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Logarithmic Frobenius structures and Coxeter discriminants

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Abstract

We consider a class of solutions of the WDVV equation related to the special systems of covectors (called $\vee$-systems) and show that the corresponding logarithmic Frobenius structures can be naturally restricted to any intersection of the corresponding hyperplanes. For the Coxeter arrangements the corresponding structures are shown to be almost dual in Dubrovin's sense to the Frobenius structures on the strata in the discriminants discussed by Strachan. For the classical Coxeter root systems this leads to the families of $\vee$-systems from the earlier work by Chalykh and Veselov. For the exceptional Coxeter root systems we give the complete list of the corresponding $\vee$-systems. We present also some new families of $\vee$-systems, which cannot be obtained in such a way from the Coxeter root systems.

Keywords: Frobenius manifolds; Coxeter discriminants; Hyperplanes arrangements

1. Introduction

The space of orbits M_G of a finite Coxeter group G is probably the most remarkable example of the Frobenius manifolds [4,5,14]. In fact it is a unique in some sense according to the Dubrovin conjecture [5] proved by Hertling [8]. It has the only disadvantage that in general the corresponding prepotential (known to be polynomial) cannot be written explicitly in a simple way.

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On the other hand one can show that for any Coxeter root system $R \subset V$ the function

$$F_R(x) = \sum_{\alpha \in R} (\alpha, x)^2 \log(\alpha, x)^2 \tag{1}$$

satisfies the so-called WDVV equations (see the next section) and thus determines some multiplication structure on the tangent bundle of the complement Σ_R to the mirror hyperplanes $(\alpha, x) = 0$ (see [7,19]). This type of solutions for the WDVV equation came from the Seiberg–Witten investigations of $N = 2$ SUSY Yang–Mills theory [12].

It turned out that in this respect the Coxeter root systems are not unique: in [19,20] it was shown that the deformed root systems discovered by Chalykh and the authors in [2,21] also give the solutions of the WDVV equations. This led to the notion of the $\vee$-systems [19], which can be considered as a proper extension of the Coxeter root systems for this problem. Some new families of the $\vee$-systems generalising classical Coxeter cases were found later in [1].

The answer to a natural question what is the relation between the two types of Frobenius structures related to Coxeter groups was found recently by Dubrovin [6]. He showed that the original Frobenius structure on M_G and the corresponding product on Σ_R (denoted by star) are related by the following remarkably simple formula:

$$u * v = E^{-1} \cdot u \cdot v, \tag{2}$$

where E is the Euler vector field.

In this paper we show that this relation (2) between two multiplications can be extended to any stratum of the *Coxeter discriminant* $\Sigma \subset M_G$. The restrictions of the Frobenius structure on M_G to the strata of the discriminant were considered first by Strachan [18], who investigated the natural submanifolds of the Frobenius manifolds.

Thus we extend Dubrovin's duality to the duality between Strachan's structures on the natural submanifolds and logarithmic Frobenius structures with the prepotentials of the form (1), where R is to be replaced by certain $\vee$-systems. These systems can be described as projections (or restrictions in the dual picture) of the Coxeter root systems to the corresponding intersection subspace of the mirror hyperplanes.

Remarkably enough in this way we get a geometric explanation of the $\vee$-systems found in [1], which can be considered therefore as a closure (in Zariski sense) of the discrete infinite set of the restricted classical Coxeter root systems.

We use the results of Orlik and Solomon [13] and Shcherbak [17], who classified the strata in the Coxeter discriminant, to give the complete list of the $\vee$-systems, which are the restrictions of the exceptional Coxeter root systems. The case of the root system of type F_4 leads to new interesting examples of the $\vee$-systems, which we denote F_3, F_5, F_6 (see Section 5 below).

Finally, we present some new families of the $\vee$-systems, which can not be obtained as the restrictions of the Coxeter root systems.

2. WDVV equation, $\vee$-systems and logarithmic Frobenius structures

The (generalised) WDVV equation is the following overdetermined system of nonlinear partial differential equations in $\mathbf{R}^n$:

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad i, j, k = 1, \dots, n, \tag{3}$$

where F_m is the $n \times n$ matrix constructed from the third partial derivatives of the unknown function $F = F(x^1, \dots, x^n)$:

$$(F_m)_{pq} = \frac{\partial^3 F}{\partial x^m \partial x^p \partial x^q}. \tag{4}$$

Let us introduce for any vector field $a = a^i \partial_i$ the matrices $F_a = a^i F_i$ (here and below the summation over repeated indices is assumed). It is known [12] that (3) are equivalent to the equations

$$F_i G^{-1} F_j = F_j G^{-1} F_i, \quad i, j = 1, \dots, n, \tag{5}$$

where $G = F_\eta$ for some vector field η, which is assumed to be invertible. It is easy to see that one can rewrite (5) as the commutativity relations

$$[\widehat{F}_a, \widehat{F}_b] = 0, \tag{6}$$

where $\widehat{F}_a = G^{-1} F_a$, $\widehat{F}_b = G^{-1} F_b$ for any two vector fields a and b.

Consider the (pseudo-Riemannian) metric

$$\langle a, b\rangle = G_{ij} a^i b^j = F_{ijk} a^i b^j \eta^k$$

and define the multiplication on the tangent bundle by the formula

$$a * b = \widehat{F}_a(b) = \widehat{F}_b(a). \tag{7}$$

It has the following properties:

1. Commutativity: $a * b = b * a$.
2. Associativity: $(a * b) * c = a * (b * c)$.
3. Frobenius property: $\langle a * b, c\rangle = \langle a, b * c\rangle$.

The first and the last properties immediately follow from the symmetry of the partial derivatives, but the associativity imposes a non-trivial condition on F, which is nothing else but the WDVV relation (3), (6). Comparing all this with Dubrovin's definition of the Frobenius manifold [6] we see that we lack some properties: in general metric G is not flat and the vector field η is not covariantly constant.

Let us consider now the following particular class of the solutions of the WDVV equation [20].

Let V be a real linear vector space of dimension n, V^* be its dual space consisting of the linear functions on V (covectors), A be a finite set of covectors $\alpha \in V^*$. One can always assume them to be noncollinear, although sometime it is convenient not to do this (see below).

Consider the following function on V:

$$F^A = \sum_{\alpha \in A} (\alpha, x)^2 \log(\alpha, x)^2, \tag{8}$$

where $(\alpha, x) = \alpha(x)$ is the value of covector $\alpha \in V^*$ on a vector $x \in V$. It is defined on the complement $\Sigma_A = V \setminus \bigcup_{\alpha\in A} \Pi_\alpha$ to the union of all hyperplanes Π: $\alpha(x) = 0$. One can check that the corresponding F_a is (up to a constant) the matrix of the following bilinear form on V:

$$F_a^A = \sum_{\alpha\in A} \frac{(\alpha, a)}{(\alpha, x)} \alpha \otimes \alpha,$$

where $\alpha \otimes \beta(u, v) = \alpha(u)\beta(v)$ for any $u, v \in V$ and $\alpha, \beta \in V^*$.

If we choose the vector field η to be the Euler vector field $\eta = x^i \partial_i$ we come to the constant matrix $G^A = F_x^A$, corresponding to the following bilinear form:

$$G^A = \sum_{\alpha\in A} \alpha \otimes \alpha. \tag{9}$$

We will assume now that this form is non-degenerate, which in the real case means that the covectors $\alpha \in A$ generate V^*. Then the natural linear map $\varphi_A : V \to V^*$ defined by the formula

$$\big(\varphi_A(u), v\big) = G^A(u, v), \quad u, v \in V,$$

is invertible. We will denote $\varphi_A^{-1}(\alpha)$, $\alpha \in V^*$ as $\alpha^\vee$. By definition the operator

$$\sum_{\alpha\in A} \alpha^\vee \otimes \alpha = \mathrm{Id}$$

is an identity operator in V, or equivalently

$$(\alpha, v) = \sum_{\beta\in A} \big(\alpha, \beta^\vee\big)(\beta, v) \tag{10}$$

for any $\alpha \in V^*$, $v \in V$. In this notation the operators $\widehat{F}_a^A$ can be written as

$$\widehat{F}_a^A = \sum_{\alpha\in A} \frac{(\alpha, a)}{(\alpha, x)} \alpha^\vee \otimes \alpha.$$

This leads to the following multiplication for the tangent vectors u and v on Σ_A:

$$u * v = \sum_{\alpha\in A} \frac{\alpha(u)\alpha(v)}{\alpha(x)} \alpha^\vee. \tag{11}$$

A simple calculation [20] shows that the associativity of this multiplication (which is the same as WDVV relation for (8) or commutativity of $\widehat{F}_a^A$) can be rewritten as

$$\sum_{\alpha\neq\beta,\, \alpha,\beta\in A} \frac{G^A(\alpha^\vee, \beta^\vee) B_{\alpha,\beta}(a, b)}{(\alpha, x)(\beta, x)} \alpha \wedge \beta \equiv 0, \tag{12}$$

where

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$$

and

$$B_{\alpha,\beta}(a,b) = \alpha \wedge \beta(a,b) = \alpha(a)\beta(b) - \alpha(b)\beta(a).$$

Another interpretation of these relations is the commutativity condition of the following differential operators of the Knizhnik–Zamolodchikov type:

$$\nabla_a = \partial_a - \sum_{\alpha \in A} \frac{(\alpha, a)}{(\alpha, x)} \alpha^\vee \otimes \alpha, \tag{13}$$

which therefore define a flat connection on Σ_A (see [20]).

