On Degenerations of Cohen–Macaulay Modules

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In relation to degenerations of modules, we introduce several partial orders on the set of isomorphism classes of finitely generated modules over a noetherian commutative local ring. Our main theorem says that, under several special conditions, any degenerations of maximal Cohen–Macaulay modules are essentially obtained by the degenerations of Auslander–Reiten sequences.

Key Words: degeneration; Cohen–Macaulay module; Auslander–Reiten sequence; Auslander–Reiten quiver.

1. INTRODUCTION

The purpose of this paper is to provide, in several cases, a way to describe the degenerations of maximal Cohen–Macaulay modules. More precisely, let $R$ be a complete Cohen–Macaulay commutative local ring with only a finite number of indecomposable maximal Cohen–Macaulay modules. Our main Theorem 5.3 says that if $R$ has dimension 2 or $R$ is an integral domain of dimension 1, then any degenerations of maximal Cohen–Macaulay modules over $R$ are generated by the degenerations of Auslander–Reiten sequences, and hence getting degeneration is just a matter of combinatorial computation from the data of an Auslander–Reiten quiver of the category of maximal Cohen–Macaulay modules over $R$.

If $R$ is an artinian ring that is not necessarily a commutative ring, then the degeneration problem of modules is vigorously studied by Bongatz [4] in relation to the Auslander–Reiten quiver. An idea he proposes is to consider an order relation called the “hom” order for modules. By definition, for finite $R$-modules $M$ and $N$, we denote $M \leq_{\text{hom}} N$ if $\text{length}(\text{Hom}_R(M, Y)) \leq \text{length}(\text{Hom}_R(N, Y))$ for all finite $R$-modules $Y$.
This is well defined by a theorem of Auslander [3]. If a module $M$ degenerates to $N$, then the order relation $M \leq_{\text{hom}} N$ holds. Bongartz [4] proved that if the category of finite $R$-modules is directed, then any degeneration is obtained by a composition of degenerations given by Auslander–Reiten sequences, and therefore $M$ degenerates to $N$ if and only if $M \leq_{\text{hom}} N$.

One of the main motivations of this paper is to consider how to generalize the Bongartz theory to maximal Cohen–Macaulay modules. We introduce the following definition for the hom order for finitely generated modules over a commutative noetherian local ring $R$: “$M \leq_{\text{hom}} N$ if and only if $\text{length}(\text{Hom}_R(M, Y)) \leq \text{length}(\text{Hom}_R(N, Y))$ for all $R$-modules $Y$ of finite length.”

Our first task is to verify that this defines a well-defined order on the set of isomorphism classes of finitely generated $R$-modules, which will be done in Section 2 in a slightly more general setting (Theorem 2.2). We shall also give several properties of this ordering in Section 3.

The second is to show the relationship of this ordering with Deformation Theory. Actually we can show that if $M$ degenerates to $N$ then $M \leq_{\text{hom}} N$, as might have been expected. For this, we shall provide in Section 4 a precise definition of the deformation in our setting and several properties of deformations.

Finally we will show that under some special conditions, the order relation of maximal Cohen–Macaulay modules $M \leq_{\text{hom}} N$ holds only when $N$ is essentially obtained from $M$ after a succession of degenerations given by Auslander–Reiten sequences.

We should note that there is a large difference between this and the theory of Bongartz. The main reason for this is that the Auslander–Reiten quiver of the category of maximal Cohen–Macaulay modules generally has a lot of directed cycles and hence is not directed, even if the category has only a finite number of indecomposable objects. Therefore we need several ideas different from those of Bongartz. One of our main tools for this is the Cohen–Macaulay approximation of modules that is defined and developed in the paper [1] of Auslander and Buchweitz. Because of this, we can show the desired results in the case where either $R$ has dimension 2 or $R$ is an integral domain of dimension 1. See Theorem 5.3 and its proof. At this moment, we cannot give any proof of this theorem in higher dimensional cases, though one might expect that it holds for rings of dimension larger than 2.

2. \(\lambda\) AND \(\rho\) FUNCTIONS

In this paper $R$ always denotes a commutative noetherian ring, and $\mathcal{L}_R$ is the set of all isomorphism classes of $R$-modules of finite length. We blur
the distinction between an isomorphism class and a representative of it. Associated to a given finitely generated $R$-module $M$, we consider functions on $\mathcal{L}_R$:

**Definition 2.1.**

\[
\lambda_M(Y) := \text{length}_R(\text{Hom}_R(M, Y)), \\
\rho_M(Y) := \text{length}_R(M \otimes_R Y) \quad \text{for } Y \in \mathcal{L}_R.
\]

The first theorem states that these functions determine the local-isomorphism class of a module $M$.

**Theorem 2.2.** The following conditions are equivalent for finitely generated $R$-modules $M$ and $N$:

1. $\lambda_M(Y) = \lambda_N(Y)$ for all $Y \in \mathcal{L}_R$.
2. $\rho_M(Y) = \rho_N(Y)$ for all $Y \in \mathcal{L}_R$.
3. $M_p \cong N_p$ for all $p \in \text{Spec}(R)$ (i.e., $M$ and $N$ are locally isomorphic).

This is a generalization of a theorem of Auslander, who proved the theorem for a finite-dimensional (noncommutative) algebra and a theorem of Bongarz, who proved it for artinian (noncommutative) rings. (In their cases, the third condition of course should read $M \cong N$.) See [3] for the details.

**Proof.** (1) $\iff$ (2): For a given module $Y \in \mathcal{L}_R$, let $\text{Supp}(Y) = \{m_1, m_2, \ldots, m_r\}$ be the support of $Y$ in $\text{Spec}(R)$. Note that each $m_i$ is a maximal ideal of $R$ and, letting $Y_i = Y_{m_i}$, we have $Y \cong \bigoplus_{i=1}^r Y_i$, since $Y$ is a module of finite length. Thus it is easy to see from the definition that we have the equalities

\[
\lambda_M(Y) = \sum_{i=1}^r \lambda_M(Y_i), \quad \rho_M(Y) = \sum_{i=1}^r \rho_M(Y_i).
\]

Now let $E_i$ be the injective hull of $R/m_i$ as an $R_{m_i}$-module, and we have the following equalities by the Matlis duality theorem:

\[
\text{length}_R(\text{Hom}_R(M, Y_i)) = \text{length}_{R_{m_i}}(\text{Hom}_{R_{m_i}}(M_{m_i}, Y_i)) \\
= \text{length}_{R_{m_i}}(\text{Hom}_{R_{m_i}}(M_{m_i} \otimes_{R_{m_i}} \text{Hom}_{R_{m_i}}(Y_i, E_i), E_i)) \\
= \text{length}_{R_{m_i}}(M_{m_i} \otimes_{R_{m_i}} \text{Hom}_{R_{m_i}}(Y_i, E_i)) \\
= \text{length}_R(M \otimes_R \text{Hom}_R(Y_i, E_i)).
\]

As a result of this, we have the equality $\lambda_M(Y_i) = \rho_M(\text{Hom}_R(Y_i, E_i))$, from which the equivalence of (1) and (2) follows.
(1) $\iff$ (3): For a maximal ideal $m$, we consider $\lambda_{M_m}$, that is, the $\lambda$-function of $M_m$ as an $R_m$-module. Then we see that, for $Y \in \mathcal{L}_R$, $$\lambda_M(Y) = \sum_m \lambda_{M_m}(Y_m),$$
where $m$ runs through for all of the maximal ideals of $R$.

