A Forelli–Rudin Construction and Asymptotics of Weighted Bergman Kernels

Miroslav Engliš

MÚ AV ČR, Zitná 25, 115 67 Prague 1, Czech Republic
E-mail: englis@math.cas.cz

Communicated by R. Melrose
Received June 21, 1998; revised August 4, 1999; accepted June 13, 2000

Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^N$ with smooth boundary, $-\phi$, $-\psi$ two smooth defining functions for $\Omega = \{ \phi > 0 \}$ such that $-\log \phi$, $-\log \psi$ are plurisubharmonic, $z \in \Omega$ a point at which $-\log \phi$ is strictly plurisubharmonic, and $M \geq 0$ an integer. We show that as $k \to \infty$, the Bergman kernels with respect to the weights $\phi^M \psi^M$ have the asymptotic expansion

$$ K_{\phi^M \psi^M}(z, z) = \frac{k^N}{\pi^N |\phi(z) \psi(z)|^2} \sum_{j=0}^{\infty} b_j(z) k^{-j}, \quad b_0 = \det \left( \frac{\partial^2 \log \phi}{\partial z_j \partial \bar{z}_k} \right). $$

For $\Omega$ strongly pseudoconvex with real-analytic boundary, $\phi$, $\psi$ real analytic and $-\log \phi$, $-\log \psi$ strictly plurisubharmonic on $\Omega$, we obtain also the analogous result for $K_{\phi^M \psi^M}(x, y)$ for $(x, y)$ near the diagonal and discuss consequences for the asymptotics of the Berezin transform and for the Berezin quantization. The proofs rely on Fefferman’s expansion for the Bergman kernel in a certain Forelli–Rudin type domain over $\Omega$; as another application, they also yield a generalization of the cited Fefferman’s expansion to a class of weighted Bergman kernels.

Let $\Omega$ be a domain in $\mathbb{C}^N$, $\rho$ a positive continuous function on $\Omega$, and $K_\rho$ the reproducing kernel of the weighted Bergman space of all holomorphic functions on $\Omega$ square-integrable with respect to the measure $\rho(z)\, dz$, $dz$ being the Euclidean volume element in $\mathbb{C}^N$; we call $K_\rho$ the weighted Bergman kernel corresponding to $\rho$, and for $\rho \equiv 1$ we will speak simply of the Bergman kernel $K_\Omega$ of $\Omega$. The Berezin transform $B_\rho$ is the integral operator defined by

$$ B_\rho f(y) = \int_\Omega f(x) \frac{|K_\rho(x, y)|^2}{K_\rho(y, y)} \rho(x) \, dx $$

The author’s research was supported by GA AV ČR grant A1019701.
for all $y$ for which $K_p(y, y) \neq 0$. In terms of the operator $M_f$ of multiplication by $f$ on the space $L^2(\Omega, \rho \, dz)$ this can be rewritten as

$$B_p f(y) = \frac{\langle M_f K_p(\cdot, y), K_p(\cdot, y) \rangle}{\|K_p(\cdot, y)\|^2}.$$  

from which it is immediate that the integral (1) converges, for instance, for any bounded measurable function $f$.

The Berezin transform was first introduced by F. A. Berezin [Ber] in the context of quantization of Kähler manifolds. More specifically, let $\phi$ be a positive function on $\Omega$ such that $\Phi = -\log \phi$ is strictly plurisubharmonic, and set $g_{\Phi} = \partial^2 \Phi / \partial z_j \partial \bar{z}_k$ and $\gamma = \det(g_{\Phi})$ (so that $ds^2 = g_{\Phi} \, dz_j \, d\bar{z}_k$ is the Kähler metric with potential $\Phi$ and $\gamma$ the corresponding volume density). For $\Omega$ a bounded symmetric domain in $\mathbb{C}^N$ and $\phi(z) = 1/K_\Omega(z, z)$ (so that $ds^2$ is the Bergman metric), Berezin showed that for all $m \geq 1$ it holds that

$$K_{\phi^m}(z, z) = p(m) \phi(z)^{-m},$$

or more precisely

$$K_{\phi^m}(x, y) = p(m) \phi(x, y)^{-m}$$

where $\phi(x, y)$ is a function on $\Omega \times \Omega$ holomorphic in $x, \bar{y}$ such that $\phi(x, x) = \phi(x)$, and $p$ is a polynomial of degree $N$ which depends only on $\Omega$; and that

