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Intersections of Hyperconics in Projective Planes of Even Order

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We show how to lift the even intersection equivalence relation from the hyperovals of $PG(2, 4)$ to an equivalence relation amongst sets of hyperconics in $\pi = PG(2, F)$. Here, F is any finite or infinite field of characteristic two that contains a subfield of order 4, but does not contain a subfield of order 8. Moreover, we are able to determine the number of points that two hyperconics in π will have in common provided some projective subplane of order 4 intersects both of them in hexads. © 2001 Academic Press

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1. INTRODUCTION

Throughout this paper, π denotes the projective plane $PG(2, F)$, where F is any field, finite or infinite, of characteristic two which contains a subfield of order 4, but does not contain a subfield of order 8. In π, a *hyperconic* is a conic together with its nucleus. A *hexad* in π is a set of six points that forms a hyperoval in some projective subplane $PG(2, 4)$ of order 4 of π. Two hexads are *coplanar* if they are contained in the same $PG(2, 4)$ subplane of π. The 168 hexads in a fixed $PG(2, 4)$ subplane π_0 of π satisfy the much-studied even intersection equivalence relation whereby two hexads are equivalent if they intersect in an even number of points. There are three equivalence classes each of size 56 amongst the hexads in π_0. We denote these three classes by I-hexads, II-hexads, and III-hexads.

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Let H_1 and H_2 be two hyperconics in π such that some $PG(2, 4)$ subplane π_0 of π intersects them both in hexads, say $G_i = H_i \cap \pi_0$. Then the number of common points of H_1 and H_2 depends on the number of common points of the two hexads G_1 and G_2. *Our main result is that* $|H_1 \cap H_2|$ *is even if and only if* $|G_1 \cap G_2|$ *is even.*

A hexad G in a $PG(2, 4)$ subplane π_0 of π contains six points, each being the nucleus of the conic through the remaining five points. Thus, by extending the scalars from $GF(4)$ to the field F, each hexad can be *lifted* to give six hyperconics in π. Using this lifting, the 168 hyperovals in any $PG(2, 4)$ subplane π can be lifted to give 1008 hyperconics in π. *Remarkably, each of these sets of* 1008 *hyperconics satisfies an even intersection equivalence relation.* There are many such systems of 1008 hyperconics. In fact, each hyperconic in $PG(2, F) = PG(2, q)$ is contained in $\binom{q+1}{3}/\binom{5}{3}$ such systems. This is because a hexad contained in a hyperconic H must contain the nucleus of H, and the nucleus together with three other points of H determines a unique hexad contained in H. Hexads are discussed in detail in [11]. Basic facts about the projective plane of order 4 and its hyperovals can be found in [8] and [7].

2. THE EQUIVALENCE RELATION

THEOREM 2.1. *Let* $\pi = PG(2, F)$, *where* F *is any field, finite or infinite, of characteristic two that contains a subfield of order* 4, *but which does not contain a subfield of order* 8. *Let* H_1 *and* H_2 *be hyperconics in* π *and let* G_1 *and* G_2 *be coplanar hexads contained in* H_1 *and* H_2, *respectively. Then* $|G_1 \cap G_2|$ *is even if and only if* $|H_1 \cap H_2|$ *is even.*

COROLLARY 2.2. *Let* $\pi = PG(2, F)$, *where* F *is any field, finite or infinite, of characteristic two that contains a subfield of order* 4, *but which does not contain a subfield of order* 8. *Let* π_0 *be any projective subplane of order* 4 *of* π. *Then the even intersection equivalence relation amongst the* 168 *hexads of* π_0 *can be lifted to an even intersection equivalence relation amongst a set of* 1008 *hyperconics in* π.

