NOTE

A Note on "Uniqueness of Limit Cycles in a Liénard-Type System"

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\[ \frac{dx}{dt} = h(y) - F(x), \quad \frac{dy}{dt} = -g(x), \]

(1.1)

where the functions in (1.1) are assumed to be continuous and such that uniqueness for solutions of initial value problems is guaranteed.

1. INTRODUCTION

Huang and Sun [5] proposed a theorem to guarantee the uniqueness of limit cycles for the generalized Liénard system

\[ \frac{dx}{dt} = h(y) - F(x), \quad \frac{dy}{dt} = -g(x), \]

(1.1)

where the functions in (1.1) are assumed to be continuous and such that uniqueness for solutions of initial value problems is guaranteed.
If we define, as usual, \( G(x) = \int_0^x g(\xi) d\xi \) and \( H(y) = \int_0^y h(\tau) d\tau \), then the conditions stated by Huang and Sun are as follows:

(i) \( h(0) = 0 \), \( h(y) \) is strictly increasing, and \( h(\pm \infty) = \pm \infty \);
(ii) \( xg(x) > 0 \) when \( x \neq 0 \) and \( G(\pm \infty) = \pm \infty \);
(iii) there exist constants \( x_1, x_2 \) with \( x_1 < 0 < x_2 \) such that \( F(x_1) = F(0) = F(x_2) = 0 \) and \( xF(x) < 0 \) for \( x \in (x_2, x_2) \setminus \{0\} \);
(iv) there exist constants \( M > 0 \), \( K \), \( K_0 \) with \( K > K_0 \), such that \( F(x) > K \) for \( x \geq M \) and \( F(x) < K_0 \) for \( x \leq -M \);
(v) one of the following holds:
   (a) \( G(x_1) = G(x_2) \), or
   (b) \( G(-x) \geq G(x) \) for \( x > 0 \). Furthermore let \( W(x) = \int_0^{x^{-1} F(a) h(y)} dy \), where \( x^{-1} \) is the inverse function of \( h \). Then
      (a) if \( x_2 \leq |x_1| \) then \( \max_{0 \leq x \leq x_2} \{G(x) + W(x)\} \geq G(x_1) \),
      (b) if \( 0 < |x_1| < x_2 \) then \( \max_{x_1 \leq x \leq 0} \{G(x) + W(x)\} \geq G(x_2) \).

Conditions (i)–(iii) imply that system (1.1) has a unique singularity, which will be an antisaddle, i.e., a critical point at which the product of the eigenvalues of the Jacobian matrix is positive. It has been proved by Huang and Sun [5] that

- it follows from conditions (i)–(iii) that this singularity is located at \( O(0,0) \) and is unstable.

- If (iv) holds then there exists a closed curve \( \Gamma \) such that every trajectory intersecting it crosses it in the exterior-to-interior direction, hence implying the existence of at least one stable limit cycle, by the Poincaré–Bendixson theorem; see, for instance, Andronov et al. [2].

- Finally, condition (v) ensures that all closed trajectories of system (1.1) have to intersect both \( x = x_1 \) and \( x = x_2 \).

Huang and Sun claimed that if conditions (i)–(v) hold then system (1.1) has exactly one limit cycle. We will point out that this claim is incorrect. In fact, in their proof Huang and Sun compare the values of the differential of the function \( G(x) + H(y) \) integrated along two limit cycles. However, this comparison is valid only if the following condition is added:

\[
F(x) \text{ is nondecreasing for } x \in (-\infty, x_1) \cup (x_2, \infty). \quad (1.2)
\]

We will give an example due to Zhang and Shi [12], which satisfies conditions (i)–(v) but violates (1.2), which has three limit cycles.

If (i)–(v) and (1.2) do hold we will give a short proof that system (1.1) has exactly one limit cycle, not by using a comparison method but by
estimating the divergence of system (1.1) integrated along a limit cycle. By this we can show that the limit cycle is hyperbolic. A limit cycle is hyperbolic, or simple, if for any arbitrarily small analytic perturbation of the system there is no other limit cycle in a sufficiently small neighbourhood of the limit cycle; see, for instance, [2].