The corresponding sets A, for which all these equivalent properties hold, are called $\vee$-*systems* [19]. They satisfy the following relations, called $\vee$-*conditions*:

$$\sum_{\beta \in \Pi \cap A} \beta(\alpha^\vee)\beta^\vee = \lambda \alpha^\vee, \tag{14}$$

for any two-dimensional plane $\Pi \subset V^*$, $\alpha \in \Pi \cap A$ and some λ, which may depend on Π and α (see [20]).

To give a more geometric description let us introduce the Euclidean structure on V using the form G^A. We say that a finite set A in the Euclidean vector space is *well-distributed* if

$$\sum_{\alpha \in A} (\alpha, x)(\alpha, y) = \lambda(x, y)$$

for some λ and *reducible* if $A = A_1 \cup A_2$ is a union of two non-empty orthogonal subsystems. Then A is a $\vee$-system if it is well-distributed and any its two-dimensional subsystem is either reducible or well-distributed in the corresponding plane (see [20]).

For any $\vee$-system A formula (11) defines what we call *logarithmic Frobenius structure on* Σ_A *with prepotential* (8). It satisfies all the properties of the Frobenius structure in Dubrovin's sense [6] except the covariant constancy of the unit vector field. In particular, logarithmic Frobenius structure defines an F manifold with flat structure compatible with the multiplication in the sense of [11].

The full classification of the $\vee$-systems is still to be done. Here are the examples known so far:

(1) any two-dimensional system (trivial examples);
(2) Coxeter $\vee$-systems [7,19];
(3) deformed A_n and B_n families [1];
(4) deformed root systems related to simple Lie superalgebras [15].

By a *Coxeter root system* $\mathcal{R}$ we will mean a finite set of non-zero vectors in Euclidean space such that for any root $\alpha \in \mathcal{R}$ the reflection $s_\alpha : x \to x - \frac{2(\alpha,x)}{(\alpha,\alpha)}\alpha$ leaves $\mathcal{R}$ invariant: $s_\alpha \mathcal{R} = \mathcal{R}$. We assume also that the only root from $\mathcal{R}$ collinear to $\alpha \in \mathcal{R}$ apart from α itself is $-\alpha$. The reflections s_α, $\alpha \in \mathcal{R}$, generate a finite *Coxeter group* G.

In other words the Coxeter root systems consist of the normals to the mirror hyperplanes (two for each mirror), normalised in such a way that the length is a constant on each orbit of the corresponding Coxeter group. Note that these systems depend on the parameters whose number is equal to the number of orbits of the group on its root system (or one less if we consider these systems up to a dilation).

For a given Coxeter root system $\mathcal{R}$ we define the corresponding $\vee$-system $A \subset \mathcal{R}$ as a subset of non-collinear roots such that $\mathcal{R} = A \cup (-A)$. The standard choice is so-called *positive part* of root system $\mathcal{R}_+$, consisting of vectors positive with respect to some linear form.

The deformed A_n system has the form

$$A_n(c) = \left\{\sqrt{c_i c_j}(e_i - e_j),\ 1 \leqslant i < j \leqslant n+1\right\}$$

in $\mathbb{R}^{n+1}$, where $c_1, \dots, c_{n+1}$ are arbitrary parameters. The deformed B_n family depends on $n+1$ parameters $(\gamma; c_1, \dots, c_n)$ and has the form

$$B_n(\gamma, c) = \begin{cases} \sqrt{c_i c_j}(e_i \pm e_j), & 1 \leqslant i < j \leqslant n, \\ \sqrt{2c_i(c_i + \gamma)}e_i, & i = 1, \dots, n \end{cases} \tag{15}$$

(see [1]). The construction of the next section gives some explanation of these two families (see also Section 5).

3. Restrictions of the logarithmic Frobenius structures on the intersection subspaces

Let $A = \{\alpha\} \subset V^*$ be a $\vee$-system, $F(x) = \sum_{\alpha \in A} \alpha(x)^2 \log \alpha(x)^2$ be the prepotential of the corresponding logarithmic Frobenius structure.

By a *subsystem* $B \subset A$ we will mean the intersection of A with any subspace $U \subset V^*$. Choose any subsystem B in A and consider the corresponding subspace $L = L_B \subset V$, which is the intersection of the hyperplanes $\Pi_\beta = \{x \in V\colon\ \beta(x) = 0\}$ for all $\beta \in B$. On the first sight the prepotential F and the corresponding multiplication (11) can not be restricted to L, but in fact this can be done in the following natural way.

Let $C = A \setminus B$ be the complement to B in A. Consider the corresponding space $L \setminus \Sigma_L$, which is a complement in L to the union Σ_L of the intersection hyperplanes $\widetilde{\Pi}_\gamma = \Pi_\gamma \cap L$ for all $\gamma \in C$.

Theorem 1. *The logarithmic Frobenius structure* (8), (11) *has a natural restriction to the space* $L \setminus \Sigma_L$ *with the prepotential*

$$F_B = \sum_{\gamma \in A \setminus B} \gamma(x)^2 \log \gamma(x)^2, \quad x \in L \setminus \Sigma_L, \tag{16}$$

which also satisfies the WDVV equation.

Corollary 1. *The restriction of a* $\vee$*-system* A *to the intersection subspace* $L = L_B$ *for any subsystem* $B \subset A$ *is a* $\vee$*-system as well.*

Note that if we would assume the covectors in the original system A to be noncollinear, the same will not be true in general for the restricted system. This is however not a problem since a group of the collinear covectors $\gamma_i = \lambda_i \gamma$, $i = 1, \dots, k$, can be replaced by a single covector $\tilde{\gamma} = \lambda\gamma$, where $\lambda^2 = \sum_{i=1}^k \lambda_i^2$.

Let us now prove the theorem. Consider a point $x_0 \in \Sigma_L$ and two tangent vectors u, v at x_0 to Σ_L. Let us extend vectors u and v to two local analytic vector fields $u(x), v(x)$ in the whole space V tangent to the subspace L. Outside L we have a well-defined multiplication $u(x) * v(x)$, so the question is what happens when we approach $x_0 \in L$. The answer is given by the following

Lemma 1. *The product $u(x) * v(x)$ has a limit when x tends to x_0 given by*

$$u * v = \sum_{\alpha \in A \setminus B} \frac{\alpha(u)\alpha(v)}{\alpha(x_0)} \alpha^\vee.$$

In particular, the limit is determined by u and v only.

Proof. It is enough to analyse the singular part of the product $u(x) * v(x)$ near a hyperplane Π_β: $\beta(x) = 0$, $\beta \in B$. Consider Euclidean local coordinates (t, s) near the hyperplane such that $t = \beta(x)$ is the coordinate along the normal direction, and s is vector of $n - 1$ coordinates of the orthogonal projection of x onto Π_β. The vector fields can be written as

$$u(x) = u(t, s) = a(t, s)\partial_t + \xi(t, s),$$
$$v(x) = v(t, s) = b(t, s)\partial_t + \eta(t, s),$$

where vector fields ξ and η are parallel to the hyperplane Π_β: $\beta(\xi) = \beta(\eta) = 0$. Since the fields are assumed to be tangential to Π_β we have $a(0, s) = b(0, s) = 0$. The coefficients a, b are analytic, so

$$\lim_{t \to 0} \frac{a(t, s)b(t, s)}{t} = 0.$$

This means that $u(x) * v(x)$ is non-singular at $\beta(x) = 0$, and that the β term disappears when calculating the product at Π_β. As β is an arbitrary element from the subset B the lemma follows. $\square$

Thus the $*$-product is defined for two tangent vectors to Σ_L. The next question is whether it belongs to the tangent space to Σ_L. This is true and follows immediately from the following statement.