To prove Theorem 2.1, we consider two coplanar hexads G_1 and G_2 in a $PG(2, 4)$ subplane π_0 and two hyperconics $H_1 = C_1 \cup \{N_1\}$, $H_2 = C_2 \cup \{N_2\}$ in π with $G_1 \subset H_1$ and $G_2 \subset H_2$. We wish to show that $|H_1 \cap H_2|$ is even if and only if $|G_1 \cap G_2|$ is even. We will consider separately the cases $|G_1 \cap G_2| = 0, 1, 2, 3, 6$. (It is not possible for two hexads to have exactly four or exactly five common points as a quadrangle in π_0 determines a unique hexad.) For each of these cases, we determine all possible values of $|H_1 \cap H_2|$. This will be done by a careful coordinatization of π_0 and π so as to force C_1 to be a nice conic, usually $Y^2 = XZ$. To do this, we will choose a certain quadrangle and use the fact that $PGL(3, F)$ is transitive on quadrangles. Then, when finding the common points of H_1 and H_2, the common affine points of C_1 and C_2 can be found from the roots of a polynomial of degree at most four.

Given $C_1: Y^2 = XZ$ and $C_2: aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, the common affine points of C_1 and C_2 are of the form $(X, Y, 1)$, where $X = Y^2$ and Y is a root of the polynomial $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. We will call the polynomial $p(t)$ the *intersection polynomial* for the conics C_1 and C_2.

Recall that in any plane $\pi = PG(2, F)$, a unique conic can be drawn through five points with no three collinear. Using this, one can show (see [11]) that, if G is a hexad of a $PG(2, 4)$ subplane π_0, and H is a hyperconic of π that contains G, then the nucleus of H is in G.

We are now ready to embark on a proof of Theorem 2.1.

Proof. Let $H_i = C_i \cup \{N_i\}$ be a hyperconic consisting of the conic C_i together with its nucleus N_i, $i = 1, 2$. Suppose that G_1 and G_2 are coplanar hexads in the $PG(2, 4)$ subplane π_0, and suppose that G_1 and G_2 are contained in H_1 and H_2 respectively. We write $GF(4) = \{0, 1, \omega, \omega^2\}$ and we will always coordinatize π_0 and π so that $l_\infty: Z = 0$ is the line at infinity. We consider separately the cases $|G_1 \cap G_2| = 6, 2, 0, 1, 3$.

$|\mathbf{G}_1 \cap \mathbf{G}_2| = \mathbf{6}$: Two hyperconics can have at most six common points since two conics can have at most four common points. (In [11] it is proved that if two hyperconics do have six common points, then those six points must be a hexad in some projective subplane of order 4.) Thus $H_1 \cap H_2 = G_1 \cap G_2$.

$|\mathbf{G}_1 \cap \mathbf{G}_2| = \mathbf{2}$: The number of common points of H_1 and H_2 will depend on whether N_1 and N_2 are in $G_1 \cap G_2$. We will break up the case $|G_1 \cap G_2| = 2$ into five subcases: (i) $N_1 = N_2$, (ii) $G_1 \cap G_2 = \{N_1, N_2\}$, (iii) one of N_1, N_2 (say N_1) is in $G_1 \cap G_2$, (iv) $N_1, N_2 \notin G_1 \cap G_2$ and three of $\{N_1, N_2, P_1, P_2\}$ are collinear, where $G_1 \cap G_2 = \{P_1, P_2\}$, and finally (v) N_1, $N_2 \notin G_1 \cap G_2$ and N_1, N_2 together with the two points of $G_1 \cap G_2$ from a quadrangle.

In each of (i), (ii), and (iii), the point N_1 is on $G_1 \cap G_2$. Since $PGL(3, F)$ is transitive on quadrangles, we coordinatize π so that $N_1 = (0, 1, 0)$ and the other point of $G_1 \cap G_2$ is $(1, 0, 0)$. Furthermore, if one picks any point P of $G_2 \setminus G_1$, we may choose the coordinates of two other points of G_1 to be $(0, 0, 1)$ and $(1, 1, 1)$ in such a way that P is on the line through $(0, 0, 1)$ and $(0, 1, 0)$ and also on the line through $(1, 1, 1)$, $(1, 0, 0)$. This forces one point of G_2 (the point P) to be $(0, 1, 1)$. In the subcases (i), (ii), and (iii) we have $N_1 = (0, 1, 0)$, and N_2 varies amongst $\{(0, 1, 0), (1, 0, 0,), (0, 1, 1)\}$. Thus $C_1: Y^2 = XZ$, and the corresponding equations for C_2 can be determined by using the fact that $(1, 1, 1)$, $(\omega^2, \omega, 1)$, and $(\omega, \omega^2, 1)$ are *not* points of C_2. The conic C_2 will have equation $C_2: aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ with nucleus (f, e, d), for some $a, b, \ldots, f \in GF(4)$.