Suppose that $M$ and $N$ are locally isomorphic. Then, since $\lambda_{M_m} = \lambda_{N_m}$, it follows that $\lambda_M = \lambda_N$. This shows the implication $(3) \implies (1)$.

Now to prove the converse, we assume that $\lambda_M = \lambda_N$ as functions on $\mathcal{L}_R$, and we want to prove that $M$ and $N$ are locally isomorphic. We divide its proof into several steps.

(i) As a first step we note that we may assume $R$ is a local ring. In fact, to show the local isomorphism, we have only to show the isomorphism $M_m \cong N_m$ for each maximal ideal $m$. For a maximal ideal $m$ we may regard the set $\mathcal{L}_R$ as the subset of $\mathcal{L}_R$ consisting of $R$-modules which are annihilated by $m$-primary ideals. Therefore we have $\lambda_{M_m} = \lambda_{N_m}$ as functions on $\mathcal{L}_R$.

In the following we assume $R$ is a local ring with maximal ideal $m$.

(ii) We may assume that $R$ is a complete local ring. Let $\hat{R}$ be the $m$-adic completion of $R$. Noting that $\mathcal{L}_R = \mathcal{L}_{\hat{R}}$, we can easily see that $\lambda_M = \lambda_N$ implies $\lambda_{\hat{M}} = \lambda_{\hat{N}}$. If we can prove that $\hat{M} \cong \hat{N}$, then $M \cong N$ will follow from Proposition (2.5.8) in [8, Chap. IV, Sect. 2].

(iii) For each natural number $n$, we have an isomorphism $M/m^nM \cong N/m^nN$. Note that $\mathcal{L}_{R/m^n}$ is a subset of $\mathcal{L}_R$ consisting of modules annihilated by $m^n$. Thus as functions on $\mathcal{L}_{R/m^n}$, we have the equality $\lambda_M = \lambda_{M/m^nM}$. Then from the assumption we see $\lambda_{M/m^nM} = \lambda_{N/m^nN}$ as functions on $\mathcal{L}_{R/m^n}$. Since $R/m^n$ is an artinian ring, it follows from a theorem of Bongartz [3] that $M/m^nM \cong N/m^nN$.

Now to complete the proof of the theorem, it is sufficient to note the following lemma, which is proved by Guralnick in [9, Corollary 1]. (One can also prove the lemma by using the separated ultraproducts that are developed in [11].)

**Lemma 2.3.** Let $(R, m)$ be a complete local ring and let $M$ and $N$ be finitely generated $R$-modules. Suppose there is an isomorphism of $R$-modules $M/m^nM \cong N/m^nN$ for each natural number $n$. Then $M$ is isomorphic to $N$ as $R$-modules.

3. THE $\text{hom}$ ORDERING

In this section we always assume that $(R, m, k)$ is a local ring. Then Theorem 2.2 says that the $\lambda$ (or $\rho$) function determines completely the
isomorphism classes of finitely generated $R$-modules. Moreover, using the \( \lambda \) function, we can measure the largeness of a module.

**Definition 3.1 ("hom" Ordering).** For finitely generated $R$-modules $M$ and $N$, we denote $M \leq_{\text{hom}} N$ if $\lambda_M(Y) \leq \lambda_N(Y)$ for all $Y \in \mathcal{L}_R$.

Theorem 2.2 says that $\leq_{\text{hom}}$ gives a well-defined partial order on the set of isomorphism classes of finitely generated $R$-modules; that is, $M \leq_{\text{hom}} N$ and $N \leq_{\text{hom}} M$ imply $M \cong N$. It is easy to see from the proof of (1) $\iff$ (2) of Theorem 2.2 that $M \leq_{\text{hom}} N$ if and only if $\rho_M(Y) \leq \rho_N(Y)$ for all $Y \in \mathcal{L}_R$.

The hom ordering $\leq_{\text{hom}}$ for modules over artinian algebras or, equivalently, representations of quivers, has been considered by Bongartz in his sequence of works. See [4, 7]. Our main motivation in this paper is to show how to describe this partial ordering for maximal Cohen–Macaulay modules. As we will show in the next section, the hom ordering is deeply related to the degeneration problem of modules.

We provide here several cases in which the hom ordering actually occurs.

**Lemma 3.2.** Suppose that there is a short exact sequence of finitely generated $R$-modules $0 \to N' \to M \to N'' \to 0$. Then $N'' \leq_{\text{hom}} M \leq_{\text{hom}} N' \oplus N''$.

**Proof.** This is easy because we have an exact sequence $0 \to \text{Hom}_R(N'', Y) \to \text{Hom}_R(M, Y) \to \text{Hom}_R(N', Y)$ for $Y \in \mathcal{L}_R$ and hence $\lambda_{N''}(Y) \leq \lambda_M(Y) \leq \lambda_{N'}(Y) + \lambda_{N''}(Y)$.

**Definition 3.3 ("ext" Ordering).** We denote by $\leq_{\text{ext}}$ the transitive relation between finitely generated $R$-modules, which is generated by the following rule: If there is a filtration $M_0 = (0) \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ such that $N \cong M_1 \oplus (M_2/M_1) \oplus \cdots \oplus (M_n/M_{n-1}) \oplus (M_n/M_n)$, then $M \leq_{\text{ext}} N$.

It follows from Lemma 3.2 that $M \leq_{\text{ext}} N$ implies $M \leq_{\text{hom}} N$. Therefore we can show that $M \leq_{\text{ext}} N$ and $N \leq_{\text{ext}} M$ imply $M \cong N$; hence the extension ordering is also a well-defined partial order on the set of isomorphism classes of $R$-modules.

Note that the hom ordering satisfies the cancellation property in the following sense (the proof is obvious).

**Remark 3.4.** (1) For any finitely generated $R$-module $L$, if $M \oplus L \leq_{\text{hom}} N \oplus L$ then $M \leq_{\text{hom}} N$.

(2) For an integer $n$, if $M^n \leq_{\text{hom}} N^n$ then $M \leq_{\text{hom}} N$. (Here $M^n$ denotes the direct sum of $n$-copies of $M$.)

Note, however, that the extension ordering does not have this property.

**Example 3.5.** (1) Let $R$ be a two-dimensional regular local ring and let $\mathfrak{m}$ be its maximal ideal. Then we have an exact sequence $0 \to R \to
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$R^2 \rightarrow \mathfrak{m} \rightarrow 0$, and thus $R \oplus R \leq_{\text{ext}} R \oplus \mathfrak{m}$. But $R \leq_{\text{ext}} \mathfrak{m}$ can never happen because $\mathfrak{m}$ is indecomposable.