Next, we will state an additional condition to guarantee the uniqueness of the limit cycle in case (1.2) is violated. If the functions in (1.1) are all odd then system (1.1) exhibits symmetry with respect to the origin and the conditions of our theorem can be weakened.

Finally, we provide some examples that illustrate our results.

2. THREE UNIQUENESS THEOREMS FOR SYSTEM (1.1)

We will first state a theorem in case both (i)–(v) and (1.2) hold.

**Theorem 2.1.** If conditions (i)–(v) and (1.2) hold then system (1.1) has exactly one closed orbit, a hyperbolic stable limit cycle.

This theorem will be proved by showing that if \( \gamma \) is a closed orbit then its characteristic exponent \( \mathcal{R} = \int_{\gamma} f(x) dt \) satisfies \( \mathcal{R} < 0 \), where \( f(x) = (d/dx) F(x) \). This shows that \( \gamma \) is hyperbolic and stable; see, for instance, Andronov et al. [2]. Because two adjacent limit cycles cannot both be stable, the uniqueness of \( \gamma \) follows. In order to estimate the characteristic exponent the following lemma by Zeng et al. [10] appears to be useful.

**Lemma 2.2.** Let \( \gamma \) be an arc of an orbit of the system (1.1), described by \( y(x), \alpha \leq x \leq \beta \). Then

\[
\int_{\gamma} f(x) dt = \text{sgn}(h(y(\alpha)) - F(\alpha)) \left[ \ln \frac{F(\beta) - h(y(\alpha))}{F(\alpha) - h(y(\alpha))} \right] + \int_{\alpha}^{\beta} \frac{g(x)(dh/dy)}{(F(\beta) - h(y(x))(F(x) - h(y(x)))^2} dx.
\]

**Proof of Theorem 2.1.** It was shown by Huang and Sun [5] that it follows from conditions (i)–(v) that system (1.1) has at least one limit cycle \( \Gamma \) and it intersects both \( x = x_1 \) and \( x = x_2 \); see Figure 2.1. Denote the intersection point of \( \Gamma \) with the positive \( y \)-axis by \( A \). Let \( B \) and \( C \) be the intersection points of \( \Gamma \) with \( x = x_2 \) in the first and fourth quadrant, respectively. If we denote the arc of \( \Gamma \) between \( A \) and \( B \) by \( \gamma_1 \), then
applying Lemma 2.2 with $\alpha = 0$ and $\beta = x_2$ yields

$$\int_{\gamma_1} -f(x)\,dt = \int_0^{x_2} \frac{F(x)g(x)(dh/dy)}{h(y(x))(F(x) - h(y(x)))^2} \,dx.$$  

This integral is negative because the integrand is negative by virtue of (i)–(iii). Thus we have proved

$$\int_{\gamma_1} -f(x)\,dt < 0.$$  

For $\gamma_2$, the arc of $\Gamma$ between $B$ and $C$, we obtain by (1.2) and $f(x) = (d/dx)F(x)$

$$\int_{\gamma_2} -f(x)\,dt < 0.$$  

Proceeding in this way we can prove that $\int_\Gamma -f(x)\,dt < 0$. This completes the proof.

If the monotonicity of $F(x)$ is assumed only on the intervals $(\tilde{x}_1, x_1)$ and $(x_2, \tilde{x}_2)$ then we can obtain the following:

**Corollary 2.3.** If conditions (i)–(v) hold and $F(x)$ is nondecreasing on $(\tilde{x}_1, x_1)$ and $(x_2, \tilde{x}_2)$ then in the strip $\tilde{x}_1 \leq x \leq \tilde{x}_2$ system (1.1) has at most one closed orbit, a hyperbolic stable limit cycle.
Proof. If system (1.1) has a closed orbit then its uniqueness can be proved as in Theorem 2.1. However, in the strip $x_1 \leq x \leq x_2$ the existence of a closed orbit is no longer guaranteed.