(i) With the above coordinatization, if $N_1 = N_2$, then we have $C_2: Y^2 + Z^2 + XZ = 0$ giving intersection polynomial $p(t) = 1$. Therefore the only common points of H_1 and H_2 are those on l_∞ and $H_1 \cap H_2 = G_1 \cap G_2$.

(ii) With the above coordinatization, if $G_1 \cap G_2 = \{N_1, N_2\}$, then we have $C_2: X^2 + Z^2 + YZ = 0$ giving intersection polynomial $p(t) = t^4 + t + 1$. This polynomial has degree 4 and is irreducible over $GF(2)$. Therefore it has four simple distinct roots in $GF(2^4)$. Also, on l_∞, H_1 and H_2 both contain the points (0, 1, 0) and (1, 0, 0). Therefore $|H_1 \cap H_2| = 6$ if F contains a subfield of order 16, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise.

(iii) With the above coordinatization, with N_1 (say) on $G_1 \cap G_2$, but $N_2 = (0, 1, 1)$ not on $G_1 \cap G_2$, then we have $C_2: Z^2 + XY + XZ = 0$ giving intersection polynomial $p(t) = t^3 + t^2 + 1$. The polynomial $p(t)$ is an irreducible polynomial over $GF(2)$, and therefore it contains three simple distinct roots in $GF(8)$. Since we are assuming that F does not contain a subfield of order 8, the only common points of H_1 and H_2 are those on l_∞, namely (0, 1, 0) and (1, 0, 0). Thus $H_1 \cap H_2 = G_1 \cap G_2$.

(iv) For this case, we have $N_1, N_2 \notin G_1 \cap G_2$, but three of N_1, N_2, P_1, P_2 are collinear, where $G_1 \cap G_2 = \{P_1, P_2\}$. Coordinatize π similar to above, with the same four points (0, 1, 0), (1, 0, 0), (0, 0, 1), (1, 1, 1) on G_1, but with $G_1 \cap G_2 = \{(1, 0, 0), (0, 0, 1)\}$ and with the line through (0, 1, 0), (1, 0, 0) and the line through (1, 1, 1), (0, 0, 1) meeting in the point N_2 of G_2, which then has coordinates $N_2 = (1, 1, 0)$. Then we have $C_2: Y^2 + XZ + YZ = 0$ giving intersection polynomial $p(t) = t$. Thus $H_1 \cap H_2 = G_1 \cap G_2$.

(v) In this case, we have $N_1, N_2 \notin G_1 \cap G_2$. Also, N_1, N_2, P_1, P_2 is a quadrangle, where $G_1 \cap G_2 = \{P_1, P_2\}$. Coordinatize π similar to above, with the same four points on G_1, but with $G_1 \cap G_2 = \{(1, 0, 0), (0, 0, 1)\}$, $N_1 = (0, 1, 0)$, and N_1N_2 meeting G_1 in $\{N_1, (1, 1, 1)\}$. This forces $N_2 = (1, e, 1)$, for some $e \in GF(4)$. Note that $b + e \neq 0, 1$ since (1, 1, 1) and $(\omega, \omega^2, 1)$ are not in G_2. Thus $C_2: bY^2 + XY + eXZ + YZ = 0$ with $b + e \neq 0, 1$. This gives intersection polynomial $p(t) = t^3 + (b + e)t^2 + t = t(t^2 + (b + e)t + 1)$. The polynomial $t^2 + (b + e)t + 1$ is a polynomial of degree 2 which is irreducible over $GF(4)$ and which has two simple distinct roots in $GF(4^2)$ (see [9, p. 52]). Thus $p(t)$ has exactly one root in $GF(4)$; $p(t)$ has exactly three roots in F if F contains a subfield of order 16 and exactly one root otherwise. In addition, H_1 and H_2 have one common point, (1, 0, 0), on the line l_∞. Thus $|H_1 \cap H_2| = 4$ if F contains a subfield of order 16, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise.

