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Fluctuation and Stability in a Diffusive Predator-Prey System

G.P. SAMANTA

Department of Applied Mathematics, University of Calcutta

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Abstract. Recently a deterministic Predator-Prey model which takes into account low predator densities has been discussed by Smith [1]. In the present paper we have studied the stochastic behavior of this model with diffusion.

PREDATOR-PREY MODEL: BASIC STOCHASTIC DIFFERENTIAL EQUATIONS

We consider a model of two interacting species distributed over a one-dimensional space. Let $N_1(x,t)$ and $N_2(x,t)$ be the prey population and the predator population densities respectively and are assumed to satisfy the system of deterministic differential reaction-diffusion equations [2]

$$\frac{\partial N_1}{\partial t} = (r_1 - 1)(1 - N_1)N_1 - N_1N_2 + D\frac{\partial^2 N_1}{\partial x^2} \tag{1a}$$

$$\frac{\partial N_2}{\partial t} = r_2N_1N_2 - N_2 + D\frac{\partial^2 N_2}{\partial x^2} \tag{1b}$$

where $r_i(> 1)$ denote the maximum reproductive rate of species i and D is the diffusion coefficient which we have assumed to be the same for both the species.

Now to take into account of the fluctuating environment we modify system (1) to the form of the partial stochastic differential equations

$$\frac{\partial N_1}{\partial t} = r(1 - N_1)N_1 - N_1N_2 + D\frac{\partial^2 N_1}{\partial x^2} \tag{2a}$$

$$\frac{\partial N_2}{\partial t} = r_2N_1N_2 - N_2 + D\frac{\partial^2 N_2}{\partial x^2} \tag{2b}$$

where $r = (r_1 - 1) + \phi(x,t), \phi(x,t)$ is an environmental parameter and we assume that it is a white noise of unit spectral density.

If the spectral densities of the population fluctuations of the Prey and Predator species are S_{N_1} and S_{N_2} respectively, then [3]

$$S_{N_m}(k,\omega) = | T_m(k,\omega) |^2 \qquad (m = 1,2) \tag{3}$$

where the transfer functions $T_m(k,\omega)$ are given by

$$| T_1(k,\omega) |^2 = \frac{(r_2 - 1)^2(D^2k^4 + \omega^2)}{r_2^4\{w^4 + (\beta^2 - 2\omega_0^2)\omega^2 + \omega_0^4\}}$$

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$$| T_2(k,\omega) |^2 = \frac{(r_1-1)^2(r_2-1)^4}{r_2^4\{\omega^4+(\beta^2-2\omega_0^2)\omega^2+\omega_0^4\}} \tag{4}$$

where $\omega_0^2 = D^2k^4 + \frac{d(r_1-1)}{r_2}k^2 + \frac{(r_1-1)(r_2-1)}{r_2}$,

$$\beta^2 = 4D^2k^4 + \frac{4D(r_1-1)}{r_2}k^2 + \frac{(r_1-1)^2}{r_2^2}.$$

Here we assume that

$$r_1 - 1 < 4(r_2-1)r_2. \tag{5}$$

This is motivated by the fact that since there is no "crowding effect" term in (1*b*), therefore (1) represents a more realistic mathematical model for those cases of low predator densities and hence of high growth rates.

Population Fluctuations: Intensity and Auto-Correlation Function

The intensity (variance) of the population fluctuations are given by [3]

$$\sigma_m^2 = \frac{1}{4\pi^2}\int^{\infty}\int_{-\infty} | T_m(k,\omega) |^2 \, dk d\omega \qquad (m=1,2) \tag{6}$$

σ_1^2, σ_2^2 represent the intensity of the prey and predator populations respectively. Therefore,

$$\sigma_1^2 = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(r_2-1)^2(D^2k^4+\omega^2)}{r_2^4\{\omega^4+(\beta^2-2\omega_0^2)\omega^2+\omega_0^4\}} dk \, d\omega \tag{7}$$

We put $\beta^2 - 2\omega_0^2 = 2\omega_0^2 cos2\phi$, this is permissible since we have assumed that $r_1 - 1 \leq 4(r_2-1)r_2$.