Next we discuss the example of Zhang and Shi [12] which shows that if the conditions (i)–(v) hold but (1.2) does not, then system (1.1) can have more than one limit cycle.

Consider the following system

$$\frac{dx}{dt} = y - \varepsilon F(x), \quad \frac{dy}{dt} = -x,$$

with $0 < \varepsilon \ll 1$ and $F(x) = \frac{1}{35}x^7 - \frac{3}{40}x^5 + \frac{299}{4800}x^3 - \frac{99}{4800}x$.

For $\varepsilon = 0$ all trajectories of (2.1) are closed and satisfy $H(x, y) = x^2 + y^2 = R^2$. To find the closed orbits for $0 < \varepsilon \ll 1$ we have to study $I(r) = \int_0^r \sigma y \cos tF(r \cos t)dt + O(\varepsilon^2)$, whose zeros correspond with limit cycles for system (2.1); see Pontryagin [7]. An elementary calculation reveals that

$$I(r) = \int_0^r \sigma y \cos tF(r \cos t)dt = \frac{\pi}{64}r^2(r^2 - 1)(r^2 - \frac{9}{10})(r^2 - \frac{11}{10}).$$

Because the nonzero roots of $I(r)$ are simple it follows that (2.1) has three hyperbolic limit cycles, located in the vicinity of the circles, $x^2 + y^2 = R$, with $R = (12 - i)/10$, $i = 1, 2, 3$.

It is easy to check that conditions (i)–(v) hold but (1.2) is not satisfied, as can be seen by plotting the graph of $F(x)$.

If (1.2) is violated then we need an additional condition to guarantee the uniqueness of the limit cycle. In order to formulate this condition we will use a lemma by Zhang et al. [11].

Lemma 2.4. Let $F_1(x) = F(x)$ and $F_2(x) = F(-x)$, both for $0 \leq x \leq d$, where either $d \in \mathbb{R}^+$ or $d = \infty$. Suppose that conditions (i)–(iii) are satisfied and in addition assume that the following holds:

(i) $g(-x) = -g(x)$ and $g(x)$ is nondecreasing as $x$ increases;
(ii) $y = F_1(x)$ intersects $y = F_2(x)$ at two points, $(0, 0)$ and $(a, b)$ with $0 < a < d$;
(iii) $F_2(x) \geq F_1(x)$ for $x \in (0, a)$;
(iv) For $j = 1, 2$ there exist $\tau_j, \xi_j \in [a, d]$ with $\tau_j \leq \xi_j$ such that

(a) $(-1)^j F_1(x) \leq 0$ for $x \in (\tau_j, r \cup [a, d]$, where $r = \max_{j=1,2} \{\tau_j + \xi_j\}$;
(b) $(-1)^j F_1(x) + (-1)^{j+1} F_2(x - \xi_j) + \xi_j \leq 0, \neq 0$, for $x \in [0, \tau_j]$;
(c) $F_1(x) > 0$ and $F_2(x) < 0$ for $x \in (r, d]$. 


Then for all \( x_0 \in [r, d] \) the backward and forward orbits passing through \((x_0, h^{-1}(F(x_0)))\) cross the \( y \)-axis in \( A \) and \( B \), respectively. Similarly, the forward and backward orbits passing through \((-x_0, h^{-1}(F(-x_0)))\) cross the \( y \)-axis in \( C \) and \( D \), respectively, where \( y_A > y_C \) and \( y_B > y_D \); see Fig. 2.2.

The proof of Lemma 2.4 can be found in Zhang et al. [11, Theorem 7.9, Chap. 4], where conditions that guarantee the existence of at least \( n \) limit cycles are derived. In fact, Lemma 2.4 corresponds with the case \( n = 1 \).

For a geometrical interpretation of \( F(x) \) in Lemma 2.4 we refer to Fig. 2.3.

We will now show an important implication of Lemma 2.4.