$|\mathbf{G}_1 \cap \mathbf{G}_2| = \mathbf{0}$: Choose coordinates so that $N_1 = (0, 1, 0)$ and (1, 0, 0) are the points of G_1 on N_1N_2 and so that (0, 0, 1) and (1, 1, 1) are the points of G_1 on another line through N_2. Thus, $N_2 = (1, 1, 0)$, $C_1: Y^2 = XZ$, $C_2: aX^2 + bY^2 + cZ^2 + XZ + YZ = 0$, for some $a, b, c \in GF(4)$, with intersection polynomial $p(t) = at^4 + (b + 1)t^2 + t + c$.

Also, $G_1 = \{(0, 1, 0), (1, 0, 0), (0, 0, 1), (1, 1, 1), (\omega, \omega^2, 1), (\omega^2, \omega, 1)\}$. In π_0, there are exactly ten hexads skew to a fixed hexad. Of the ten hexads skew to G_1, exactly four contain (1, 1, 0). Two of these contain $(1, \omega, 0)$ and the other two contain $(1, \omega^2, 0)$. The other six hexads skew to G_1 consist of two containing both $(1, \omega, 0)$ and $(1, \omega^2, 0)$ and four which miss l_∞. Let

$$D_1 = \{(1, 1, 0), (1, \omega^2, 0), (1, 0, 1), (\omega^2, \omega^2, 1), (\omega^2, 0, 1), (1, \omega^2, 1)\}$$

$$D_2 = \{(1, 1, 0), (1, \omega^2, 0), (0, 1, 1), (\omega, \omega, 1), (\omega, 1, 1), (0, \omega, 1)\}$$

$$D_3 = \{(1, 1, 0), (1, \omega, 0), (0, 1, 1), (0, \omega^2, 1), (\omega^2, 1, 1), (\omega^2, \omega^2, 1)\}$$

$$D_4 = \{(1, 1, 0), (1, \omega, 0), (1, 0, 1), (1, \omega, 1), (\omega, 0, 1), (\omega, \omega, 1)\}$$

be these four which miss l_∞.

If $G_2 = D_1$, then C_1 and C_2 have intersection polynomial $p(t) = \omega^2(t^4 + t^2 + \omega t + \omega^2)$. If $G_2 = D_2$, then C_1 and C_2 have intersection polynomial $p(t) = \omega^2(t^4 + t^2 + \omega t + 1)$. If $G_2 = D_3$, then $p(t) = \omega(t^4 + t^2 + \omega^2 t + 1)$. If $G_2 = D_4$, then $p(t) = \omega(t^4 + t^2 + \omega^2 t + \omega)$. For each of these we have that $p(t)$ is an irreducible polynomial of degree 4 over $GF(4)$. Such a polynomial has four simple distinct roots in $GF(4^4)$. Note that H_1 and H_2 do not contain any common points on l_∞. Therefore, $|H_1 \cap H_2| = 4$ if F contains a subfield in order 256, and $H_1 \cap H_2 = \emptyset$ otherwise.