(2) Let $R = k[[x, y, z]]/(x^2 + yz)$ and let $\mathfrak{p}$ be the prime ideal generated by $(x, y)$. Then it is easy to see that there is an exact sequence $0 \rightarrow \mathfrak{p} \rightarrow R^2 \rightarrow \mathfrak{p} \rightarrow 0$, and therefore $R \oplus R \leq_{\text{ext}} \mathfrak{p} \oplus \mathfrak{p}$. However, since $\mathfrak{p}$ is indecomposable, $R \leq_{\text{ext}} \mathfrak{p}$ never holds.

To add the cancellation property to the extension ordering it will be reasonable to make the following definition.

Definition 3.6 (“EXT” Ordering). The relation $\leq_{\text{EXT}}$ between finitely generated modules is a partial order generated by the following rules:

1. If $M \leq_{\text{ext}} N$ then $M \leq_{\text{EXT}} N$.
2. $M \leq_{\text{EXT}} N$ if and only if $M \oplus L \leq_{\text{EXT}} N \oplus L$, for all finitely generated $R$-modules.
3. $M \leq_{\text{EXT}} N$ if and only if $M^n \leq_{\text{EXT}} N^n$, for all natural numbers $n$.

The following implications are clear.

Proposition 3.7. $M \leq_{\text{ext}} N \Rightarrow M \leq_{\text{EXT}} N \Rightarrow M \leq_{\text{hom}} N$.

In Example 3.5 (1), we have $R \leq_{\text{EXT}} \mathfrak{m}$ but not $R \leq_{\text{ext}} \mathfrak{m}$. Thus the converse of the first implication does not hold in general. There are also a lot of examples that fail the converse of the second implication. For the easiest example, consider the fact that $R \leq_{\text{hom}} R^2$. It never happens that $R \leq_{\text{EXT}} R^2$ because $R$ and $R^2$ have different ranks. (See the comments after Definition 4.13.)

4. DEGENERATIONS OF MODULES

In this section we always assume that $(R, \mathfrak{m})$ is a local ring such that $R$ contains a coefficient field $k \cong R/\mathfrak{m}$ that is an algebraically closed field.

Definition 4.1. Let $M$ and $N$ be finitely generated $R$-modules and let $t$ be a variable over $R$. One says that $M$ degenerates to $N$ if there is a finitely generated $R[t]$-module $Q$ which is $k[t]$-flat such that, denoting $Q_c := Q \otimes_{k[t]} k[t]/(t - c)$ for $c \in k$, we have $Q_c \cong M$ if $c \neq 0 \in k$ and $Q_0 \cong N$.

First of all we note that the extension ordering implies degeneration.

Proposition 4.2. Suppose $M$ and $N$ are finitely generated $R$-modules and that there is a filtration $M_0 = (0) \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$ such that $N \cong M_1 \oplus (M_2/M_1) \oplus \cdots \oplus (M_{n-1}/M_{n-2}) \oplus (M_n/M_{n-1})$. Then $M$ degenerates to $N$. 
Proof. Consider the $R[t]$-module $Q$ that is the submodule of $M \otimes_R R[t]$ defined by

$$Q := M_0 \oplus M_1 t \oplus M_2 t^2 \oplus \cdots \oplus M_{n-1} t^{n-1} \oplus M_n t^n \oplus M_{n+1} t^{n+1} \oplus \cdots,$$

where $M_i = M$ for $i \geq n$. Then it is easy to see that $Q$ is a finitely generated $R[t]$-module that is flat over $k[t]$. Localizing $Q$ by $t$, we have $Q[\frac{1}{t}] = M \otimes_R R[t, t^{-1}]$. Hence, for $c(\neq 0) \in k$,

$$Q_c = Q/(t - c)Q \cong Q\left[\frac{1}{t}\right] \left/(t - c)Q\left[\frac{1}{t}\right] \cong M.$$

On the other hand, since $tQ = M_0 t \oplus M_1 t^2 \oplus M_2 t^3 \oplus \cdots$, we have

$$Q_0 = Q/tQ \cong M_0 \oplus (M_1/M_0) \oplus \cdots \oplus (M_n/M_{n-1}) \cong N$$

as $R$-modules. Therefore, from the definition, $M$ degenerates to $N$.

We should notice that the converse of the proposition does not hold in general.

Example 4.3. Let $R = k[[x]]$ be the formal power series ring and let $Q$ be the $R[t]$-module $R[t]/(x(x - t))$. Then it is easy to see that $Q$ is $k[t]$-flat and $Q_c \cong R/(x)$ if $c \neq 0$ and $Q_0 \cong R/(x^2)$. Therefore $R/(x)$ degenerates to $R/(x^2)$, but the order relation $R/(x) \preceq_{\text{ext}} R/(x^2)$ does not hold, because $R/(x^2)$ is indecomposable.

Next we should remark that the degeneration implies hom order.

Proposition 4.4. If $M$ degenerates to $N$, then $M \preceq_{\text{hom}} N$.

To prove this proposition it will be convenient to prepare the following preliminary result.

Lemma 4.5. Let $Q$ be a finitely generated $R[t]$-module that is flat over $k[t]$ and let $Y \in \mathcal{L}_R$. For $c \in k$ and $n \in \mathbb{N}$ we consider

$$\phi^n(c) := \text{length}_R(\text{Ext}^n_R(Q_c, Y)).$$

Then for any $n \in \mathbb{N}$, the function $\phi^n$ is an upper semicontinuous function on $k = \mathbb{A}_k^1$.

If this lemma is proved and if $M$ degenerates to $N$, then for $Y \in \mathcal{L}_R$, we have

$$\text{length}_R(\text{Hom}_R(M, Y)) = \phi^0(c) \leq \phi^0(0) = \text{length}_R(\text{Hom}_R(N, Y)),$$

and hence $M \preceq_{\text{hom}} N$ as in the proposition.

Before proceeding to the proof of Lemma 4.5, we make the following remark.
Remark 4.6. For a finitely generated $R[t]$-module $Q$, the following conditions are equivalent:

(a) $Q$ is flat over $k[t]$.
(b) An element $t-c \in R[t]$ is a nonzero divisor on $Q$ for any $c \in k$.
(c) Let

$$
\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to Q \to 0
$$

be an $R[t]$-projective resolution of $Q$. For any $c \in k$, taking the tensor product with $k[t]/(t-c)$ over $k[t]$, we have the exact sequence of $R$-modules:

$$
\cdots \to (P_{n+1})_c \to (P_n)_c \to (P_{n-1})_c \to \cdots \to (P_1)_c \to (P_0)_c \to Q_c \to 0.
$$

For the proof of this, we have just to notice that $Q$ is $k[t]$-flat if and only if $\text{Tor}_i^{k[t]}(Q, k[t]/(t-c)) = 0$ for $i > 0$ and for $c \in k$, which is also equivalent to $\text{Tor}_i^{k[t]}(Q, k[t]/(t-c)) = 0$ for $c \in k$.