**Corollary 2.5.** Lemma 2.4 implies that for all \( x_0 \in [r, d] \) system (1.1) has no closed orbits in the strip \(|x| \leq d\) which cross \( x = x_0 \) or \( x = -x_0 \).

**Proof.** First the nonexistence of closed orbits intersecting both \( x = -x_0 \) and \( x = x_0 \) is shown. Assume by contradiction that system (1.1) has a closed orbit \( \Gamma_1 \) intersecting \( y = h^{-1}(F(x)) \) in \( S(x_i, h^{-1}(F(x_i))) \) and \( T(x_i, h^{-1}(F(x_i))) \), with \( x_i > x_0 \) and \( x_i < -x_0 \); see Fig. 2.4.

First assume that \( x_i > -x_i \). Let \( R \) denote the intersection of \( \Gamma_1 \) with the positive \( y \)-axis. Then by Lemma 2.4 the forward orbit \( \gamma \) passing through \((-x_i, h^{-1}(F(-x_i)))\) will cross the positive \( y \)-axis, say in \( U \), such that \( y_U < y_R \). This is impossible because obviously \( \gamma \) cannot intersect \( \Gamma_1 \). The case \( x_i < -x_i \) can be dealt with in a similar way.

Fig. 2.2. The implication of Lemma 2.4.
Finally we exclude the possibility of a closed orbit intersecting only $x = -x_0$ or $x = x_0$. An oscillatory orbit intersecting $x = -x_0$ but not $x = x_0$ has to cross the $y$-axis between $A$ and $C$; see Fig. 2.2. But then, because $y_B > y_D$, this trajectory cannot intersect $x = -x_0$ again so it cannot be closed. The same argument holds for trajectories crossing $x = x_0$ but not $x = -x_0$. This completes the proof.
Remark 2.1. If \( g(x) \) does not satisfy condition (I) of Lemma 2.4 then we can apply the transformation \( u = \sqrt{2G(x)} \sgn x \) to reduce system (1.1) to

\[
\frac{du}{dt} = h(y) - F(x(u)), \quad \frac{dy}{dt} = -u,
\]

where \( x(u) \) is the inverse function of \( u = \sqrt{2G(x)} \sgn x \). Now (2.2) satisfies condition (I) of Lemma 2.4, because \( g(u) = u \), but in general it will be quite cumbersome to check the other conditions.

Theorem 2.6. Suppose that system (1.1) satisfies conditions (i)–(iii), (v), and (I)–(IV) and in addition assume that

\[
F'(x) \geq 0 \quad \text{for} \ x \in (-r, x_1) \cup (x_2, r).
\]

Then in the strip \( |x| \leq d \) system (1.1) has exactly one closed orbit, a hyperbolic stable limit cycle.

Proof. Consider the backward and forward trajectories passing through \( B_0(r, h^{-1}(F(r))) \) and suppose that they cross the \( y \)-axis in \( A_0 \) and \( C_0 \), respectively. Similarly, suppose that the forward and backward trajectories passing through \( E_0(-r, h^{-1}(F(-r))) \) cross the \( y \)-axis in \( F_0 \) and \( D_0 \), respectively. Then by Lemma 2.4 every trajectory of (1.1) intersecting the curve \( A_0 B_0 C_0 D_0 E_0 F_0 A_0 \) crosses it in the exterior-to-interior direction, because \( y_{A_0} > y_{F_0} \) and \( y_{C_0} > y_{D_0} \); see Fig. 2.5.

Because \( O(0,0) \) is an unstable antisaddle it follows from the Poincaré–Bendixson theorem that system (1.1) has at least one limit cycle in the strip \( |x| < r \). By condition (v) any such limit cycle will have to intersect both \( x = x_1 \) and \( x = x_2 \). Because (2.3) holds it follows from Corollary 2.3 that the limit cycle is hyperbolic and stable and hence unique. It follows from applying Corollary 2.5 with \( x_0 = r \) that there are no limit cycles in the strip \( |x| \leq d \) that cross \( x = -r \) or \( x = r \). This completes the proof.