Denote by D the hexad

$$\{(0, 1, 0), (1, 0, 0), (0, 0, 1), (1, 1, 1), (\omega, \omega^2, 1), (\omega^2, \omega, 1)\}.$$

Also, let

$$E_1 = \{(0, 1, 0), (1, 1, 0), (0, 1, 1), (1, \omega^2, 1), (\omega, 1, 1), (\omega^2, \omega^2, 1)\}$$

$$E_2 = \{(0, 1, 0), (1, 1, 0), (0, \omega, 1), (1, 0, 1), (\omega, \omega, 1), (\omega^2, 0, 1)\}$$

$$E_3 = \{(0, 1, 0), (1, \omega, 0), (0, 1, 1), (1, 0, 1), (\omega, 0, 1), (\omega^2, 1, 1)\}$$

$$E_4 = \{(0, 1, 0), (1, \omega, 0), (0, \omega^2, 1), (1, \omega, 1), (\omega, \omega, 1), (\omega^2, \omega^2, 1)\}$$

$$E_5 = \{(0, 1, 0), (1, \omega^2, 0), (0, \omega^2, 1), (1, \omega^2, 1), (\omega, 0, 1), (\omega^2, 0, 1)\}$$

$$E_6 = \{(0, 1, 0), (1, \omega^2, 0), (0, \omega, 1), (1, \omega, 1), (\omega, 1, 1), (\omega^2, 1, 1)\}$$

$$F_1 = \{(0, 1, 0), (1, 1, 0), (0, 1, 1), (1, \omega, 1), (\omega, \omega, 1), (\omega^2, 1, 1)\}$$

$$F_2 = \{(0, 1, 0), (1, 1, 0), (0, \omega^2, 1), (1, 0, 1), (\omega, 0, 1), (\omega^2, \omega^2, 1)\}$$

$$F_3 = \{(0, 1, 0), (1, \omega, 0), (0, \omega, 1), (1, \omega, 1), (\omega, 0, 1), (\omega^2, 0, 1)\}$$

$$F_4 = \{(0, 1, 0), (1, \omega, 0), (0, \omega^2, 1), (1, \omega^2, 1), (\omega, 1, 1), (\omega^2, 1, 1)\}$$

$$F_5 = \{(0, 1, 0), (1, \omega^2, 0), (0, \omega, 1), (1, \omega^2, 1), (\omega, \omega, 1), (\omega^2, \omega^2, 1)\}$$

$$F_6 = \{(0, 1, 0), (1, \omega^2, 0), (0, 1, 1), (1, 0, 1), (\omega, 1, 1), (\omega^2, 0, 1)\}.$$

These are the 12 hexads that intersect D only in the point (0, 1, 0). Say D is a I-hexad, $E_1, \dots, E_6$ are II-hexads, and $F_1, \dots, F_6$ are III-hexads.

$|\mathbf{G_1} \cap \mathbf{G_2}| = \mathbf{1}$: The number of common points of H_1 and H_2 will depend on whether or not N_1 and N_2 are the common point $G_1 \cap G_2$. We will break up the case $|G_1 \cap G_2| = 1$ into four cases: (i) $N_1 = N_2$, (ii) one of N_1, N_2 (N_1 say) is the point $G_1 \cap G_2$, (iii) N_1, N_2 are not the point $G_1 \cap G_2$, and the three points N_1, N_2, and $G_1 \cap G_2$ are collinear, (iv) N_1, N_2, and $G_1 \cap G_2$ are three distinct noncollinear points.

In each of these cases we will coordinatize so that the common point P of G_1 and G_2 is (0, 1, 0). For cases (i), (ii), and (iii), we will coordinate as follows: Given a fixed point Q of $G_2 \setminus G_1$, we can choose the coordinates of π so that (1, 0, 0) is the other point of G_1 on PQ and so that (0, 0, 1) and (1, 1, 1) are the points of G_1 on another line through Q. This forces Q to be (1, 1, 0), G_1 to be D, $C_1 : Y^2 = XZ$, and G_2 to be one of E_1, E_2, F_1, F_2.