Now we proceed to the proof of Lemma 4.5:

Let $Q$ be an $R[t]$-module that is $k[t]$-flat and is such that $Q_c \cong M$ for $c \neq 0$ and $Q_0 \cong N$. Take a free resolution of $Q$ over $R[t]$,

$$
\cdots \to P_{n+1} \xrightarrow{\varphi_{n+1}(t)} P_n \xrightarrow{\varphi_n(t)} P_{n-1} \to \cdots \to P_1 \to P_0 \to Q \to 0,
$$

where $P_n \cong R[t]^{\beta_n}$ and each $\varphi_n(t)$ can be regarded as a matrix over $R[t]$.

Then by Remark 4.6, the sequence

$$
\cdots \to (P_{n+1})_c \xrightarrow{\varphi_{n+1}(c)} (P_n)_c \xrightarrow{\varphi_n(c)} (P_{n-1})_c \to \cdots \to (P_1)_c \to (P_0)_c \to Q_c \to 0
$$

is an $R$-free resolution of $Q_c$ for $c \in k$, where we should note that $(P_n)_c = R^{\beta_n}$. Thus, for $Y \in \mathcal{D}_R$, $\text{Ext}^{\bullet}_R(Q_c, Y)$ is the homology of the complex

$$
Y^{\beta_{n-1}} \xrightarrow{\psi_n(c)} Y^{\beta_n} \xrightarrow{\psi_{n+1}(c)} Y^{\beta_{n+1}}.
$$

Now letting $\ell = \text{length}_R(Y)$, we note that $Y \cong k^\ell$ as $k$-vector spaces, since $k$ is an algebraically closed field. Note also that regarding the above complex as the complex of $k$-vector spaces, we have

$$
k^{\ell\beta_{n-1}} \xrightarrow{\psi_n(c)} k^{\ell\beta_n} \xrightarrow{\psi_{n+1}(c)} k^{\ell\beta_{n+1}},
$$

where each $\psi_i(c)$ is a polynomial of $c$ with coefficients being matrices over $k$. Thus, for $r \in \mathbb{N}$, we have an equivalence:

$$
\text{length}_R \text{Ext}^\bullet_R(Q_c, Y) \geq r \iff \text{rank } \psi_n(c) + \text{rank } \psi_{n+1}(c) \leq \ell \beta_n - r.
$$

Therefore it is enough to show that the set $D_j = \{ c \in k \mid \text{rank } \psi_n(c) + \text{rank } \psi_{n+1}(c) \leq j \}$ is a closed subset of $k$ for any $j \in \mathbb{N}$. But this is clear because $F^i_n = \{ c \in k \mid \text{rank } \psi_n(c) \leq i \}$ is a closed subset of $k$ and $D_j = \bigcup_{i=0}^j (F^i_n \cap F^{j-1}_n)$.
As a corollary of this lemma we note the following:

**Corollary 4.7.** Suppose $M$ degenerates to $N$. Then,

(a) $\text{pd}_RM \leq \text{pd}_RN$.
(b) $\text{depth}_RM \geq \text{depth}_RN$.

**Proof.** Let $Q$ be an $R[t]$-module that is $k[t]$-flat and is such that $Q_c \cong M (c \neq 0)$ and $Q_0 \cong N$. Then, note that $\text{pd}_RQ_c < r$ if and only if $\text{Ext}^r_R(Q_c, k) = 0$. Hence (a) follows from the lemma.

To prove claim (b), we first show that we may assume $R$ is a complete local ring. For this let $\hat{R}$ be the completion of $R$ and let $\hat{Q}$ be the $\hat{R}[t]$-module $Q \otimes_R \hat{R}$. Then, for $c \in k$, we have $(\hat{Q})_c = Q \otimes_R \hat{R}/(t-c) \cong Q_c \otimes_R \hat{R}$. Therefore we can see that $\hat{M} = M \otimes_R \hat{R}$ degenerates to $\hat{N} = N \otimes_R \hat{R}$ as an $\hat{R}$-module. Since $\text{depth}_{\hat{R}}\hat{M} = \text{depth}_RM$, we may assume that $R$ is complete.

Now let $R$ be a complete local ring. Then, since $R$ contains a field, there is a complete regular local subring $T$ of $R$ such that $R$ is a module-finite over $T$. Since $M$ degenerates to $N$ as a $T$-module, it follows from (a) that $\text{pd}_TM \leq \text{pd}_TN$. Therefore (b) follows from the Auslander–Buchsbaum equality: $\text{depth} T - \text{pd}_TM = \text{depth}_TM = \text{depth}_RN$. $\blacksquare$

As a further corollary of 4.7(b), we have

**Corollary 4.8.** Assume that $M$ degenerates to $N$. If $N$ is a maximal Cohen–Macaulay module, then so is $M$.

Note in this corollary that a maximal Cohen–Macaulay module may degenerate to a non-Cohen–Macaulay module. For example, let $R$ be any Cohen–Macaulay local domain of positive dimension and let $M$ be a maximal Cohen–Macaulay module of rank $r$. Then we can embed the free module $R'$ in $M$ and form an exact sequence $0 \to R' \to M \to M' \to 0$. Putting $N = R' \oplus M'$, we see by Proposition 4.2 that $M$ degenerates to $N$, but $N$ is not a maximal Cohen–Macaulay module if $M' \neq (0)$.

For a finitely generated $R$-module $M$, we denote the multiplicity of $M$ with respect to the maximal ideal $m$ of $R$ by $e_R(M)$. Note that $e_R(M)$ is essentially given by the leading coefficient of the Hilbert polynomial, more precisely,

$$\text{length}_R(M/m^nM) = e_R(M) \frac{n^d}{d!} + o(n^d) \quad \text{for } n \gg 0,$$

where $d = \dim R$. Note also that if $R$ is an integral domain then the rank of a module is well defined and $e_R(M) = (\text{rank } M) \cdot e_R(R)$.

**Proposition 4.9.** If $M$ degenerates to $N$, then $e_R(M) = e_R(N)$. In particular, if $R$ is an integral domain and if $M$ degenerates to $N$, then $\text{rank } M = \text{rank } N$. 

Proof. As in the proof of Corollary 4.7(b) we may assume that $R$ is a complete local ring, since $e_R(M) = e_R(\hat{M})$. Then again in the same manner we can take a regular local subring $T$ on which $R$ is module-finite. In this case, as remarked previously, $M$ degenerates to $N$ as $T$-modules. Since $e_R(M)/e_R(N) = \text{rank}_T M/\text{rank}_T N$, to prove the proposition we may assume that $R$ is a complete regular local ring.