Lemma 2. *Let A be a $\vee$-system, $\alpha \in A$ be its element, $\Pi_\alpha = \{x{:}\ \alpha(x) = 0\}$ be the corresponding hyperplane. Then if u, v are tangent to Π_α then the same is true for $u * v$.*

Proof. We have to show that

$$\sum_{\substack{\beta \in A \\ \beta \neq \alpha}} \frac{\beta(u)\beta(v)}{\beta(x)} \alpha(\beta^\vee) = 0$$

if $x \in \Pi_\alpha$; $u, v \in T_x\Pi_\alpha$. From relation (12), which is true for any $\vee$-system, we have the following identity:

$$\sum_{\substack{\beta\in A\\ \beta\neq\alpha}} \frac{G^A(\alpha^\vee,\beta^\vee)(\alpha(a)\beta(b)-\alpha(b)\beta(a))(\alpha(z)\beta(y)-\alpha(y)\beta(z))}{\beta(x)} \equiv 0$$

holding on the hyperplane $\alpha(x)=0$, where a, b, y, z are arbitrary vectors in V. We take $b = u \in \Pi_\alpha$, $y = v \in \Pi_\alpha$. Then $\alpha(b)=\alpha(y)=0$ and we get

$$\alpha(a)\alpha(z)\sum_{\substack{\beta\in A\\ \beta\neq\alpha}} \frac{\alpha(\beta^\vee)\beta(u)\beta(v)}{\beta(x)} = 0,$$

which implies the lemma. □

The fact that the restricted structure has the prepotential (16) is a simple check now. The theorem is proved.

We should mention that a related result was recently found also by Couwenberg, Heckman and Looijenga [3]. In fact, the main object of their paper (Dunkl system) is very close to the notion of the ∨-system.

In the next two sections we discuss the restricted logarithmic Frobenius structures for the Coxeter root systems.

4. Dubrovin's duality on the strata of the Coxeter discriminant

Let G be a finite Coxeter group generated by reflections in $V=\mathbb{R}^n$ and $M_G = V/G$ be the corresponding space of orbits. Let $y_1,\dots,y_n$ be a set of free homogeneous generators of the algebra of the G-invariant polynomials with degrees $d_1, d_2, d_3, \dots, d_n = 2$ assumed to be in the decreasing order. It is known that M_G can be equipped with the Frobenius manifold structure [4]. The identity vector field is $e = \frac{\partial}{\partial y_1}$, the Euler vector field is proportional to

$$E = \sum_{i=1}^n d_i y_i \frac{\partial}{\partial y_i}.$$

Dubrovin introduced the following notion of almost dual Frobenius manifold [6]. For a given Frobenius manifold the dual $\star$ product is defined as follows:

$$u \star v = E^{-1}\cdot u \cdot v \tag{17}$$

on the set where the operator of multiplication by E is invertible.

Dubrovin has also shown that for the Coxeter orbit space M_G the dual structure is in fact the logarithmic Frobenius structure with the prepotential

$$F(x) = \sum_{\alpha\in R} (\alpha,x)^2 \log(\alpha,x)^2, \tag{18}$$

where R is the set of normals (of the same length) to the mirrors $(\alpha,x)=0$ of the group G.

Now we are going to show that Dubrovin duality relation (17) can be extended to any strata of the Coxeter discriminant. The restrictions of the Frobenius structure on M_G to the strata of the discriminant variety were considered by Strachan [18]. We will show now that Dubrovin's duality can be extended to Strachan's structures and the corresponding dual structures are nothing else but the logarithmic Frobenius structures related to the $\vee$-systems, which are the restrictions of the Coxeter root systems.

To make this precise recall that the Coxeter discriminant $\Sigma \subset M_G$ consists of irregular orbits, which are orbits of G consisting of less than $|G|$ points. It is the image of the union of the reflection hyperplanes under the natural projection map $\pi : V \to M_G$. Consider a stratum $S = \pi(L)$ in Σ, which is the image of the intersection subspace $L = \bigcap_{\beta\in B} \Pi_\beta$, B is a subsystem in the root system R of G.

According to Strachan [18] S is a *natural submanifold* of the Frobenius manifold M_G, which means that the tangent space at any regular point of S is closed under multiplication and contains the Euler vector field. The restricted structure does not satisfy Dubrovin's axioms [4] of the Frobenius manifold, since the corresponding metric is not flat anymore, see [18].

Consider a point $x \in L \setminus \Sigma_L$, where as before Σ_L is the union of the hyperplanes $\widetilde{\Pi}_\gamma = \Pi_\gamma \cap L$, $\gamma \in R \setminus B$. Let u, v be two vectors tangent to L at x. Since the natural projection π restricted to L is a local diffeomorphism at x we can consider u, v also as tangent vectors at $y = \pi(x) \in S$. We know that the logarithmic Frobenius structure with prepotential (18) can be restricted to L and the corresponding prepotential on $L \setminus \Sigma_L$ is given by

$$F_B = \sum_{\gamma\in R\setminus B} \gamma(x)^2 \log \gamma(x)^2, \tag{19}$$

where $\gamma(x) = (\gamma, x)$. The following theorem is a corollary of the previous results and the results of Dubrovin [6] and Strachan [18].

Theorem 2. *The restriction of the Frobenius structure on M_G to the stratum S in the Coxeter discriminant Σ and the logarithmic Frobenius structure on $L \setminus \Sigma_L$ with prepotential* (19) *are related by Dubrovin's duality formula*

$$u * v = E^{-1} \cdot u \cdot v. \tag{20}$$

We would like to note that all the roots in R in this claim are normalised to have the same length according to Dubrovin's duality theorem. It is not clear what kind of structure on M_G corresponds to the case when the lengths depend on the choice of orbit.

5. Restrictions of the Coxeter root systems

In this section we discuss $\vee$-systems which can be obtained as the restrictions of the Coxeter $\vee$-systems.

5.1. A_n-type systems

We start with the standard A_n root system consisting of the covectors $e_i - e_j$ with $i, j = 1, 2, \dots, n+1$.

Let $c = (c_1, c_2, \dots, c_m)$ be any partition of $n+1$:

$$\sum_{i=1}^m c_i = n+1, \quad c_1 \geqslant c_2 \geqslant \cdots \geqslant c_m \geqslant 1.$$

It defines a natural representation of the set $I_{n+1} = \{1, 2, \dots, n+1\}$ as a union of the subsets $I_{n+1} = \bigcup_{k=1}^{m} C_k$ with $|C_k| = c_k$. The set $B_c \subset A_n$ of the covectors $e_i - e_j$, where i and j belong to the same subset C_k is a subsystem of A_n and any subsystem can be represented in such a way modulo permutation, which is the action of the corresponding group $G = S_{n+1}$.

The corresponding subspace L_c is given by the condition that all the coordinates with indices inside the same group C_k, $k = 1, \dots, m$ are equal. The restrictions of the covectors $e_1, e_2, \dots, e_{c_1}$ to L_c are the same; let us denote the corresponding covector f_1. Similarly we define f_k, $k = 2, \dots, m$ for all other groups. In these notation the restricted system consists of the covectors $f_i - f_j$ with multiplicities $c_i c_j$. Therefore we get the following

Proposition 1. *The $\vee$-systems, which can be obtained by the restriction of the A_n system, have the form*

$$\alpha_{ij} = \sqrt{c_i c_j}(f_i - f_j), \tag{21}$$

where $1 \leqslant i < j \leqslant m \leqslant n+1$ and $c = (c_1, c_2, \dots, c_m)$ is a partition of $n+1$.

Note that system (21) is a $\vee$-system for general, not necessarily integer values of the parameters c_i. The corresponding solution to the WDVV equation was first found in the paper [1], where it was also proved that for $n = 3$ this is the most general solution of A-type.

Remark. The extension of the values of the parameters from integers to the real (complex) is actually automatic. Indeed the $\vee$-conditions are equivalent to the set of algebraic relations on the parameters of the solution. If they are satisfied for all integers they must be valid for all values of the parameters. In particular, the same is true for the families coming from all the classical Coxeter root systems.

5.2. BCD_n-type systems

The Coxeter group of B_n-type has two orbits on its root system, which leads to an extra parameter in our construction, which we will denote Λ. Consider the following set of covectors in $\mathbb{R}^n$:

$$e_i \pm e_j, \ \Lambda e_i, \tag{22}$$

for $1 \leqslant i < j \leqslant n$. When $\Lambda = 0, 1, 2$ one gets the positive part of the root systems D_n, B_n, C_n respectively. The set defines a $\vee$-system for any value of the parameter Λ (see [7,19]), which we will denote as $B_n(\Lambda)$.

Let c_0 be an integer from the set $\{0, 1, \dots, n\}$ and $c = \{c_1, c_2, \dots, c_m\}$ be a partition of $n - c_0$: $c_1 + \dots + c_m = n - c_0$, $c_1 \geqslant c_2 \geqslant \dots \geqslant c_m \geqslant 1$.

Let $I_n = \bigcup_{k=0}^{m} C_k$, $|C_k| = c_k$ be the corresponding partition of the set $I_n = \{0, 1, \dots, n\}$. Consider the subsystem consisting of the covectors Λe_s, $s \in C_0$ and $e_i - e_j$, where i and j belong to the same subset C_k, $k = 1, \dots, m$. This is the most general subsystem in the B_n, C_n case modulo action of the corresponding group G. In the D_n-case there are other types but they lead to the same $\vee$-systems. We should add also that $c_0 \neq 1$ in the D_n-case.