If \( h(y) = y \) and \( g(x) = x \) then system (1.1) reduces to

\[
\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -x.
\]

If an additional condition is satisfied then for system (2.4) the conditions of Theorem 2.6 can be weakened.
**Theorem 2.7.** Suppose that system (2.4) satisfies conditions (iii), (v), and (II)–(IV) and in addition assume that there exist $\alpha_2 > x_2$, $\alpha_1 < x_1$ such that

\[
\begin{align*}
F'(x) &\geq 0 \text{ for } x \in [x_2, \alpha_2], & F'('2) &\leq 0, \\
F''(x) &\leq 0 \text{ for } x \in [\alpha_2, r], \\
F'(x) &\geq 0 \text{ for } x \in [\alpha_1, x_1], & F'('1) &\leq 0, \\
F''(x) &\geq 0 \text{ for } x \in [-r, \alpha_1],
\end{align*}
\]  

and

\[
F(\alpha_2) \geq M + \hat{x} \quad \text{and} \quad F(\alpha_1) \leq -M - \hat{x},
\]

where $\hat{x} = \max(-\alpha_1, \alpha_2)$, $M = \max(\max_{0 \leq x \leq x_1}(-F(x)), \max_{x_1 \leq x \leq 0}(F(x)))$. Then in the strip $|x| \leq d$ system (2.4) has exactly one closed orbit, a hyperbolic stable limit cycle.

For the proof of Theorem 2.7 we will use a modification of a result by Rychkov [8]; see also Ye et al. [9, Theorem 7.2].

**Lemma 2.8.** Suppose that there exist constants $\beta_1 < \alpha_1 < x_1 < 0 < x_2 < \alpha_2 < \beta_2$ such that

(a) $F(x_1) = F(0) = F(x_2) = 0$ and $xF(x) < 0$ for $x \in (x_2, x_1) \setminus \{0\}$;

(b) $F'(x) \geq 0$ for $x \in (\alpha_1, x_1) \cup (x_2, \alpha_2)$.
(c) $F'(x) \leq 0$ for $x \in (\alpha_2, \beta_2)$ and $F'(x) \geq 0$ for $x \in (\beta_1, \alpha_1)$;
(d) every closed orbit of system (2.4) intersects both $x = x_1$ and $x = x_2$;
(e) every closed orbit of system (2.4) that is not inside the strip $\alpha_1 \leq x \leq \alpha_2$ intersects both $x = \alpha_1$ and $x = \alpha_2$.

Then system (2.4) has at most two limit cycles in the strip $\beta_1 \leq x \leq \beta_2$. If two limit cycles exist then the inner one is stable and the outer one unstable.

The method of proof for Lemma 2.8 is exactly the same as in Rychkov's theorem; see Ye et al. [9, Theorem 7.2]. This is because the limit cycles of (2.4) are either intersecting both $x = \alpha_1$ and $x = \alpha_2$, or they are inside the strip $\alpha_1 \leq x \leq \alpha_2$ while intersecting both $x = x_1$ and $x = x_2$.

**Proof of Theorem 2.7.** It was proved by Huang [3], see also Ye et al. [9, Theorem 5.2], that it follows from condition (2.7) that the backward orbits passing through $P_2(\alpha_2, F(\alpha_2))$ and $P_1(\alpha_1, F(\alpha_1))$ intersect $x = \alpha_1$ and $x = \alpha_2$ in $B_1$ and $A_1$ respectively with $y_{B_1} > F(\alpha_1)$ and $y_{A_1} < F(\alpha_2)$; see Fig. 2.6.