Now assume that $R$ is a regular local ring. And let $Q$ be an $R[t]$-module that is $k[t]$-flat, and let $Q_c \cong M$ ($c \neq 0$) and $Q_0 \cong N$. Then, since $R[t]$ is a regular ring, we have an $R[t]$-projective resolution of finite length:

$$0 \to P_r \to P_{r-1} \to \cdots \to P_1 \to P_0 \to Q \to 0.$$ 

For $c \in k$, taking the tensor product of this sequence with $k[t]/(t - c)$ over $k[t]$, we have the following exact sequence of $R$-modules from Remark 4.6:

$$0 \to (P_r)_c \to (P_{r-1})_c \to \cdots \to (P_1)_c \to (P_0)_c \to Q_c \to 0.$$ 

Therefore we have rank $Q_c = \sum_i (-1)^i \text{rank} (P_i)_c$. Note here that, in general, for a finite projective $R[t]$-module $P$, the ranks of $P_c$ ($c \in k$) as $R$-modules are independent of $c$. In fact they equal the rank of $P$ as $R[t]$-modules. Hence rank $Q_c$ is independent of $c$. As a result, we have rank $M = \text{rank} N$ as desired.

More generally than in Proposition 4.9, we can prove that a degeneration of modules occurs only among modules in the same class in the Grothendieck group. To be precise, let $G_0(R)$ be the Grothendieck group of the category of finitely generated $R$-modules, that is, $G_0(R)$ is an abelian group generated by the classes $[M]$ of finitely generated $R$-modules $M$ with relations $[M] = [M'] + [M'']$ if there is an exact sequence $0 \to M' \to M \to M'' \to 0$. We can prove the following result.

**Proposition 4.10.** If $M$ degenerates to $N$, then $[M] = [N]$ in $G_0(R)$.

Proof. Let $\varphi: G_0(R) \to G_0(R[t])$ be a map defined by $\varphi([M]) = [M \otimes_R R[t]]$ for all finitely generated $R$-modules $M$. It is easy to see that $\varphi$ is a well-defined group homomorphism. On the other hand, for any $c \in k$ we define a map $\psi_c: G_0(R[t]) \to G_0(R)$ by

$$\psi_c([N]) := \sum_i (-1)^i \left[ \text{Tor}^i_{R[t]}(N, R[t]/(t - c)) \right]$$

$$= [N/(t - c)N] - [(0 : t - c)N],$$

which is also a well-defined homomorphism between Grothendieck groups.

It is easy to see that $\psi_c \cdot \varphi = 1_{G_0(R)}$. It is also known that $\varphi$ is an isomorphism of abelian groups. See [2, Chap. VII, Sect. 4, Theorem 4.1]. Hence, for any $c, c' \in k$, we have $\psi_{c'} = \varphi^{-1} = \psi_c$. Thus the maps $\psi_c$ are independent of $c \in k$. Therefore if $Q$ is an $R[t]$-module that is $k[t]$-flat with
$M \cong Q_{r}(c \neq 0)$ and $N \cong Q_{0}$, then we have $[M] = [Q_{r}] = \psi_{r}([Q]) = \psi_{0}([Q]) = [Q_{0}] = [N]$ in $G_{0}(R)$ as desired.

Now we define the degeneration order.

DEFINITION 4.11 ("deg" Order). We write $M \leq_{\text{deg}} N$ if there is a sequence of finitely generated $R$-modules $M = M_{0}, M_{1}, \ldots, M_{n} = N$ such that each $M_{i}$ degenerates to $M_{i+1}$ for $0 \leq i < n$.

It follows from Propositions 4.2 and 4.4 that there are implications $M \leq_{\text{ext}} N \implies M \leq_{\text{deg}} N \implies M \leq_{\text{hom}} N$. And we can verify (cf. Theorem 2.2) that $\leq_{\text{deg}}$ defines a well-defined partial order on the set of isomorphism classes of finitely generated $R$-modules.

Note that if $M$ and $N$ are comparable in the degeneration order, then $e_{R}(M) = e_{R}(N)$ or, more strongly, $[M] = [N]$ in $G_{0}(R)$.

Note also that the order $\leq_{\text{deg}}$ does not satisfy the cancellation properties in Remark 3.4.

EXAMPLE 4.12. Let $R$ and $\varphi$ be the same as in Example 3.5 (2). We have shown that $R \oplus R \leq_{\text{ext}} \varphi \oplus \varphi$ and hence $R \oplus R \leq_{\text{deg}} \varphi \oplus \varphi$. However, since we can show that $[R] \neq [\varphi]$ in $G_{0}(R)$, it never occurs that $R \leq_{\text{deg}} \varphi$.

We thus should consider the following ordering.

DEFINITION 4.13 ("DEG" Order). The relation $\leq_{\text{DEG}}$ between finitely generated $R$-modules is a partial order generated by the following rules:

(a) If $M \leq_{\text{deg}} N$ then $M \leq_{\text{DEG}} N$.
(b) $M \leq_{\text{DEG}} N$ if and only if $M \oplus L \leq_{\text{DEG}} N \oplus L$, for all finitely generated $R$-modules $L$.
(c) $M \leq_{\text{DEG}} N$ if and only if $M^{n} \leq_{\text{DEG}} N^{n}$, for all natural numbers $n$.

One should notice from Propositions 4.9 and 4.10 that if $M$ and $N$ are comparable in one of the orders $\leq_{\text{ext}}, \leq_{\text{EXT}}, \leq_{\text{deg}}, \leq_{\text{DEG}}$, then $M$ and $N$ have the same multiplicity (or rank), or, more strongly, they define the same class in the rational Grothendieck group $G_{0}(R)_{Q} := G_{0}(R) \otimes_{Z} Q$. Moreover, the following implications always hold.

PROPOSITION 4.14. $M \leq_{\text{EXT}} N \implies M \leq_{\text{DEG}} N \implies M \leq_{\text{hom}} N$

5. AR ORDERING AND A MAIN THEOREM

In the rest of the paper we assume that $(R, m)$ is a Cohen–Macaulay complete local ring with only an isolated singularity such that $R$ contains a coefficient field $k$ that is an algebraically closed field; thus, by the Serre conditions, if $\dim R \geq 2$, then $R$ is a normal domain. Furthermore, we denote
by CM(R) the category of maximal Cohen–Macaulay modules over R. It is known in this case that the category CM(R) admits Auslander–Reiten sequences. See Yoshino [12, Chap. 3] and [12, (4.22)] for more detail.

We are interested in how to describe the DEG order for maximal Cohen–Macaulay modules. To do this it seems to be natural to consider the restricted extension order defined only by Auslander–Reiten sequences.

**Definition 5.1 ("AR" Ordering).** We define the order $\leq_{AR}$ on isomorphism classes of maximal Cohen–Macaulay $R$-modules as the partial order generated by the following rules:

1. If $0 \to \tau X \to E \to X \to 0$ is an AR-sequence in CM(R), then $E \leq_{AR} X \oplus \tau X$.
2. $M \leq_{AR} N$ if and only if $M \oplus L \leq_{AR} N \oplus L$, for all maximal Cohen–Macaulay modules $L$.
3. $M \leq_{AR} N$ if and only if $M^n \leq_{AR} N^n$, for all natural numbers $n$.