The corresponding subspace L_c is defined by the condition that the first c_0 coordinates are equal to zero and the coordinates with the indices within the same subset C_k, $k = 1, \dots, m$ are equal. All the covectors from a subset C_k have the same restriction to L_c, which we will denote as f_k, $k = 1, \dots, m$. The restriction gives the following set of covectors with multiplicities:

$$\begin{aligned} f_i \pm f_j&, \text{ multiplicity } c_i c_j, \\ f_i&, \text{ multiplicity } 2c_i c_0, \\ 2f_i&, \text{ multiplicity } \frac{c_i(c_i-1)}{2}, \\ \Lambda f_i&, \text{ multiplicity } c_i, \end{aligned}$$

where $1 \leqslant i < j \leqslant m$. Equivalently, in the resulting $\vee$-system we may change all the covectors proportional to f_i to just one covector

$$\sqrt{2c_i^2 + c_i\big(\Lambda^2 + 2c_0 - 2\big)}\, f_i.$$

Proposition 2. *The $\vee$-systems, which are the restrictions of the $B_n(\Lambda)$ system* (22)*, have the form*

$$\sqrt{c_i c_j}(f_i \pm f_j),\ \sqrt{2c_i(c_i+\gamma)}\, f_i \quad (1 \leqslant i < j \leqslant m), \tag{23}$$

where $\gamma = \frac{1}{2}(\Lambda^2 + 2c_0 - 2)$, $\sum_{i=0}^m c_i = n$ with integer $c_0 \geqslant 0$, $c_1 \geqslant c_2 \geqslant \dots \geqslant c_m \geqslant 1$.

These $\vee$-systems (for general values of the parameters) were found in [1]. We denote them $B_m(\gamma; c_1, \dots, c_m)$. As before the Coxeter restrictions corresponding to the integer values of the parameters form a subset, which is dense in Zariski sense.

5.3. Exceptional systems: $\vee$-systems of F_n-type

In this section we discuss the restrictions of the root system F_4 and the analogues of the corresponding $\vee$-systems in higher dimensions.

The Coxeter $\vee$-system of F_4-type is the following set in $\mathbb{R}^4$:

$$e_i \pm e_j,\ 2\Lambda e_i,\ \Lambda(e_1 \pm e_2 \pm e_3 \pm e_4), \tag{24}$$

where $1 \leqslant i < j \leqslant 4$, $\Lambda \in \mathbb{R}$, and the signs can be chosen arbitrarily. Again an additional parameter Λ is due to the existence of two orbits of the corresponding group $G = F_4$ on its root system.

There are two non-equivalent choices of the one-dimensional subsystems: $B = \{2\Lambda e_1\}$ and $B = \{e_1 - e_2\}$, leading to the following $\vee$-systems. In the first case we get the system $F_3^1(\Lambda)$:

$$\begin{gathered} e_1 \pm e_2,\ e_2 \pm e_3,\ e_1 \pm e_3,\ \sqrt{4\Lambda^2+2}\, e_1,\ \sqrt{4\Lambda^2+2}\, e_2, \\ \sqrt{4\Lambda^2+2}\, e_3,\ \Lambda\sqrt{2}\,(e_1 \pm e_2 \pm e_3). \end{gathered} \tag{25}$$

In the second case the system $F_3^2(\Lambda)$ consists of the covectors

$$\begin{gathered} \sqrt{2\Lambda^2+1}\,(e_1 \pm e_2),\ \sqrt{2}\,(e_2 \pm e_3),\ \sqrt{2}\,(e_1 \pm e_3), \\ 2\sqrt{2\Lambda^2+1}\, e_3,\ 2\Lambda e_1,\ 2\Lambda e_2,\ \Lambda(e_1 \pm e_2 \pm 2e_3). \end{gathered}$$

Both systems contain 13 covectors. It turns out that these two families are equivalent as the families of $\vee$-systems. More precisely, the following statement holds.

Proposition 3. *The $\vee$-system $F_3^1(\lambda)$ is equivalent to the $\vee$-system $F_3^2(\mu)$ with $\mu = \frac{1}{2\lambda}$.*

Indeed, a linear transformation $\mathcal{A}$, which transforms $F_3^1(\lambda)$ to $F_3^2(\mu)$, has the form

$$\mathcal{A}\big(\sqrt{4\lambda^2+2}\,e_1\big) = \sqrt{2\mu^2+1}(e_1+e_2), \qquad \mathcal{A}\big(\sqrt{4\lambda^2+2}\,e_2\big) = \sqrt{2\mu^2+1}(e_1-e_2),$$
$$\mathcal{A}\big(\sqrt{4\lambda^2+2}\,e_3\big) = 2\sqrt{2\mu^2+1}\,e_3.$$

This equivalence is related to the symmetry $\Lambda \to (2\Lambda)^{-1}$ of the initial F_4-system (24), which can be easily checked.

We will denote the family $F_3^1(\Lambda)$ simply as $F_3(\Lambda)$. As it follows from the proposition the parameter Λ here is natural to be considered as a point on a projective line $\mathbb{R}P^1$ rather than in $\mathbb{R}$.

We would like to mention that the one-parameter families F_4, F_3 cannot be generalised to two-parametric families of $\vee$-systems. More precisely, a simple calculation shows that if we put two non-trivial arbitrary coefficients at the vectors of type e_i and at $e_1 \pm e_2 \pm e_3 \pm e_4$ in (24) we will have a $\vee$-system only in the case (24) (similarly for (25)).

This calculation suggests to consider the following systems, which we will call the systems of F_n-type:

$$e_i \pm e_j,\ \Lambda e_i,\ M(e_1 \pm e_2 \pm \cdots \pm e_n), \tag{26}$$

where M is assumed to be non-zero.

Theorem 3. *In dimension $n=5$ the set* (26) *is a $\vee$-system if and only if $M^2=1$ and $\Lambda^2=6$. In dimension $n=6$ this is true if and only if $\Lambda^2=4$ and $M^2=\frac{1}{2}$. There are no $\vee$-systems of F_n-type for $n>6$.*

Proof. The system is invariant under the action of the Coxeter group B_n, therefore the corresponding $\vee$-inner product must be proportional to the standard Euclidean one. The $\vee$-conditions are non-trivial only for planes, containing 2 covectors of the form $M(e_1 \pm e_2 \pm \cdots \pm e_n)$.

If $n=5$ there are only two different cases. For the plane, containing the covectors

$$M(e_1+e_2+e_3+e_4+e_5),\ M(-e_1+e_2+e_3+e_4+e_5),\ \Lambda e_1$$

the $\vee$-condition gives $\Lambda^2 = 6M^2$. For the plane containing

$$M(e_1+e_2+e_3+e_4+e_5),\ M(-e_1-e_2+e_3+e_4+e_5),\ e_1+e_2$$

the $\vee$-condition is $M^2=1$.

Similarly in dimension 6 we have two types of planes with non-trivial $\vee$-conditions: those containing the covectors

$$M(e_1+e_2+e_3+e_4+e_5+e_6),\ M(-e_1+e_2+e_3+e_4+e_5+e_6),\ \Lambda e_1$$

and

$$M(e_1+e_2+e_3+e_4+e_5+e_6),\ M(-e_1-e_2+e_3+e_4+e_5+e_6),\ e_1+e_2.$$

The corresponding $\vee$-conditions are $8M^2=\Lambda^2$ and $2M^2=1$ respectively.

One can check that the $\vee$-conditions for systems (26) with $n>6$ leads to $M=0$. For example, for $n=7$ the covectors $M(e_1+e_2+e_3+e_4+e_5+e_6+e_7)$ and $M(e_1+e_2+e_3-e_4-e_5-e_6-e_7)$ are the only covectors from the corresponding plane and therefore must be orthogonal, which is the case only if $M=0$. The theorem is proved. □

We denote the corresponding $\vee$-systems in dimension 5 and 6 as F_5 and F_6 respectively. The system F_6 contains 68 covectors. Its restriction along e_6 gives the system F_5, containing 41 covector. The restriction of F_5 along e_5 gives a system from the F_4 family.

Remark. In $\mathbb{R}^8$ there is a possibility to have non-zero M by considering (26) with the additional requirement that the numbers of negative signs in the covectors

$$M(e_1\pm e_2\pm\cdots\pm e_8)$$

are even. Then the choice $M=1/2$, $\Lambda=0$ does lead to a $\vee$-system, but this is simply the (positive part of) root system E_8.

It turns out that the system F_6 itself is a restriction of the root system E_8 along its subsystem $e_7\pm e_8$ as we will see in the next section.

5.4. Other exceptional systems

To list all the $\vee$-systems coming from the exceptional root systems we need the results of Orlik and Solomon [13] and Shcherbak [17], who classified the strata in the Coxeter discriminants. We will use the standard terminology of the theory of Coxeter groups, for which we refer to [9].

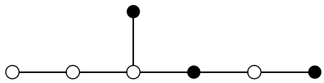
Let G be a finite group generated by reflections in a real vector space V. The reflection hyperplanes divide V into several connected components called *chambers*. Choose a chamber C and consider the set S of the reflections in the hyperplanes, which bound C. The set S can be identified with the vertices of the corresponding *Coxeter graph* Γ (see [9]). For any subset $J\subset S$ the corresponding *parabolic subgroup* G_J is generated by the reflections from J.

