac{dx_i}{x_i}\right)^{1/q_1}.$$

This fact follows at once by applying our Theorem 3.1(a) with $\varphi(u) = u^{q_0/q_1}$ and $\mathbf{g}(\mathbf{x}) = \{x_i^{a_i q_0}\}$, $i = 1, 2, \ldots, m$, and f replaced by f^{q_0}.

Final Remarks 4.9. (a) Our proof of Theorem 2.2 shows in particular that this theorem holds even if the assumption that f is C-increasing or C-decreasing is replaced by a weaker corresponding (mean C-monotonity) condition (2.5)–(2.8). Analogously we find that Theorems 2.4, Theorem 3.1, and Corollary 4.6 hold under similar somewhat weaker mean monotonity conditions.

(b) The notions of C-increasing and C-decreasing are related to the definitions of the upper index $\alpha(f)$ and the lower index $\beta(f)$, respectively, defined for f: $[a, b] \to \mathbb{R}_+$, by

$$\alpha(f) = \sup\{p \in \mathbb{R}: \text{there exists } C > 0 \text{ such that}$$
$$f(s)s^{-p} \leq Cf(t)t^{-p},\ s \leq t,\ s, t \in [a, b]\},$$
$$\beta(f) = \inf\{p \in \mathbb{R}: \text{there exists } C > 0 \text{ such that}$$
$$f(t)t^{-p} \leq Cf(s)s^{-p},\ s \leq t,\ s, t \in [a, b]\}.$$

Obviously, if f is C-increasing, then $\alpha(f) = 0$ and if f is C-decreasing, then $\beta(f) = 0$ (but the reversed implications do of course not hold in general).

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