Note that we think of the AR order only for maximal Cohen–Macaulay modules. It is obvious that $M \leq_{AR} N \iff M \leq_{EXT} N$. If CM(R) contains only a finite number of indecomposable maximal Cohen–Macaulay modules, then, since there are only a finite number of AR-sequences, we can easily describe in a combinatorial way the poset structure of CM(R) in the AR ordering.

**Example 5.2.** Let $R = k[[x, y, z]]/(x^2 + yz)$ and let $\mathfrak{p}$ be the prime ideal generated by $(x, y)$. In this case it is known (cf. [12, Proposition (14.10)]) that $\mathfrak{p}$ is the unique nonfree indecomposable maximal Cohen–Macaulay module over $R$, and there is a unique AR sequence $0 \to \mathfrak{p} \to R^2 \to \mathfrak{p} \to 0$ in CM(R). Therefore $R \oplus R \leq_{AR} \mathfrak{p} \oplus \mathfrak{p}$, and hence $R \leq_{AR} \mathfrak{p}$. Compare with Example 3.5(2).

The following theorem is the main result of this paper.

**Theorem 5.3.** Let $R$ be a Cohen–Macaulay complete local ring that is of finite Cohen–Macaulay representation type. And suppose one of the following conditions holds:

1. $R$ is an integral domain of dimension 1.
2. $R$ is of dimension 2.

Then, for any $M$ and $N$ in CM(R) with the same rank, we have the following equivalences:

$$M \leq_{AR} N \iff M \leq_{EXT} N \iff M \leq_{DEG} N \iff M \leq_{\text{hom}} N.$$
When \(R\) is a finite-dimensional (noncommutative) algebra, several results similar to Theorem 5.3 have been proved by Bongartz. For example, if \(R\) is a quiver algebra that is representation direct, then the same equivalences as in Theorem 5.3 hold. See [4, 7]. In the case of maximal Cohen–Macaulay modules, even if \(\text{CM}(R)\) is representation-finite, its Auslander–Reiten quiver may not be direct. Therefore to prove Theorem 5.3 we need several ideas different from those of Bongartz.

In the rest of the paper, we assume that \(R\) is of finite Cohen–Macaulay representation type, that is, there are only a finite number of isomorphism classes of indecomposable maximal Cohen–Macaulay modules. We denote the canonical module of \(R\) by \(K_R\) and write the set of nonisomorphic indecomposable maximal Cohen–Macaulay modules as \(\{X_0, X_1, X_2, \ldots, X_r\}\), where \(X_0 = K_R\).

Then, for each \(X_i (i \neq 0)\), we denote the Auslander–Reiten sequence starting from \(X_i\) by

\[
0 \to X_i \to E_i \to \tau^{-}X_i \to 0. \quad (1)
\]

Note that, by definition (cf. [12, (2.15)]), \(E_i\) and \(\tau^{-}X_i\) are maximal Cohen–Macaulay modules; this is used later on. Our main idea to prove the theorem is to consider the following functions on \(\mathcal{L}_R\) associated with the Auslander-Reiten sequences:

\[
\alpha_i := \lambda_{X_i} + \lambda_{\tau^{-}X_i} - \lambda_{E_i} \quad \text{for } i = 1, 2, \ldots, r.
\]

Recall that, for a finitely generated \(R\)-module \(Y\), the type of short exact sequence

\[
0 \to I(Y) \to C(Y) \to Y \to 0 \quad (2)
\]

is called a Cohen–Macaulay approximation of \(Y\), where \(C(Y) \in \text{CM}(R)\) and \(I(Y)\) is of finite injective dimension. It is known from [1] that, for a given \(Y\), there always exists a Cohen–Macaulay approximation of \(Y\).

**Lemma 5.4.** Let \(Y \in \mathcal{L}_R\) and let \(1 \leq i \leq r\). Then \(\alpha_i(Y)\) is equal to the number of \(X_i\)-summands in the direct decomposition of \(C(Y)\). Equivalently, \(C(Y)\) is isomorphic to \(\bigoplus_{i=1}^{r} X_i^{\alpha_i(Y)}\) up to a \(K_R\)-summand.
Proof. For each $i$, from the above exact sequences (1) and (2) we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \rightarrow \text{Hom}(\tau X, I) \rightarrow \text{Hom}(E_i, I) \rightarrow \text{Hom}(X, I) \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \rightarrow \text{Hom}(\tau X, C) \rightarrow \text{Hom}(E_i, C) \rightarrow \text{Hom}(X, C) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \rightarrow \text{Hom}(\tau X, Y) \rightarrow \text{Hom}(E_i, Y) \rightarrow \text{Hom}(X, Y) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
$$

Here the first row and all of the columns are exact, since $\text{Ext}^1(X, I) = 0$ if $X \in \text{CM}(R)$ and if $I$ is a module of finite injective dimension. (See [6, Proposition 3.3.3(b)].) Now from this diagram we see that $\alpha_i(Y) = \text{length}_R(\text{Coker}(\varphi(C)))$. Note now from the property of Auslander–Reiten sequences that, for an indecomposable maximal Cohen–Macaulay module $X$,

$$
\text{Coker}(\text{Hom}(E_i, X) \rightarrow \text{Hom}(X, X)) = \begin{cases} 
\text{End}(X)/\text{rad}(\text{End}(X)) & (X \cong X) \\
0 & (X \not\cong X).
\end{cases}
$$

Here we should note that $\text{End}(X)/\text{rad}(\text{End}(X)) \cong k$. In fact, since $X_i$ is an indecomposable module, $\text{End}(X_i)$ is a noncommutative local ring and thus $\text{End}(X_i)/\text{rad}(\text{End}(X_i))$ is a skew field that is finite over $k$. But, since $k$ is an algebraically closed field, we have $\text{End}(X_i)/\text{rad}(\text{End}(X_i)) = k$. Therefore if we denote $C(Y) \cong \oplus_{i=0}^i X_i^\beta$, then we have

$$
\text{Coker}(\text{Hom}(E_i, C(Y)) \rightarrow \text{Hom}(X, C(Y))) \cong k^\beta 
$$

for $i = 1, 2, \ldots, r$.

Thus we conclude that $\alpha_i(Y) = \beta_i$ for $i = 1, 2, \ldots, r$.  

Lemma 5.5. (1) Let $R$ be an integral domain of dimension 1. Then there are modules $Y_1, Y_2, \ldots, Y_r$ in $\mathcal{R}$ such that $\alpha_i(Y_j) = \delta_{ij}$ for $1 \leq i, j \leq r$.

(2) Let $R$ be of dimension 2. Then there are modules $Y_1, Y_2, \ldots, Y_r$ in $\mathcal{R}$ and an integer $n$ such that $\alpha_i(Y_j) = n \cdot \delta_{ij}$ for $1 \leq i, j \leq r$. 