An oscillatory orbit intersecting $x = \alpha_1$ but not $x = \alpha_2$ has to cross $x = \alpha_1$ between $B_1$ and $P_1$. But then this trajectory cannot cross $x = \alpha_1$ again (because it will first intersect $y = F(x)$; see Fig. 2.6); hence it cannot be closed. The same argument holds for trajectories crossing $x = \alpha_2$ but not $x = \alpha_1$. Therefore (2.7) implies that condition (e) of Lemma 2.8 is

![Fig. 2.6. Backward orbits through $P_1$ and $P_2$.](image-url)
satisfied. Because $O(0,0)$ is an unstable antisaddle the existence of a limit cycle $\Gamma_{in}$ in the strip $\alpha_1 \leq x \leq \alpha_2$ also follows by the Poincaré–Bendixson theorem. Condition (d) is satisfied because (v) holds; see Huang and Sun [5]. The limit cycle $\Gamma_{in}$ in the strip $\alpha_1 \leq x \leq \alpha_2$ is unique, hyperbolic, and stable by Corollary 2.3. As in Theorem 2.6 there are no limit cycles intersecting $x = -r$ or $x = r$. Because all conditions of Lemma 2.8 are satisfied it follows that if system (2.4) has a limit cycle $\Gamma_{out}$ intersecting $x = \alpha_1$ and $x = \alpha_2$ then it has to be unstable and unique. This is impossible because as a result of the Poincaré–Bendixson theorem the number of stable and unstable limit cycles in the strip $|x| \leq r$ intersecting both $x = \alpha_1$ and $x = \alpha_2$ has to be equal; see, for instance, [2]. This completes the proof.

Remark 2.2. It is easy to see that if we assume that $F'(x) \geq 0$ on $[-r, x_1]$ instead of (2.6) then we do not need (2.7) to prove the uniqueness of the limit cycle.

Remark 2.3. If both on $[x_1, 0]$ and on $[0, x_2]$ $F'(x)$ has only one zero, say $a_1$ and $a_2$, respectively, then condition (2.7) can be weakened, see Huang and Yang [4], to

$$F(\alpha_2) \geq M + \hat{x} - \frac{\sigma}{2M + \hat{x}} \quad \text{and} \quad F(\alpha_1) \leq -M - \hat{x} + \frac{\sigma}{2M + \hat{x}},$$

(2.8)

where $M = \max(-F(a_1), F(a_2))$, $\hat{x} = \max(-\alpha_1, \alpha_2)$, and $\sigma = \min(\frac{1}{2}a_2^2 - \frac{1}{4}a_2^2, \frac{1}{2}a_2^2 - \frac{1}{4}a_2^2)$.  

3. UNIQUENESS THEOREMS WHEN \( h(y), g(x), \) AND \( F(x) \) ARE ODD

If the functions $h(y)$, $g(x)$, and $F(x)$ are odd then system (1.1) is symmetric with respect to the origin. This means that the conditions of Theorems 2.6 and 2.7 can be weakened. For this case we will not use Lemma 2.4 but a more general result by Alsholm [1, Corollary 3].

Lemma 3.1. Consider system (1.1) and suppose that (i) and (ii) are satisfied. Furthermore assume that

1. $h(-y) = -h(y)$, $g(-x) = -g(x)$, and $F(-x) = -F(x)$;
2. there exists $x_2 > 0$ such that $F(0) = F(x_2) = 0$ and $F(x) < 0$ for $x \in (0, x_2)$;
(γ) let \( I = [0, x_2] \) and \( J = [x_2, d] \) with \( x_2 < d \). \( \varphi: I \to J \) is weakly increasing, continuous and

\[
g(\varphi(x)) \varphi'(x) \geq g(x) \quad \text{for} \quad x \in I;
\]

(δ) with \( \varphi \) satisfying (γ) we have \( F(\varphi(x)) \geq -F(x) \) for \( x \in I \);

(ε) \( F(x) > 0 \) for all \( x_0 \in [x_2, d] \).

Then for all \( x_0 \in [\varphi(x_2), d] \) the backward and forward orbits passing through \( (x_0, h^{-1}(F(x_0))) \) cross the \( y \)-axis, in \( A \) and \( B \) respectively and \( y_A > -y_B \).