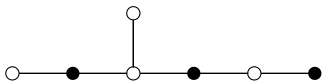
There exists the following natural correspondence between the strata and the conjugacy classes of the parabolic subgroups of the corresponding Coxeter group. Let L be a linear subspace of V, which is an intersection of some reflection hyperplanes. If R is the Coxeter root system of G then $L=L_B$ for some subsystem $B\subset R$ (see Section 4). Consider the subgroup H generated by reflections s_β, $\beta\in B$, which leaves the subspace L fixed. It is known that H is conjugated to G_J for some $J\subset S$. This gives us a one-to-one correspondence between the orbits of G on the set of all such L (or equivalently, the strata in the Coxeter discriminant) and the parabolic subgroups of G considered up to a conjugation.

The explicit description of the corresponding subgraphs (with the set of vertices J) of the Coxeter graphs for the exceptional groups can be found in [13,17]. Typically all the subgroups of a given type are conjugate besides the following exceptions listed below.

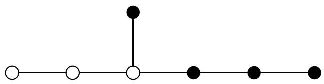
As $\vee$-systems are non-trivial only in dimension greater than two we are interested in the parabolic subgroups of corank at least 3. Then there are only two pairs of non-conjugate isomorphic subgroups in the group E_7, and one more pair in the group F_4.

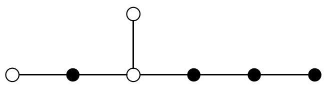
Namely, in the case of the group E_7 there are two non-conjugate classes of subgroups of type A_1^3. The Coxeter subgraphs corresponding to them can be chosen as follows:

$(E_7, A_1^3)_1$: 

$(E_7, A_1^3)_2$: 

Also there are two non-conjugate classes of subgroups of type $A_1 \times A_3$ inside E_7. The corresponding graphs can be taken as follows:

$(E_7, A_1 \times A_3)_1$: 

$(E_7, A_1 \times A_3)_2$: 

Finally, the system F_4 has 2 types of non-conjugate subgraphs of type A_1 given by the two roots of different length.

Applying the restriction procedure from Section 3 we can construct a $\vee$-system for each pair (G, H), where G is a Coxeter group and H is its parabolic subgroup. The complete list of the corresponding $\vee$-systems for the exceptional Coxeter groups is given in Appendix A.

The $\vee$-systems F_3, F_5, F_6 found in Section 5.3 correspond to the following $\vee$-systems from Appendix A:

$$F_6 = (E_8, A_1 \times A_1), \qquad F_5 = (E_8, A_3), \qquad F_3(\Lambda) = \big(F_4(\Lambda), A_1\big)_1.$$

In particular, we see that F_5 and F_6 are indeed the restrictions of E_8-system.

6. Non-Coxeter families of $\vee$-systems

A natural question is if there exist $\vee$-systems, which can not be obtained through the restriction of the Coxeter root systems. In this section we are going to show that the answer is positive by presenting some new one-parameter families of $\vee$-systems, which contain only a finite number of restrictions of Coxeter root systems. Note that the restrictions of the exceptional Coxeter root systems (unlike the classical ones) form a set, which is already closed in Zariski sense.

Theorem 4. *The following set of covectors in* $\mathbb{R}^4$:

$$e_1 \pm e_2,\ e_1 \pm e_3,\ e_2 \pm e_3,\ \Lambda e_1,\ \Lambda e_2,\ \Lambda e_3,\ K e_4,\ M(e_1 \pm e_2 \pm e_3 \pm e_4) \tag{27}$$

with $M \neq 0$ *is a* $\vee$*-system if and only if the parameters satisfy*

$$\Lambda^2 = 2\big(2M^2 + 1\big), \qquad K^2 = \frac{2M^2(2M^2 - 1)}{M^2 + 1}. \tag{28}$$

The corresponding $\vee$*-system is a restriction of a Coxeter root system if and only if* $M^2 = 1$ *or* $M^2 = \frac{1}{2}$.

Proof. The $\vee$-quadratic form in this case is

$$G=(\Lambda^2+8M^2+4)(x_1^2+x_2^2+x_3^2)+(K^2+8M^2)x_4^2.$$

The $\vee$-conditions are non-trivial for the following three types of the two-dimensional planes:

$$\begin{aligned}
\pi_1&=\langle M(e_1+e_2+e_3+e_4),\Lambda e_1\rangle, &\quad G_1&=(2M^2+\Lambda^2)x_1^2+2M^2(x_2+x_3+x_4)^2,\\
\pi_2&=\langle M(e_1+e_2+e_3+e_4),Ke_4\rangle, &\quad G_2&=2M^2(x_1+x_2+x_3)^2+(2M^2+K^2)x_4^2,\\
\pi_3&=\langle M(e_1+e_2+e_3+e_4),e_1+e_2\rangle, &\quad G_3&=(2M^2+1)(x_1+x_2)^2+2M^2(x_3+x_4)^2,
\end{aligned}$$

and have the form

$$\begin{gathered}
\Lambda^2+2M^2=2M^2(\Lambda^2+8M^2+4)\left(\frac{2}{\Lambda^2+8M^2+4}+\frac{1}{K^2+8M^2}\right),\\
(\Lambda^2+8M^2+4)(2M^2+K^2)=6M^2(K^2+8M^2),\\
2M^2+1=M^2\left(1+\frac{\Lambda^2+8M^2+4}{K^2+8M^2}\right).
\end{gathered}$$

One can check that these relations are equivalent to the parametrisation (28).

When $M^2=1$ or $M^2=\frac{1}{2}$ these $\vee$-systems are equivalent to the Coxeter restrictions (E_7,A_3) and $(E_6,A_1\times A_1)$ respectively. Note that these are the only 4-dimensional Coxeter restrictions, containing 18 and 17 covectors respectively. One can check that other values of M do not correspond to any of these two systems. This completes the proof of the theorem. □

More examples of the non-Coxeter families one can get by the restrictions of the $\vee$-systems (27), (28) to the corresponding 3-dimensional hyperplanes determined by one of the covectors. In particular, for the covector $e_i\pm e_j$ we have the one-parameter family of $\vee$-systems

$$\begin{gathered}
\sqrt{2(2M^2+1)}e_1,\ 2\sqrt{2(M^2+1)}e_2,\ M\sqrt{\frac{2(2M^2-1)}{M^2+1}}e_3,\\
\sqrt{2}(e_1\pm e_2),\ M\sqrt{2}(e_1\pm e_3),\ M(e_1\pm 2e_2\pm e_3).
\end{gathered}$$

This family contains (for non-zero M) only two Coxeter restrictions: $(E_7,A_1\times A_3)_2$ and (E_6,A_1^3) when $M^2=1$ and $M^2=\frac{1}{2}$ respectively.

For the covector $e_1+e_2+e_3-e_4$ we have the following family of $\vee$-systems:

$$\begin{gathered}
e_1+e_2,\ e_1+e_3,\ e_2+e_3,\ \sqrt{2}e_1,\ \sqrt{2}e_2,\ \sqrt{2}e_3,\ \frac{M\sqrt{2}}{\sqrt{M^2+1}}(e_1+e_2+e_3),\\
\frac{1}{\sqrt{4M^2+1}}(e_1-e_2),\ \frac{1}{\sqrt{4M^2+1}}(e_1-e_3),\ \frac{1}{\sqrt{4M^2+1}}(e_2-e_3).
\end{gathered}$$

This $\vee$-system is equivalent to Coxeter restriction (E_7,A_4) when $M^2=1$ and to $(E_6,A_1\times A_2)$ at $M^2=\frac{1}{2}$, for other $M\neq 0$ it can not be obtained in such a way.

Note that when $M=0$ all these systems reduce to some Coxeter root systems.

Remark. The $\vee$-systems in Theorem 4 are equivalent to the deformed root systems related to Lie superalgebra of type $AB(1,3)$, which were described by Sergeev and Veselov [15]. One can show that other exceptional simple Lie superalgebras also give non-Coxeter families of $\vee$-systems.

7. Concluding remarks

The restrictions of the Coxeter root systems on the mirror hyperplanes and more generally to any their intersections were investigated within the general theory of the hyperplanes arrangements by Orlik and Solomon [13]. We have shown that the corresponding complements admit a natural logarithmic Frobenius structure, dual in Dubrovin's sense to the Strachan's structures on the strata of the Coxeter discriminants.

As we know the general logarithmic Frobenius structures are related to the $\vee$-systems, which can be considered as a proper extension of the Coxeter root systems. We have shown that this class of systems is closed under the operation of restriction to any intersection of the corresponding hyperplanes. The restrictions of the Coxeter root systems give us many examples of the $\vee$-systems but as we have shown not all of them.

Thus the classification of the $\vee$-systems still remains one of the most important open problems in this area. Probably the next step in this direction should be classification of all families of $\vee$-systems passing through these restrictions. It would also be natural to study $\vee$-systems in the complex space, in particular, in relation with complex reflection groups.