After some calculations (using Gradshteyn et al. [4]), we have

$$\sigma_1^2 = \frac{(r_2-1)^2}{4r_2^4\sqrt{D}}$$

$$\times \left[\frac{4r_2(r_2-1)}{4r_2(r_2-1)-(r_1-1)}\sqrt{\frac{r_2}{2(r_1-1)}} + \left\{\frac{(r_1-1)(r_2-1)}{r_2}\right\}^{-\frac{3}{4}}\left\{\sqrt{\frac{r_1-1}{r_2(r_2-1)}}+2\right\}^{-\frac{1}{2}}\right]$$

$$\times \left[\left\{\frac{(r_1-1)(r_2-1)}{(r_1-1)-4r_2(r_2-1)} + \left\{1 - \frac{2r_2(r_2-1)}{4r_2(r_2-1)-(r_1-1)}\right\}\left\{\frac{(r_1-1)(r_2-1)}{r_2}\right\}^{\frac{1}{2}}\right\}\right] \tag{8}$$

Similarly, we have

$$\sigma_2^2 = \frac{(r_1-1)(r_2-1)^4}{4r_2^2\{(r_1-1)-4(r_2-1)r_2\}\sqrt{D}}$$

$$\times \left[-4\sqrt{\frac{r_2}{2(r_1-1)}} + \left\{\frac{(r_1-1)(r_2-1)}{r_2}\right\}^{-\frac{3}{4}}\left\{\sqrt{\frac{(r_1-1)}{r_2(r_2-1)}}+2\right\}^{-\frac{1}{2}}\right]$$

$$\times \left[\left\{2\left\{\frac{(r_1-1)(r_2-1)}{r_2}\right\}^{\frac{1}{2}} + \frac{(r_1-1)}{r_2}\right\}\right] \tag{9}$$

The effect of diffusion in the Predator-Prey system is, in general, to stabilize an initially unstable system, with exception of diffusive instability [5]. From (7) and (8) we see that the fluctuations σ_1^2, σ_2^2 remain finite for finite value of diffusive coefficient D. This implies the stability of the diffusive system under consideration under the random perturbation of the parameter $(r_1 - 1)$. This is an extension of the usual deterministic model. We also see that as $D \to 0+$, σ_1^2, $\sigma_2^2 \to \infty$. This leads to the conclusion that in a fluctuating environment the Predator-Prey system with small diffusion will be unstable. These results are in excellent agreement with those of Smeach et al. [2].

The auto-correlation functions ρ_{N_1}, ρ_{N_1} of the population fluctuations of the prey and predator species respectively are given by [3] .

$$\rho_{N_m}(\xi, \tau) = \int_{-\infty}^{\infty} W_m e^{-\frac{1}{2}\beta|\tau|} \cos\omega_1\tau \cos k\xi dk \quad (m = 1, 2) \tag{10}$$

where

$$\omega_1 = \sqrt{(\omega_0^2 - \frac{\beta^2}{4})}, \tag{11}$$

$$W_1 = \frac{(r_2 - 1)^2(D^2k^4 + \beta)}{4\pi\sigma_1^2 r^4 \omega_0^2 \beta}, \; W_2 = \frac{(r_1 - 1)^2(r_2 - 1)^4}{4\pi\sigma_2^2 r^4 \omega_0^2 \beta} \tag{12}$$

and ξ, τ represent the space lag and the time lag respectively. We have calculated these auto-correlation functions numerically, taking $D = 1$, $r_1 = 6$ and $r_2 = 10$, the graphs of which are given below.