Remark 3.1. If the functions in system (1.1) are all odd then Lemma 2.4 corresponds to a special case of Lemma 3.1 with \( \varphi(x) = x + \xi \), with \( \xi = \xi_1 = \xi_2 \), by symmetry.

**Theorem 3.2.** Suppose that system (1.1) satisfies conditions (i), (ii), (α), (β), (γ), (δ), and (ε). Furthermore assume that

\[
F'(x) \geq 0 \quad \text{for} \quad x \in (x_2, \varphi(x_2)).
\] (3.1)

Then in the strip \( |x| \leq d \) system (1.1) has exactly one closed orbit, a hyperbolic stable limit cycle.

The proof of Theorem 3.2 is basically the same as that of Theorem 2.6 and is therefore omitted. Note that we have dropped condition (v) because if (ε) holds then if \( (x(t), y(t)) \) is a solution of (1.1) then so is \( (-x(t), -y(t)) \). Therefore every closed orbit is symmetric with respect to the origin.

For a geometrical interpretation of Theorem 3.2 we refer to Fig. 3.1.

Finally we will state a theorem for the case that \( h(y) = y \), \( g(x) = x \), and \( F(-x) = -F(x) \), hence weakening the conditions of Theorem 2.7.

**Theorem 3.3.** Suppose that system (2.4) with \( F(-x) = -F(x) \) satisfies the conditions (β), (γ), (δ), and (ε). Furthermore assume that there exists
\(\alpha > x_2\) such that

\[F'(x) \geq 0 \quad \text{for} \quad x \in [x_2, \alpha] \quad \text{and} \quad F''(x) \leq 0 \quad \text{for} \quad x \in [\alpha, \varphi(x_2)].\]

Then in the strip \(|x| \leq d\) system (2.4) has exactly one closed orbit, a hyperbolic stable limit cycle.

The proof is basically the same as the proof of Theorem 2.7 and is therefore omitted.

**Remark 3.2.** We have dropped condition (2.7) as condition (e) of Lemma 2.8 is always satisfied because every closed orbit is symmetric with respect to the origin and \(\alpha_1 = -\alpha_2\).

**Remark 3.3.** Because (2.7) does not necessarily hold the limit cycle might not be contained in \(|x| \leq \alpha\). If this occurs then we cannot prove that the limit cycle is hyperbolic through Theorem 2.1. However, we can use a result by Odani [6, Theorem B], to arrive at the same conclusion.

### EXAMPLES

In this section we will give some examples that illustrate our results. For all systems in this section we will prove that there exists a unique, hyperbolic stable limit cycle.

**Example 4.1.** Consider the system

\[
\frac{dx}{dt} = \frac{1}{5} y - x(x - 1)(x + 1.1), \quad \frac{dy}{dt} = -x^3. \tag{4.1}
\]

This example was discussed by Huang and Sun [5, Example 3.1]. It satisfies all conditions of Theorem 2.1.

**Example 4.2.** Consider the system

\[
\frac{dy}{dt} = h(y) - kF(x), \quad \frac{dy}{dt} = -g(x), \tag{4.2}
\]

with \(k > 0\), \(F(x) = \frac{1}{20}(x^2 - 1)(20x^2 - 140x + 247)\), see Fig. 4.1, and where \(h(y)\) and \(g(x)\) satisfy (i), (ii), and (i). It is easy to check that (iii), (v), (ii), and (iii) are also satisfied with \(x_2 = -x_1 = 1\) and hence \(a = 1\). In order to apply Theorem 2.6 we only need to ascertain that (iv) and (2.3) are also satisfied. It can be shown that for \(\tau_1 = \tau_2 = 1, \xi_1 = \xi_2 = 1.1\) (IV) is also satisfied; see Fig. 4.2. Note that \(F'(x) > 0\) for \(x \in [1, \alpha]\) with \(\alpha \approx 2.27427\) and so \(r = 1 + 1.1 < \alpha\) and hence (2.3) also holds.
EXAMPLE 4.3. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= y - F(x), \\
\frac{dy}{dt} &= -x,
\end{align*}
\]