Another very important problem is to investigate the relations with Seiberg–Witten theory, see [6,12]. In particular, it would be interesting to analyse from this point of view a role of the special $\vee$-systems related to the deformed Calogero–Moser systems and Lie superalgebras [15]. The relations with the theory of (super)Jack polynomials (see [10,16]) also deserve better understanding.

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Appendix A. Restrictions of the exceptional Coxeter root systems

Below is a complete list of the $\vee$-systems A, which can be obtained as the restrictions of the exceptional Coxeter root systems. They are labeled by a pair (G, H), where G is an exceptional Coxeter group and H is its parabolic subgroup (see Section 5). When the type of the subgroup does not fix the subgroup up to a conjugation we use the index 1 or 2 following the description of all such cases given above. We give also the dimension of the space spanned by the $\vee$-system A (which is the same as corank of H) and the number $|A|$ of (noncollinear) covectors in A. The list of the equivalences between these $\vee$-systems is given after the table.

	(G, H)	Covectors of the $\vee$-system A	Dimension	$\|A\|$
1	(E_8, A_1)	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 6)$; $\sqrt{2}(e_i \pm e_7)$ $(1 \leqslant i \leqslant 6)$; $2e_7$; $\frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6)$ (odd number of minuses); $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm 2e_7)$ (even number of minuses in the first six terms)	7	91
2	$(E_8, A_1 \times A_1)$	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 6)$; $2e_i$ $(1 \leqslant i \leqslant 6)$; $\frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6)$	6	68
3	(E_8, A_2)	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 5)$; $\sqrt{3}(e_i \pm e_6)$ $(1 \leqslant i \leqslant 5)$; $2\sqrt{3}e_6$; $\frac{\sqrt{3}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6)$ (odd number of minuses); $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm 3e_6)$ (even number of minuses)	6	63
4	(E_8, A_1^3)	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 4)$; $\sqrt{2}(e_i \pm e_5), 2e_i$ $(1 \leqslant i \leqslant 4)$; $2\sqrt{3}e_5$; $e_1 \pm e_2 \pm e_3 \pm e_4$; $\frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm 2e_5)$	5	49
5	$(E_8, A_1 \times A_2)$	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 3)$; $\sqrt{2}(e_i \pm e_4), \sqrt{3}(e_i \pm e_5)$ $(1 \leqslant i \leqslant 3)$; $\sqrt{6}(e_4 \pm e_5)$; $2e_4$; $2\sqrt{3}e_5$; $\frac{\sqrt{3}}{2}(e_1 \pm e_2 \pm e_3 \pm e_5 \pm 2e_4)$ (odd number of minuses in the first four terms); $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm 3e_5 \pm 2e_4)$ (even number of minuses in the first four terms); $\frac{\sqrt{6}}{2}(e_1 \pm e_2 \pm e_3 \pm e_5)$ (even number of minuses); $\frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3 \pm 3e_5)$ (odd number of minuses)	5	46
6	(E_8, A_3)	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 5)$; $\sqrt{6}e_i$ $(1 \leqslant i \leqslant 5)$; $e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5$	5	41
7	(E_8, A_1^4)	$\sqrt{3}(e_1 \pm e_2)$, $\sqrt{2}(e_1 \pm e_3)$, $\sqrt{2}(e_2 \pm e_3)$, $\sqrt{2}(e_1 \pm e_4)$, $\sqrt{2}(e_2 \pm e_4)$, $2(e_3 \pm e_4)$; $2e_1, 2e_2, 2\sqrt{3}e_3, 2\sqrt{3}e_4$; $e_1 \pm e_2 \pm 2e_3$; $e_1 \pm e_2 \pm 2e_4$; $\frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm 2e_3 \pm 2e_4)$	4	32
8	$(E_8, A_1^2 \times A_2)$	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 3)$; $\sqrt{3}(e_i \pm e_4)$ $(1 \leqslant i \leqslant 3)$; $2e_1, 2e_2, 2e_3, 2\sqrt{6}e_4$; $\frac{\sqrt{6}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$; $\frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3 \pm 3e_4)$	4	32
9	(E_8, A_2^2)	$e_1 \pm e_2, \sqrt{3}(e_1 \pm e_3), \sqrt{3}(e_1 \pm e_4), \sqrt{3}(e_2 \pm e_3), \sqrt{3}(e_2 \pm e_4), 3(e_3 \pm e_4)$; $2\sqrt{3}e_3, 2\sqrt{3}e_4$; $\frac{1}{2}(e_1 \pm e_2 \pm 3e_3 \pm 3e_4)$ (even number of minuses); $\frac{\sqrt{3}}{2}(e_1 \pm e_2 \pm 3e_3 \pm e_4)$ (odd number of minuses); $\frac{\sqrt{3}}{2}(e_1 \pm e_2 \pm e_3 \pm 3e_4)$ (odd number of minuses); $\frac{3}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$ (even number of minuses)	4	30
10	$(E_8, A_1 \times A_3)$	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 3)$; $\sqrt{2}(e_i \pm e_4)$ $(1 \leqslant i \leqslant 3)$; $\sqrt{6}e_1, \sqrt{6}e_2, \sqrt{6}e_3, 4e_4$; $e_1 \pm e_2 \pm e_3 \pm 2e_4$; $\sqrt{2}(e_1 \pm e_2 \pm e_3)$	4	28
11	(E_8, A_4)	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 3)$; $\sqrt{5}(e_i \pm e_4)$ $(1 \leqslant i \leqslant 3)$; $2\sqrt{10}e_4$; $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm 5e_4)$ (even number of minuses); $\frac{\sqrt{5}}{2}(e_1 \pm e_2 \pm e_3 \pm 3e_4)$ (odd number of minuses); $\frac{\sqrt{10}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$ (even number of minuses)	4	25
12	(E_8, D_4)	$e_i \pm e_j$ $(1 \leqslant i < j \leqslant 4)$; $2\sqrt{2}e_i$ $(1 \leqslant i \leqslant 4)$; $\sqrt{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$	4	24
13	$(E_8, A_1^3 \times A_2)$	$\sqrt{2}(e_1 \pm e_2), \sqrt{6}(e_2 \pm e_3), \sqrt{6}(e_1 \pm e_3), 2e_1, 2\sqrt{3}e_2, 2\sqrt{6}e_3, \frac{\sqrt{2}}{2}(e_1 \pm 2e_2 \pm 3e_3), \frac{\sqrt{6}}{2}(e_1 \pm 2e_2 \pm e_3), e_1 \pm 3e_3$	3	19
14	$(E_8, A_2^2 \times A_1)$	$\sqrt{3}(e_1 + e_2), 3(e_1 + e_3), 3(e_2 + e_3), e_1 - e_2, \sqrt{3}(e_1 - e_3), \sqrt{3}(e_2 - e_3)$, $\sqrt{6}e_1$, $\sqrt{6}e_2$, $6\sqrt{2}e_3$, $\sqrt{6}(e_1 + e_2 + 3e_3)$, $\sqrt{3}(e_1 + e_2 + 4e_3)$, $3(e_1 + e_2 + 2e_3)$, $e_1 + 2e_2 + 3e_3$, $\sqrt{3}(e_1 + 3e_3)$, $\sqrt{3}(e_2 + 3e_3)$, $2e_1 + e_2 + 3e_3$, $\sqrt{6}(e_1 + 2e_3)$, $\sqrt{6}(e_2 + 2e_3)$, $\sqrt{6}(e_1 + e_2 + e_3)$	3	19
15	$(E_8, A_1^2 \times A_3)$	$2(e_1 \pm e_2), 2(e_2 \pm e_3), 2(e_1 \pm e_3), 2e_1, 2\sqrt{10}e_2, 2e_3, \frac{\sqrt{2}}{2}(e_1 \pm 4e_2 \pm e_3)$, $\sqrt{2}(e_1 \pm 2e_2 \pm e_3)$	3	17

(*continued on next page*)

(*Continued*)