(i) If $N_1 = N_2$, coordinatize as above. Then $N_1 = N_2 = P = (0, 1, 0)$ and $G_2 = E_1, E_2, F_1$, or F_2. We consider the case where $G_2 = E_1$. The others can similarly be considered, and the conclusion is the same (namely, that $|H_1 \cap H_2| = 3$ if F contains a subfield of order 16, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise). Now C_1 and C_2 give intersection polynomial $p(t) = \omega^2(t^2 + \omega t + 1)^2$. This polynomial is an irreducible polynomial of degree 2 over $GF(4)$. It has two simple distinct roots in $GF(4^2)$. Note that H_1 and H_2 also contain one common point, (0, 1, 0), on l_∞. Therefore, $|H_1 \cap H_2| = 3$ if F contains a subfield of order 16, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise.

(ii) Suppose $N_1 \neq N_2$ and one of these, N_1 say, is the common point P of G_1 and G_2. Coordinatize π as above so that N_2 is the chosen point $Q = (1, 1, 0)$ of $G_2 \setminus G_1$. Then G_2 is one of E_1, E_2, F_1, F_2. We consider the case $G_2 = E_1$. The others give the same conclusion (namely, that $|H_1 \cap H_2| = 5$ if F contains a subfield of order 256, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise). If $G_2 = E_1$, then C_1 and C_2 have intersection polynomial $p(t) = \omega^2(t^4 + \omega t^2 + \omega t + \omega)$. This polynomial is an irreducible polynomial of degree 4 over $GF(4)$, with four simple distinct roots in $GF(4^4)$. Note that H_1 and H_2 also contain one common point, (0, 1, 0), on l_∞. Therefore, $|H_1 \cap H_2| = 5$ if F contains a subfield of order 256, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise.

(iii) Suppose that N_1 and N_2 are not the common point P of G_1 and G_2. Moreover, suppose that N_1, N_2, P are collinear. Coordinatize π as above with $P = (0, 1, 0)$, $N_1 = (1, 0, 0)$, and $N_2 = Q = (1, 1, 0)$. Once again G_2 can be one of E_1, E_2, F_1, F_2 and the conclusion is the same for each (namely, that $|H_1 \cap H_2| = 3$ if F contains a subfield of order 16 and that $H_1 \cap H_2 = G_1 \cap G_2$ otherwise). We consider $G_2 = E_1$. For this particular case we have coordinatized so that $C_1 : X^2 = YZ$ instead of the usual $Y^2 = XZ$. Therefore, we need an altered intersection polynomial. The common affine points of the conics $X^2 = YZ$ and $aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ are of the form $(X, Y, 1)$, where $X^2 = Y$ and X is a root of the polynomial $q(t) = bt^4 + dt^3 + (a+f)t^2 + et + c$. If $G_2 = E_1$, then $q(t) = \omega t^2 + t + 1$. This polynomial is irreducible and of degree 2 over $GF(4)$. It contains two simple distinct roots in $GF(4^2)$. Also, H_1 and H_2 contain one common point, (0, 1, 0). Therefore, $|H_1 \cap H_2| = 3$ if F contains a subfield of order 16, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise.

(iv) If N_1, N_2, and $G_1 \cap G_2$ is a triangle, then we choose coordinates as follows: $P = (0, 1, 0)$ is the common point of G_1 and G_2, (1, 1, 1) is the other point of G_1 on the line through P and N_2, $N_1 = (1, 0, 0,)$, and (0, 0, 1) is the other point of G_1 on the line through N_1 and N_2. Thus $N_2 = (1, 0, 1)$, $G_1 = D$, and G_2 must be one of E_2, E_3, F_2, F_6. We will suppose that $G_2 = E_2$. The other cases are similar and have the same conclusion, namely, that $H_1 \cap H_2 = G_1 \cap G_2$. If $G_2 = E_2$, then since $C_1 : X^2 = YZ$, we again need an altered intersection polynomial $q(t) = t^3 + \omega$. This polynomial has degree 3 and is irreducible over $GF(4)$. It has three simple distinct roots in $GF(4^3)$. Since we are assuming that F does not contain a subfield of order 8, we have $H_1 \cap H_2 = G_1 \cap G_2$.