(4.3)

with \( F(x) = \frac{1}{1000}(x^2 - 1)(10x^2 - 46x + 53)(100x^2 + 460x + 539) \); see Fig. 4.3. We will show that Theorem 2.7 can be applied. Again it is easy to verify conditions (iii), (v), (ii), and (iii) with \( x_2 = -x_1 = 1 \) and hence \( a = 1 \). Let \( \alpha_2 = \min(x > 1 \mid F'(x) = 0) \) and \( \beta_2 = \min(x > \alpha_2 \mid F''(x) = 0) \). In a similar way define \( \alpha_1 \) and \( \beta_1 \); see also Fig. 4.3. It can be shown that for \( \tau_1 = \tau_2 = 1, \xi_1 = \xi_2 = 1, \) (IV) is also satisfied; see Fig. 4.4. Note that \( r = 1 + 1 = 2 \). Because \( \beta_2 = 2.04569 > r \) and \( \beta_1 = -2.00427 < -r \) (2.5) and (2.6) also hold. Finally we will verify (2.7). Clearly, \( M = \max(-F(\gamma_2), F(\gamma_1)) \), where \( F'(\gamma_2) = 0, 0 < \gamma_2 < 1 \) and \( F'(\gamma_1) = 0, -1 \).
< \gamma_1 < 0. \text{ In fact } M = F(\gamma_1) \approx 9.88062. \text{ With } \hat{x} = \max(-\alpha_1, \alpha_2) = -\alpha_1 \approx 1.75595 \text{ we can check that } F(\alpha_2) \approx 19.3022 > M + \hat{x} \text{ and } \bar{F}(\alpha_1) \approx -23.8457 < -M - \hat{x}.

Remark 4.1. If we multiply \( F(x) \) in system (4.3) with a constant \( k > 0 \) then all conditions of Theorem 2.7 still hold, except possibly condition (2.7). However, for

\[
    k \geq \max \left( \frac{\hat{x}}{F(\alpha_2) - M \cdot \bar{F}(\alpha_1) + M}, \frac{-\hat{x}}{F(\alpha_2) - M \cdot \bar{F}(\alpha_1) + M} \right) = 0.186375
\]
(2.7) is also satisfied. By using (2.8) instead of (2.7) this lower bound of \( k \) can be lowered. We leave this as an exercise for the reader.

**Example 4.4.** Consider the system
\[
\frac{dx}{dt} = h(y) - kF(x), \quad \frac{dy}{dt} = -x, \tag{4.4}
\]
with \( k > 0, F(x) = 4x(x^2 - 1)/(4 + 3x^4), h(y) \) satisfies (i), and \( h(-y) = -h(y) \).

We will show that the conditions of Theorem 3.2 are fulfilled. It is easy to see that (\( \alpha \)) and (\( \beta \)) are satisfied with \( x_2 = 1 \); see Fig. 4.5. Let \( \alpha > 1 \) satisfy \( F'(\alpha) = 0 \), then \( \alpha \approx 1.98273 < 2 \). Therefore we cannot use a function of the form \( \varphi(x) = x + C, C \geq 1 \), as in Lemma 2.4 to prove that (\( \gamma \)) holds such that \( \varphi(1) < \alpha \). Instead we propose to use \( \varphi(x) = \sqrt{x^2 + 2} \) with \( x \in [0, 1] \). Obviously \( \varphi: [0, 1] \to [\sqrt{2}, \sqrt{3}] \subset [1, \infty) \) and \( \varphi(x)\varphi'(x) = x \).

Because for \( x \in [0, 1] F(\varphi(x)) \geq -F(x) \), see Fig. 4.6, (\( \gamma \)) is also satisfied. As \( \varphi(1) = \sqrt{3} < \alpha \), (3.1) holds and all conditions of Theorem 3.2 are satisfied, with \( d = \infty \).

**Remark 4.2.** The choice of the function \( \varphi(x) \) is motivated by Example 1 of Odani 6.

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