	(G,H)	Covectors of the $\vee$-system A	Dimension	$\lvert A\rvert$
16	$(E_8, A_2\times A_3)$	$2\sqrt{3}(e_1\pm e_2)$, $2(e_2\pm e_3)$, $\sqrt{3}(e_1+e_3)$, $\sqrt{\frac{15}{2}}(e_1-e_3)$, $2\sqrt{3}e_1$, $2\sqrt{6}e_2$, $\frac{1}{2}(e_3+3e_1\pm 4e_2)$, $e_3-3e_1\pm 2e_2$, $\frac{\sqrt{6}}{2}(e_3+3e_1)$, $\frac{\sqrt{3}}{2}(e_3-e_1\pm 4e_2)$, $\sqrt{3}(e_3+e_1\pm 2e_2)$	3	17
17	$(E_8, A_1\times A_4)$	$\sqrt{10}(e_1\pm e_2)$, $\sqrt{2}(e_1\pm e_3)$, $\sqrt{5}(e_2+e_3)$, $\sqrt{10}(e_2-e_3)$, $2e_1$, $2\sqrt{10}e_2$, $\frac{1}{2}(e_3\pm 2e_1+5e_2)$, $\frac{\sqrt{5}}{2}(e_3\pm 2e_1-3e_2)$, $\frac{\sqrt{10}}{2}(e_3\pm 2e_1+e_2)$, $\frac{\sqrt{2}}{2}(e_3-5e_2)$, $\frac{\sqrt{10}}{2}(e_3+3e_2)$	3	16
18	$(E_8, A_1\times D_4)$	$\sqrt{2}(e_1\pm e_2)$, $\sqrt{2}(e_1\pm e_3)$, $\sqrt{5}(e_2\pm e_3)$, $2\sqrt{5}e_1$, $2\sqrt{2}e_2$, $2\sqrt{2}e_3$, $\sqrt{2}(e_2\pm e_3\pm 2e_1)$	3	13
19	(E_8, A_5)	$\sqrt{6}(e_1\pm e_2)$, $\sqrt{6}(e_2\pm e_3)$, $\sqrt{6}(e_1-e_3)$, e_1+e_3, $2\sqrt{15}e_2$, $\frac{1}{2}(e_1\pm 6e_2+e_3)$, $\frac{\sqrt{6}}{2}(e_1\pm 4e_2-e_3)$, $\frac{\sqrt{15}}{2}(e_1\pm 2e_2+e_3)$	3	13
20	(E_8, D_5)	$e_1\pm e_2$, $e_2\pm e_3$, $e_1\pm e_3$, $\sqrt{10}e_1$, $\sqrt{10}e_2$, $\sqrt{10}e_3$, $2(e_1\pm e_2\pm e_3)$	3	13
21	(E_7, A_1)	$e_i\pm e_j$ $(1\leqslant i<j\leqslant 4)$; $\sqrt{2}(e_5\pm e_i)$ $(1\leqslant i\leqslant 4)$; $2e_5$, e_6, $\frac{\sqrt{2}}{2}(e_6\pm e_1\pm e_2\pm e_3\pm e_4)$ (even number of minuses); $\frac{1}{2}(e_6\pm 2e_5\pm e_1\pm e_2\pm e_3\pm e_4)$ (odd number of minuses in the last four terms)	6	46
22	$(E_7, A_1\times A_1)$	$e_i\pm e_j$ $(1\leqslant i<j\leqslant 4)$; $2e_1$, $2e_2$, $2e_3$, $2e_4$, e_5, $\frac{\sqrt{2}}{2}(e_5\pm e_1\pm e_2\pm e_3\pm e_4)$	5	33
23	(E_7, A_2)	$e_1\pm e_2$, $e_1\pm e_3$, $e_2\pm e_3$, $\sqrt{3}(e_4\pm e_1)$, $\sqrt{3}(e_4\pm e_2)$, $\sqrt{3}(e_4\pm e_3)$, $2\sqrt{3}e_4$, e_5, $\frac{\sqrt{3}}{2}(e_5\pm e_4\pm e_1\pm e_2\pm e_3)$ (even number of minuses); $\frac{1}{2}(e_5\pm 3e_4\pm e_1\pm e_2\pm e_3)$ (odd number of minuses)	5	30
24	$(E_7, A_1^3)_1$	$2(e_i\pm e_j)$ $(1\leqslant i<j\leqslant 4)$, $2e_i$ $(1\leqslant i\leqslant 4)$, $e_1\pm e_2\pm e_3\pm e_4$	4	24
25	$(E_7, A_1^3)_2$	$2(e_1\pm e_2)$, $2(e_1\pm e_3)$, $2(e_2\pm e_3)$, $2e_1$, $2e_2$, $2e_3$, $2\sqrt{3}e_4$, $\sqrt{2}(e_4\pm e_1\pm e_2)$, $\sqrt{2}(e_4\pm e_1\pm e_3)$, $\sqrt{2}(e_4\pm e_2\pm e_3)$	4	22
26	$(E_7, A_1\times A_2)$	$\sqrt{2}(e_2\pm e_3)$, $\sqrt{3}(e_3\pm e_1)$, $\sqrt{6}(e_2\pm e_1)$, $2e_2$, $2\sqrt{3}e_1$, e_4, $\frac{1}{2}(e_4\pm 2e_2-e_3+3e_1)$, $\frac{1}{2}(e_4\pm 2e_2+e_3-3e_1)$, $\frac{\sqrt{3}}{2}(e_4\pm 2e_2+e_3+e_1)$, $\frac{\sqrt{3}}{2}(e_4\pm 2e_2-e_3-e_1)$, $\frac{\sqrt{2}}{2}(e_4+e_3+3e_1)$, $\frac{\sqrt{2}}{2}(e_4-e_3-3e_1)$, $\frac{\sqrt{6}}{2}(e_4+e_3-e_1)$, $\frac{\sqrt{6}}{2}(e_4-e_3+e_1)$	4	21
27	(E_7, A_3)	$e_1\pm e_2$, $e_1\pm e_3$, $e_2\pm e_3$, $\sqrt{6}e_1$, $\sqrt{6}e_2$, $\sqrt{6}e_3$, e_4, $e_4\pm e_1\pm e_2\pm e_3$	4	18
28	(E_7, A_1^4)	$2(e_1\pm e_2)$, $2(e_1\pm e_3)$, $2(e_2\pm e_3)$, $2\sqrt{3}e_1$, $2\sqrt{3}e_2$, $2\sqrt{3}e_3$, $\sqrt{2}(e_1\pm e_2\pm e_3)$	3	13
29	$(E_7, A_1^2\times A_2)$	$\sqrt{3}(e_1\pm e_2)$, $2e_1$, $2\sqrt{6}e_2$, e_3, $\frac{\sqrt{2}}{2}(e_3\pm e_1\pm 3e_2)$, $\frac{\sqrt{6}}{2}(e_3\pm e_1\pm e_2)$	3	13
30	(E_7, A_2^2)	$3(e_2\pm e_1)$, $2\sqrt{3}e_2$, $2\sqrt{3}e_1$, e_3, $\frac{1}{2}(e_3+3e_2-3e_1)$, $\frac{1}{2}(e_3-3e_2+3e_1)$, $\frac{\sqrt{3}}{2}(e_3+3e_2+e_1)$, $\frac{\sqrt{3}}{2}(e_3-3e_2-e_1)$, $\frac{\sqrt{3}}{2}(e_3+e_2+3e_1)$, $\frac{\sqrt{3}}{2}(e_3-e_2-3e_1)$, $\frac{3}{2}(e_3+e_2-e_1)$, $\frac{3}{2}(e_3-e_2+e_1)$	3	13
31	$(E_7, A_1\times A_3)_1$	$2\sqrt{2}(e_1\pm e_3)$, $2e_1$, $2\sqrt{6}e_3$, $2e_2$, $e_2\pm e_1\pm 2e_3$, $\sqrt{6}(e_2\pm e_1)$, $2\sqrt{2}(e_2\pm e_3)$	3	13
32	$(E_7, A_1\times A_3)_2$	$2\sqrt{2}(e_1\pm e_3)$, $2e_1$, $2\sqrt{6}e_3$, $2e_2$, $e_2\pm 2e_1\pm 2e_3$, $\frac{\sqrt{2}}{2}(e_2\pm 4e_3)$	3	11

(*Continued*)