$|\mathbf{G_1} \cap \mathbf{G_2}| = \mathbf{3}$:

(i) If $N_1 = N_2$, then coordinatize so that $N_1 = N_2 = (0, 1, 0)$, (1, 0, 0), and (0, 0, 1) are the three common points of G_1 and G_2 and also $G_1 = D$. This gives intersection polynomial $p(t) = (b+1)t^2$, for some $b \in GF(4)$. Also, $b \neq 0$ as $(0, 1, 0) = N_2$, and $b \neq 1$ as $(1, 1, 1) \notin G_2$. Therefore, the only common points of H_1 and H_2 are the common points on l_∞. Therefore, $H_1 \cap H_2 = G_1 \cap G_2$.

(ii) If N_1 and N_2 are distinct and are both common points of G_1 and G_2, then let $N_1 = (0, 1, 0)$, $N_2 = (1, 0, 0)$, and (0, 0, 1) be in $G_1 \cap G_2$ and take (1, 1, 1) to be some other point of G_1. This gives intersection polynomial $p(t) = t(at^3 + 1)$ for some $a \in GF(4)$. Also, $a \neq 0, 1$ as $N_2 = (1, 0, 0)$, and $(1, 1, 1) \notin G_2$. Therefore, $at^3 + 1$ is of degree 3 and is irreducible over $GF(4)$. It has three simple distinct roots in $GF(4^3)$. Since we are assuming that F does not contain a subfield of order 8, the only common points of H_1 and H_2 are N_1, N_2, and (0, 0, 1). Thus, $H_1 \cap H_2 = G_1 \cap G_2$.

(iii) Suppose that exactly one of N_1 and N_2, say N_1, is contained in $G_1 \cap G_2$. Coordinatize so that $N_1 = (0, 1, 0)$, (1, 0, 0), and (0, 0, 1) are the common points of G_1 and G_2 and so that $G_1 = D$. This gives intersection polynomial $p(t) = t(t^2 + et + f)$, where $e, f \in GF(4)$. Note that $e \neq 0$ as (0, 0, 1) and $(1, 0, 0) \in G_2$, and $f \neq 0$ as (0, 0, 1) and $(0, 1, 0) \in G_2$. Also, $e + f \neq 1$ since $(1, 1, 1) \notin G_2$. Therefore, e, f both cannot be equal to 1. This means that $t^2 + et + f$ is an irreducible polynomial of degree 2 over $GF(4)$. It has two simple distinct roots in $GF(4^2)$. The hyperconics H_1 and H_2 also have two common points on l_∞ as well as the point (0, 0, 1). Therefore $|H_1 \cap H_2| = 5$ if F contains a subfield of order 16, and $H_1 \cap H_2 = G_1 \cap G_2$ otherwise.

(iv) Suppose that $N_1, N_2 \notin G_1 \cap G_2$. Coordinatize so that $N_1 = (0, 1, 0)$ and so that $(1, 0, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ are the three common points of G_1 and G_2. This gives intersection polynomial $p(t) = t(dt^2 + (b + e)t + f)$ for some $b, d, e, f \in GF(4)$ satisfying $b + d + e + f = 0$. Now, the line N_1N_2 meets exactly one of the three common points of G_1 and G_2. If $(1, 0, 0)$ is on N_1N_2, then $p(t) = t(t + 1)$. If $(0, 0, 1)$ is on N_1N_2, then $p(t) = t^2(t + 1)$. If $(1, 1, 1)$ is on N_1N_2, then $p(t) = t(t + 1)^2$. Thus $H_1 \cap H_2 = G_1 \cap G_2$. ■

We make several remarks:

1. It is interesting to note that Theorem 2.1 is indeed *false* if F does contain a subfield of order 8.

To see this, suppose that $G_1 \subset H_1$ and $G_2 \subset H_2$ where G_1 and G_2 are coplanar hexads contained in the hyperconics H_1 and H_2, respectively.

There are now three cases in which Theorem 2.1 fails if F contains a subfield of order 8.