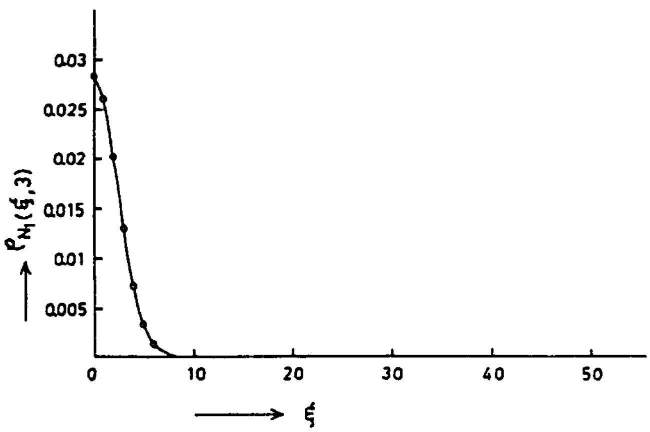

FIGURE 1

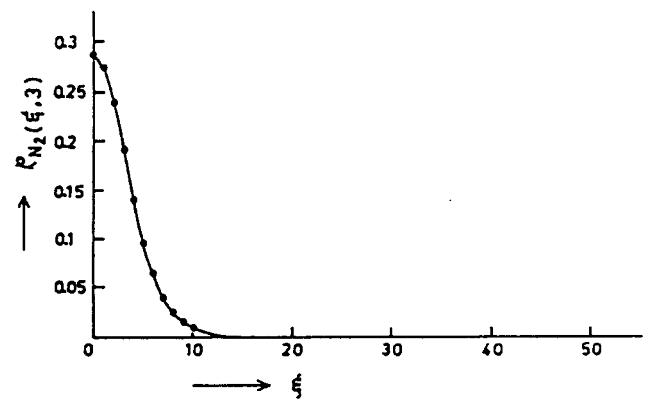

FIGURE 2

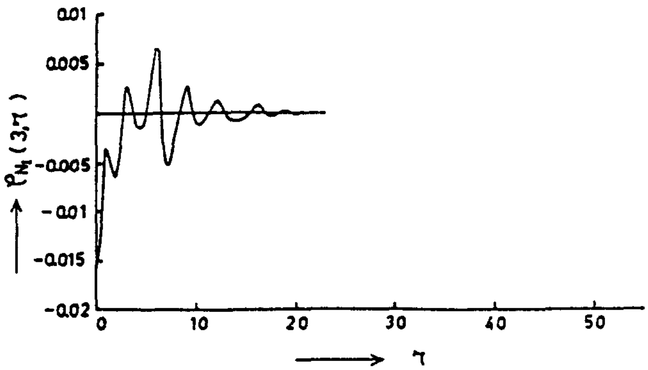

FIGURE 3

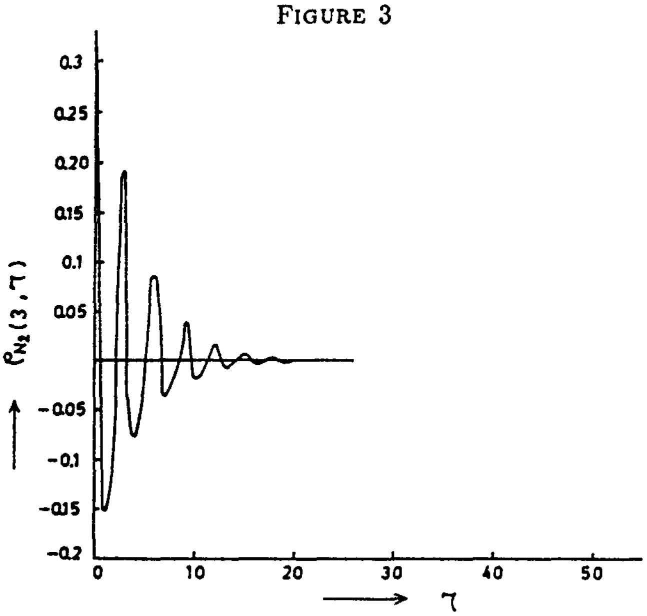

FIGURE 4

From the above Figs. 1 and 2 we see that both the populations exhibit non-cyclic fluctuations with respect to space. From Fig. 3 we note that the first population initially exhibits non-cyclic fluctuations and after some times it exhibits phase-forgetting quasi-cyclic fluctuations with respect to time. This is due to the crowding effect of the prey population at the initial time. After a very short time this crowding effect diminishes and the auto-correlation exhibits damped oscillatory behavior. As there is no crowding effect in the predator population starts from the very beginning.

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Department of Applied Mathematics, University of Calcutta, Calcutta-700009, INDIA