Proof. (1) Since $R$ is an integral domain and since every maximal Cohen–Macaulay module is torsion free, $r_i := \text{rank } X_i$ is well defined for each $i$. Note that $\text{rank } K_R = 1$. Thus the direct sum of $r_i$ copies of $K_R$ can be embedded into $X_i$ for each $i$, and we get a type of short exact sequence,

$$0 \to K_R^r \to X_i \to Y_i \to 0,$$

where $Y_i$ is a torsion module over $R$. Thus $Y_i \in \mathcal{L}_R$, since $R$ has dimension 1. Then the above sequence is a Cohen–Macaulay approximation of $Y_i$, and hence we can take as $C(Y_i) = X_i$. As a consequence of Lemma 5.4 we have $\alpha_i(Y_j) = \delta_{ij}$ for $1 \leq i, j \leq r$.

(2) First, as already remarked, note that $R$ is a normal domain. Also note that the class group $\text{Cl}(R)$ of $R$ is a finite group, because any ideal of pure height 1 is a maximal Cohen–Macaulay module and there are only a finite number of their isomorphism classes. Let $n$ be the order of the group $\text{Cl}(R)$.

For any indecomposable module $X \in \text{CM}(R)$, from Lemma 5.4 it is sufficient to find a module $Y \in \mathcal{L}_R$ with $C(Y) \cong X^n$. To show this, we apply Bruns’ theorem [10, Theorem 5.2] to the reflexive module $X^n := \text{Hom}_R(X, K_R)^n$, and we get the exact sequence

$$0 \to (X^n)^n \to R^{r+1} \xrightarrow{f} R,$$

where $r = \text{rank } (X^n)^n$. Now let $I$ be the image of $f$ that is an ideal of $R$. Considering the divisor class attached to modules in the sense of Bourbaki [5, Sect. 4.7] (or the first Chern class), we have $c(I) = c((X^n)^n) = n \cdot c(X^n) = 0$, since $|\text{Cl}(R)| = n$. Since $R$ has dimension 2, this means that $I$ is isomorphic to an $m$-primary ideal. Thus taking this $m$-primary ideal instead of $I$, we may assume that $R/I \in \mathcal{Z}_R$. Now take the $K_R$-dual of the sequence $0 \to (X^n)^n \to R^{r+1} \to R \to R/I \to 0$, and we have the exact sequence

$$0 \to K_R^{r+1} \to X^n \to \text{Ext}^2(R/I, K_R) \to 0.$$
Now suppose $M \leq_{\text{hom}} N$ and $e_R(M) = e_R(N)$. We want to show that $M \leq_{\text{AR}} N$. To do this we set

$$c_i = \lambda_N(Y_i) - \lambda_M(Y_i) \quad \text{for } 1 \leq i \leq r.$$  

Note that the $c_i$ are all nonnegative integers, because $M \leq_{\text{hom}} N$. And we put

$$M' = M^n \oplus \sum_{i=1}^{r} (X_i \oplus \tau^- X_i)^{c_i},$$

$$N' = N^n \oplus \sum_{i=1}^{r} E_i^{c_i},$$

where we should recall that $0 \to X_i \to E_i \to \tau^- X_i \to 0$ is an Auslander-Reiten sequence.

We make the following claims.

(i) To prove $M \leq_{\text{AR}} N$, it is enough to show that $M' \cong N'$. In fact, if we are able to show $M' \cong N'$, then clearly $M' \leq_{\text{AR}} N'$. And adding the basic AR order relation $\sum_{i=1}^{r} E_i^{c_i} \leq_{\text{AR}} \sum_{i=1}^{r} (X_i \oplus \tau^- X_i)^{c_i}$ to this, and deleting the common factors from both sides, we will have $M'^n \leq_{\text{AR}} N'^n,$ and hence $M \leq_{\text{AR}} N$.

(ii) We have $\lambda_M(Y_i) = \lambda_N(Y_{i'})$ for $1 \leq i \leq r$. Since $\lambda_N - \lambda_M = n(\lambda_N - \lambda_M) - \sum_{j=1}^{r} c_j \alpha_j$, evaluating this function at $Y_i$, we have $(\lambda_N - \lambda_M)(Y_i) = n(\lambda_N - \lambda_M)(Y_i) - c_i n$, which equals 0 by the choice of $c_i$.

Since $e_R(M) = e_R(N)$, it is clear that $e_R(M') = e_R(N')$. Now take a minimal reduction $x = \{x_1, x_2, \ldots, x_d\}$ of the maximal ideal $m$ of $R$ and set

$$Y_0 = K_R/xK_R,$$

which is a module of finite length, and we can prove the following equality.

(iii) $\lambda_M(Y_0) = \lambda_N(Y_0)$. In fact, for an arbitrary maximal Cohen-Macaulay module $X$, it follows from the Matlis duality that

$$\lambda_X(Y_0) = \text{length}_R(\text{Hom}_R(X, K_R/xK_R))$$

$$= \text{length}_R(\text{Hom}(X/xX, K_R/xK_R))$$

$$= \text{length}_R(X/xX) = e_R(X).$$

Combining (iii) with (ii), we have shown that $\lambda_M(Y_i) = \lambda_N(Y_i)$ holds for all $0 \leq i \leq r$.

(iv) Now we claim that

$$\det(\lambda_X(Y_j))_{0 \leq i, j \leq r} \neq 0.$$
To prove (iv), we suppose that the determinant vanishes. Then one can find an integral vector \((a_0, a_1, \ldots, a_r) \in \mathbb{Z}^{r+1}\) that is not equal to \((0, 0, \ldots, 0)\) such that \(\sum_j \lambda_{X_i}(Y_j) a_j = 0\) holds for \(i = 0, 1, \ldots, r\). Now denote the positive (resp. negative) part of \(a_j\) by \(a_j^+\) (resp. \(a_j^-\)). that is, \(a_j^+ = \max(a_j, 0)\) and \(a_j^- = \max(-a_j, 0)\), and thus it holds that \(a_j = a_j^+ - a_j^-\). Now consider the modules

\[ Y^+ = \sum_{j=0}^r Y_j^{a_j^+}, \quad Y^- = \sum_{j=0}^r Y_j^{a_j^-}, \]

which are modules in \(\mathcal{F}_R\). Note from the choice of \(Y^+\) and \(Y^-\) that \(\lambda_X(Y^+) = \lambda_X(Y^-)\) for any \(i = 0, 1, \ldots, r\). Since any maximal Cohen–Macaulay modules over \(R\) are direct sums of those \(X_i\), we have \(\lambda_X(Y^+) = \lambda_X(Y^-)\) for any \(X \in \text{CM}(R)\). Then, since \(a_i\) is just an alternative sum of \(\lambda\) functions of maximal Cohen–Macaulay modules, we see that \(a_i(Y^+) = a_i(Y^-)\) for \(i = 1, 2, \ldots, r\). On the other hand, since \(\alpha_i(Y_j) = n_i \delta_{ij}\) for \(i \geq 1\), it follows that \(n_i a_j^+ = n_i a_j^-\) and hence \(a_i = 0\) for \(i \geq 1\). Therefore we must have \(Y^+ = Y_0^{(a_0^+)}\) and \(Y^- = Y_0^{(a_0^-)}\). Since one of these is a trivial module and the other one is nontrivial, it follows that \(\lambda_X(Y^+) \neq \lambda_X(Y^-)\), and this is a contradiction. This completes the proof of (iv).