	(G,H)	Covectors of the $\vee$-system A	Dimension	$\lvert A\rvert$
33	(E_7, A_4)	$\sqrt{5}(e_1 \pm e_2), 2\sqrt{10}e_2, e_3, \frac{1}{2}(e_3 - e_1 + 5e_2), \frac{1}{2}(e_3 + e_1 - 5e_2), \frac{\sqrt{5}}{2}(e_3 + e_1 + 3e_2), \frac{\sqrt{5}}{2}(e_3 - e_1 - 3e_2), \frac{\sqrt{10}}{2}(e_3 - e_1 + e_2), \frac{\sqrt{10}}{2}(e_3 + e_1 - e_2)$	3	10
34	(E_7, D_4)	$e_1 \pm e_2, 2\sqrt{2}e_1, 2\sqrt{2}e_2, e_3, \sqrt{2}(e_3 \pm e_1 \pm e_2)$	3	9
35	(E_6, A_1)	$e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, \sqrt{2}(e_4 \pm e_1), \sqrt{2}(e_4 \pm e_2), \sqrt{2}(e_4 \pm e_3), 2e_4, \frac{\sqrt{2}}{2}(e_5 \pm e_1 \pm e_2 \pm e_3)$ (odd number of minuses); $\frac{1}{2}(e_5 \pm 2e_4 \pm e_1 \pm e_2 \pm e_3)$ (even number of minuses in the last three terms)	5	25
36	$(E_6, A_1 \times A_1)$	$e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, 2e_1, 2e_2, 2e_3, \frac{\sqrt{2}}{2}(e_4 \pm e_1 \pm e_2 \pm e_3)$	4	17
37	(E_6, A_2)	$e_1 \pm e_2, \sqrt{3}(e_3 \pm e_1), \sqrt{3}(e_3 \pm e_2), 2\sqrt{3}e_3, \frac{\sqrt{3}}{2}(e_4 \pm e_1 \pm e_2 \pm e_3)$ (odd number of minuses); $\frac{1}{2}(e_4 \pm e_1 \pm e_2 \pm 3e_3)$ (even number of minuses)	4	15
38	(E_6, A_1^3)	$\sqrt{2}(e_1 \pm e_2), 2\sqrt{3}e_1, 2e_2, \frac{\sqrt{2}}{2}(e_3 \pm 2e_1 \pm e_2), e_3 \pm e_2$	3	10
39	$(E_6, A_1 \times A_2)$	$\sqrt{6}(e_1 \pm e_2), 2e_1, 2\sqrt{3}e_2, \frac{1}{2}(e_3 \pm 2e_1 + 3e_2), \frac{\sqrt{3}}{2}(e_3 \pm 2e_1 - e_2), \frac{\sqrt{2}}{2}(e_3 - 3e_2), \frac{\sqrt{6}}{2}(e_3 + e_2)$	3	10
40	(E_6, A_3)	$2(e_1 \pm e_2), 2\sqrt{6}e_1, \frac{1}{2}(e_3 \pm 4e_1 + e_2), e_3 \pm 2e_1 - e_2, \frac{\sqrt{6}}{2}(e_3 + e_2)$	3	8
41	(H_4, A_1)	$e_1, e_2, e_3, \frac{\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3), ae_1 \pm \frac{1}{2}e_2 \pm be_3, be_1 \pm ae_2 \pm \frac{1}{2}e_3, \frac{1}{2}e_1 \pm be_2 \pm ae_3, \sqrt{2}(ae_1 \pm \frac{1}{2}e_3), \sqrt{2}(ae_1 \pm be_2), \sqrt{2}(be_1 \pm \frac{1}{2}e_2), \sqrt{2}(\frac{1}{2}e_2 \pm ae_3), \sqrt{2}(ae_2 \pm be_3), \sqrt{2}(\frac{1}{2}e_1 \pm be_3), \sqrt{2}(\frac{1}{2}e_1 \pm ae_2), \sqrt{2}(be_1 \pm ae_3), \sqrt{2}(be_2 \pm \frac{1}{2}e_3)$ $(a = \frac{\sqrt{5}+1}{4}, b = \frac{\sqrt{5}-1}{4})$	3	37

There are also the following two (equivalent) families, which can be obtained by restriction of the system $F_4(\Lambda)$ related to the exceptional group F_4. In Section 5.3 they were denoted $F_3^1(\Lambda)$ and $F_3^2(\Lambda)$ respectively.

$(F_4(\Lambda), A_1)_1$	$e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, \sqrt{4\Lambda^2+2}e_1, \sqrt{4\Lambda^2+2}e_2, \sqrt{4\Lambda^2+2}e_3, \Lambda\sqrt{2}(e_1 \pm e_2 \pm e_3)$	3	13
$(F_4(\Lambda), A_1)_2$	$\sqrt{2\Lambda^2+1}(e_1 \pm e_2), \sqrt{2}(e_2 \pm e_3), \sqrt{2}(e_1 \pm e_3), 2\sqrt{2\Lambda^2+1}e_3, 2\Lambda e_1, 2\Lambda e_2, \Lambda(e_1 \pm e_2 \pm 2e_3)$	3	13

Not all of these $\vee$-systems are different. Namely, the following equivalences among them can be established:

$$(E_8, D_4) = F_4(\sqrt{2}), \qquad (E_8, D_5) = \bigl(F_4(\sqrt{2}), A_1\bigr)_1, \qquad (E_8, A_1 \times D_4) = \bigl(F_4(\sqrt{2}), A_1\bigr)_2,$$

$$\bigl(E_7, A_1^3\bigr)_1 = F_4\left(\frac{1}{2}\right), \qquad \bigl(E_7, A_1^4\bigr) = \left(F_4\left(\frac{1}{2}\right), A_1\right)_1,$$

$$(E_7, A_1 \times A_3)_1 = \left(F_4\left(\frac{1}{2}\right), A_1\right)_2,$$

$$(E_7, D_4) = B_3\left(\frac{\sqrt{2}}{2}\right), \qquad (E_6, A_3) = B_3\left(-\frac{2}{3}; 1, 1, \frac{2}{3}\right),$$

$$\big(F_4(\Lambda), A_1\big)_1 = \left(F_4\left(\frac{1}{2\Lambda}\right), A_1\right)_2,$$

$$\big(F_4(0), A_1\big)_1 = B_3(0; 1, 1, 1) = B_3(\sqrt{2}), \qquad \big(F_4(0), A_1\big)_2 = B_3(-1; 1, 1, 2),$$

where the systems $B_n(\Lambda)$, $F_4(\Lambda)$, $B_3(\gamma; c_1, c_2, c_3)$ are defined by (22), (24), (23) respectively. We can add here also the equivalence $F_4(\Lambda) = F_4(\frac{1}{2\Lambda})$ (see Section 5.3).

References

[1] O.A. Chalykh, A.P. Veselov, Locus configurations and ∨-systems, Phys. Lett. A 285 (2001) 339–349.

[2] O.A. Chalykh, M.V. Feigin, A.P. Veselov, New integrable generalizations of Calogero–Moser quantum problem, J. Math. Phys. 39 (2) (1998) 695–703.

[3] W. Couwenberg, G. Heckman, E. Looijenga, Geometric structures on the complement of a projective arrangement, math.AG/0311404; Publ. Math. Inst. Hautes Études Sci. 101 (2005) 69–161.

[4] B. Dubrovin, Geometry of 2D topological field theories, in: Integrable Systems and Quantum Groups, Montecatini, Terme, 1993, in: Springer Lecture Notes in Math., vol. 1620, 1996, pp. 120–348.

[5] B. Dubrovin, Differential geometry of the space of orbits of a Coxeter group, Surv. Differ. Geom. IV (1999) 213–238, hep-th/9303152.

[6] B. Dubrovin, On almost duality for Frobenius manifolds, in: Geometry, Topology, and Mathematical Physics, in: Amer. Math. Soc. Transl. Ser. 2, vol. 212, 2004, pp. 75–132, math.DG/0307374.

[7] P.K.H. Gragert, R. Martini, Solutions of WDVV equations in Seiberg–Witten theory from root systems, J. Nonlinear Math. Phys. 6 (1) (1999) 1–4.

[8] C. Hertling, Frobenius Manifolds and Moduli Spaces for Singularities, Cambridge Tracts in Math., vol. 151, Cambridge Univ. Press, Cambridge, 2002.

[9] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Univ. Press, 1990.

[10] M. Kasatani, T. Miwa, A.N. Sergeev, A.P. Veselov, Coincident root loci and Jack and Macdonald polynomials for special values of the parameters, math.QA/0404079; Contemp. Math., Amer. Math. Soc., 2006, in press.

[11] Yu.I. Manin, F-manifolds with flat structure and Dubrovin's duality, Adv. Math. 198 (1) (2005) 5–26, math. DG/0402451.

[12] A. Marshakov, A. Mironov, A. Morozov, WDVV-like equations in $N = 2$ SUSY Yang–Mills theory, Phys. Lett. B 389 (1996) 43–52.

[13] P. Orlik, L. Solomon, Coxeter arrangements, in: Singularities, Part 2, Arcata, CA, 1981, in: Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983.

[14] K. Saito, On a linear structure of the quotient variety by a finite reflection group, Publ. RIMS, Kyoto Univ. 29 (1993) 535–579.

[15] A.N. Sergeev, A.P. Veselov, Deformed quantum Calogero–Moser systems and Lie superalgebras, Comm. Math. Phys. 245 (2004) 249–278.

[16] A.N. Sergeev, A.P. Veselov, Generalised discriminants, deformed quantum Calogero–Moser–Sutherland operators and super-Jack polynomials, Adv. Math. 192 (2005) 341–375.

[17] O. Shcherbak, Wavefronts and reflection groups, Russian Math. Surveys 43 (3) (1988) 149–194.

[18] I.A.B. Strachan, Frobenius manifolds: Natural submanifolds and induced bi-hamiltonian structures, Differential Geom. Appl. 20 (1) (2004) 67–99.

[19] A.P. Veselov, Deformations of root systems and new solutions to generalised WDVV equations, Phys. Lett. A 261 (1999) 297.

[20] A.P. Veselov, On geometry of a special class of solutions to generalised WDVV equations, in: Integrability: The Seiberg–Witten and Whitham equations, Edinburgh, 1998, Gordon and Breach, 2000, pp. 125–135, hep-th/0105020.

[21] A.P. Veselov, M.V. Feigin, O.A. Chalykh, New integrable deformations of quantum Calogero–Moser problem, Uspekhi Mat. Nauk 51 (3) (1996) 185–186.