(i) If $|G_1 \cap G_2| = 2$ and exactly one of the nuclei of H_1 and H_2 is a common point of G_1 and G_2, then $|H_1 \cap H_2| = 5$ if F contains a subfield of order 8.

(ii) If $|G_1 \cap G_2| = 1$ and if the nuclei of H_1 and H_2 and the point $G_1 \cap G_2$ are three distinct points that are not collinear, then $|H_1 \cap H_2| = 4$ if F contains a subfield of order 64.

(iii) If $|G_1 \cap G_2| = 3$ and the nuclei of H_1 and H_2 are two of the three points of $G_1 \cap G_2$, then $|H_1 \cap H_2| = 6$ if F contains a subfield of order 64.

2. It is an open question to determine if the even intersection equivalence relation can be extended to a larger set of hyperconics. However, the even intersection property does not give an equivalence relation in general in the plane $PG(2, q)$. For example, in $PG(2, 16)$, with $GF(16)\backslash\{0\} = \langle\alpha\rangle$, $\alpha^4 = 1 + \alpha$, let

$$H_1 : \alpha^5 X^2 + \alpha^5 Y^2 + XY + XZ + YZ = 0 \cup \{(1, 1, 1)\}$$

$$H_2 : \alpha^5 X^2 + \alpha^5 Y^2 + \alpha^9 Z^2 + XY + \alpha^4 XZ + \alpha^{13} YZ = 0 \cup \{(\alpha^{13}, \alpha^4, 1)\}$$

$$H_3 : Y^2 = XZ \cup \{(0, 1, 0)\}.$$

Then $H_1 \cap H_2 = \{(1, \alpha^3, 0), (1, \alpha^{12}, 0)\}$, $H_1 \cap H_3 = \{(1, 1, 1), (0, 0, 1)\}$, and $H_2 \cap H_3 = \{(\alpha^9, \alpha^{12}, 1)\}$.

3. While proving Theorem 2.1, we considered the case $|G_1 \cap G_2| = 0$, i.e., the case of two disjoint hexads. The proof of Theorem 2.1 for this case could have been shortened if $PGL(3, 4)$ had been transitive on pairs of pointed disjoint hexads in $PG(2, 4)$. Given a pair of disjoint hexads G_1, G_2 containing points P_1, P_2, respectively, and another pair of disjoint hexads G_1', G_2' containing points P_1', P_2', respectively, does there exist a map $\phi \in PGL(3, 4)$ such that $\phi(G_i) = G_i'$ and $\phi(P_i) = P_i'$, $i = 1, 2$? A counting argument suggests $PGL(3, 4)$ is transitive on pairs of disjoint pointed hexads (G_i, P_i), $i = 1, 2$.

We show now that this is not the case. Let the line $l = P_1P_2$ meet H_1, H_2 in Q_1 and Q_2, respectively. Similarly, the line $l' = P'_1P'_2$ meets H'_1, H'_2 in Q'_1, Q'_2, respectively. Denote the fifth point of l, l' by Y, Y', respectively.

Suppose there exists ϕ in $PGL(3, 4)$ mapping the ordered pair (G'_i, P'_i) to the ordered pair (G_i, P_i), $i = 1, 2$. Note that $\phi(P'_1) = P_1$, $\phi(P'_2) = P_2$, $\phi(Q'_1) = Q_1$, $\phi(Q'_2) = Q_2$, and $\phi(Y') = Y$. Then we claim there is no element ψ mapping the ordered pair (G'_1, P'_1) and (G'_2, Q'_2) to the ordered pair (G_1, P_1) and (G_2, P_2), respectively. For in this case, $\psi(P'_1) = P_1$, $\psi(P'_2) = Q_2$, $\psi(Q'_1) = Q_1$, $\psi(Q'_2) = P_2$, and $\psi(Y') = Y$. But then ϕ and ψ are both in $PGL(3, 4)$ and the image of each of P'_1, Q'_1, Y' is the same under both maps. This is not possible as P'_1, Q'_1, Y' are collinear. We are grateful Prof. A.E. Brouwer (see [3]) for a discussion of these matters.

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