Now to finish the proof of the theorem we must show that \(M' \cong N'\). Now write \(M' = \sum_{i=0}^r X_i^{m_i}\) and \(N' = \sum_{i=0}^r X_i^{n_i}\). Then, since we have shown \(\lambda_M(Y_j) = \lambda_{N'}(Y_j)\) for \(0 \leq i \leq r\), we have

\[ \sum_{i=0}^r m_i \lambda_X(Y_j) = \sum_{i=0}^r n_i \lambda_X(Y_j) \quad \text{for } j = 0, 1, \ldots, r. \]

But then it follows from (iv) that \(m_i = n_i\) for \(i = 0, 1, \ldots, r\). Therefore we have \(M' \cong N'\) as desired, and the proof of the theorem is completed. \(\blacksquare\)

6. SOME REMARKS

If \(R\) is of infinite Cohen–Macaulay representation type, then there is an example that fails the implication \(M \leq \text{Ext} N \implies M \leq \text{AR} N\).

**Example 6.1.** Let \(R = k[[x, y]]/(x^3 + y^3) \cong k[[t^4, t^6]]\), where \(k\) is an uncountable field and \(x = t^2, y = t^3\). For any \(\alpha \in k\), consider the \(R\)-module \(M_\alpha := (1, \alpha t^6 + t^7)R\) as a fractional ideal of \(R\). It is known from [12] that each \(M_\alpha\) is an indecomposable maximal Cohen–Macaulay module over \(R\) and that \(M_\alpha \not\cong M_\beta\) if \(\alpha \neq \beta\). See [12, Lemma 9.5] and its proof. Let \(N_\alpha\) denote the first syzygy of \(M_\alpha\), and we get the exact sequence \(0 \to N_\alpha \to R^2 \to M_\alpha \to 0\); thus \(R^2 \leq \text{Ext} M_\alpha \oplus N_\alpha\) for each \(\alpha \in k\). But in this case, for
some choice of $\alpha$, the inequality $R^2 \preceq_{AR} M_\alpha \oplus N_\alpha$ cannot hold. Because, if $R^2 \preceq_{AR} X$ holds for some maximal Cohen–Macaulay module $X$, then from the definition of the AR ordering, each of the indecomposable summands of $X$ lies in the AR-component $\Gamma^\circ$ containing the class of $R$. Since there are only countably many indecomposable modules in $\Gamma^\circ$, one can take $M_\alpha$ and $N_\alpha$ outside of $\Gamma^\circ$.

Even for several one-dimensional non-domain cases one can verify the validity of Theorem 5.3. For the easiest example, we consider the following example.

**Example 6.2.** Let $R = k[[x, y]]/(xy)$ where $k$ is a field. We can prove that the same claim as in Theorem 5.3 is true for this ring.

It is known that there are only three isomorphism classes of indecomposable maximal Cohen–Macaulay modules over $R$, which are $R, R/(x)$ and $R/(y)$, and there are only two Auslander–Reiten sequences:

$$0 \to R/(x) \to R \to R/(y) \to 0$$

$$0 \to R/(y) \to R \to R/(x) \to 0.$$

See [12, (9.9)] for the case $\text{char}(k) \neq 2$ and [12, (14.3)] for general cases. Therefore there is a unique basic AR order relation,

$$R \preceq_{AR} R/(x) \oplus R/(y),$$

and hence any order relation of $\preceq_{AR}$ is obtained from this single relation by addition or deletion of the same direct summands on both sides. Note that $e_R(R) = 2$ and $e_R(R/(x)) = e_R(R/(y)) = 1$.

Now let $M$ and $N$ be maximal Cohen–Macaulay modules over $R$ with the same multiplicity, and assume that $M \preceq_{\text{hom}} N$. We want to show that $M \preceq_{AR} N$ in this case.

We can write these modules as

$$M \cong R^{n_0} \oplus (R/(x))^{n_1} \oplus (R/(y))^{n_2}, \quad N \cong R^{n_0} \oplus (R/(x))^{n_1} \oplus (R/(y))^{n_2},$$

where we have $2m_0 + m_1 + m_2 = 2n_0 + n_1 + n_2$, because $e_R(M) = e_R(N)$.

Let $Y_{ij}$ be the module $R/(x^i, y^j)$ for $i, j \geq 1$. They are modules of finite length, and it can be seen that

$$\lambda_R(Y_{ij}) = i + j - 1, \quad \lambda_{R/(x)}(Y_{ij}) = j, \quad \text{and} \quad \lambda_{R/(y)}(Y_{ij}) = i.$$

Since $\lambda_M(Y_{ij}) \leq \lambda_N(Y_{ij})$, we have

$$(i + j - 1)m_0 + jm_1 + im_2 \leq (i + j - 1)n_0 + jn_1 + in_2,$$

for any $i, j \geq 1$. Setting $i = j = 1$ in this inequality and using $2m_0 + m_1 + m_2 = 2n_0 + n_1 + n_2$, we must have $m_0 \geq n_0$. On the other hand, as $j \to \infty$
with \( i = 1 \) in the inequality, we have \( m_0 + m_1 \leq n_0 + n_1 \). Likewise we also have \( m_0 + m_2 \leq n_0 + n_2 \). Now put \( r = m_0 - n_0 \), which is a nonnegative integer, and we have an equality \( m_1 + m_2 + 2r = n_1 + n_2 \) and inequalities \( m_1 + r \leq n_1 \) and \( m_2 + r \leq n_2 \). Therefore \( n_1 = m_1 + r \) and \( n_2 = m_2 + r \). And thus if we put \( L = R^{m_0} \oplus (R/(x))^{m_1} \oplus (R/(y))^{m_2} \), we can write \( M \cong R^r \oplus L \) and \( N \cong (R/(x)) \oplus (R/(y))^r \oplus L \). As a consequence we have \( M \preceq_{AR} N \) as desired.

Note in this example that the \( \alpha \) functions in the proof of Theorem 5.3 cannot distinguish the module \( R/(x) \) from \( R/(y) \). Actually in this case there are no such \( Y_i \)'s in Lemma 5.5.

The same method as in this example can be applied to several examples of one-dimensional reduced local rings that are not integral domains. In fact one can verify that Theorem 5.3 is still valid for the rings \( R = k[[x, y]]/(y^2 + x^{2n}) \) for small \( n \). But at this moment I have no general proof for this even in dimension 1. It seems to be natural to propose the following conjecture.

**Conjecture 6.3**. Theorem 5.3 would be valid without any assumption on \( R \) expect that \( R \) is a Cohen–Macaulay local ring of finite Cohen–Macaulay representation type.

**REFERENCES**