$$B_{\phi^m} f(y) = f(y) + \frac{1}{m} \tilde{A} f(y) + O\left(\frac{1}{m^2}\right)$$

as $m \to \infty$, where $\tilde{A}$ is the Laplace-Beltrami operator of the metric $ds^2$ on $\Omega$. Using (4), he was then able to construct a nice quantization procedure for mechanical systems whose phase-space is $\Omega$ with the Bergman metric. Later the present author showed that to get (4) it suffices that (3) hold only asymptotically as $m \to \infty$ in a certain sense and used this to extend the range of applicability of Berezin’s original procedure to all plane domains with the Poincaré metric, and to some complete Reinhardt domains in $\mathbb{C}^2$ with natural rotation-invariant Kähler metrics [E1], [E2]. The whole approach can also be adapted to arbitrary Kähler manifolds $\Omega$ in the place of domains in $\mathbb{C}^N$ [Pe].

In the present paper we show that (2) holds asymptotically for all strongly pseudoconvex domains in $\mathbb{C}^N$ with $C^\infty$ boundary, and $\phi$ a power of a defining function of $\Omega$; if the boundary is even real analytic, we also
get the appropriate analogs of (3) and (4). We even deal, in fact, with the more general setting of weights of the form \( \rho = \phi^m \psi^M \) with \( -\phi, -\psi \) two defining functions of \( \Omega \) such that \( -\log \phi, -\log \psi \) are plurisubharmonic, \( z \) a point where \( -\log \phi \) is strictly plurisubharmonic, \( M \) fixed and \( m \to \infty \).

Our method is based on the analysis of the Bergman kernel \( \tilde{K} \) of a certain Forelli-Rudin type domain \( \tilde{\Omega} \) over \( \Omega \); (2)–(3) are then obtained from Fefferman’s asymptotic expansion of \( \tilde{K} \) near the boundary. As a byproduct of this analysis we also obtain a “weighted” version of Fefferman’s asymptotic formula, i.e. a description of the behavior of \( K_\rho(x, y) \) as \( x, y \) approach a strongly pseudoconvex point of \( \partial \Omega \), for \( \rho \) a power of a defining function for \( \Omega \) such that \( -\log \rho \) is plurisubharmonic; more precisely, (2)–(3) are related to the behavior of \( \tilde{K}(x, y) \) as \( x, y \to z \in \partial \tilde{\Omega} \setminus \partial \Omega \), while the weighted Fefferman’s formula is similarly related to \( x, y \to z \in \partial \tilde{\Omega} \). This explains, by the way, why both asymptotics have somewhat similar form (the answer in both cases is

\[
K_\rho(z, z) \sim \frac{C}{\rho(z)} \det \left[ \partial \bar{\partial} \log \frac{1}{\rho(z)} \right],
\]

where either \( \rho \) is kept fixed and \( z \to \partial \Omega \), or \( \rho = \phi^m \psi^M \), \( z \) and \( M \) are fixed, and \( m \to \infty \).

The construction of the domain \( \tilde{\Omega} \) and the derivation of the weighted version of Fefferman’s formula (Theorem 4) can be found in Section 1. The asymptotic analog of (2) is established in Section 2 (Theorem 8), and those of (3) and (4) in Section 3 (Theorems 11 and 12). Unless explicitly stated otherwise, we will assume that \( \Omega \) is bounded (though most results probably extend to unbounded domains as well). Throughout the paper, “psh” is an abbreviation for “plurisubharmonic”.

1. A FORELLI–RUDIN CONSTRUCTION

Let \( \Omega \) be an arbitrary domain in \( \mathbb{C}^N \) (it need not be bounded) and \( \phi, \psi \) two positive functions on \( \Omega \). For \( d_2, d_3 = 0, 1, 2, \ldots \), define the domain \( \tilde{\Omega} = \tilde{\Omega}^{d_2, d_3} \phi, \psi \) by

\[
\tilde{\Omega} = \left\{ (z_1, z_2, z_3) \in \Omega \times \mathbb{C}^{d_2} \times \mathbb{C}^{d_3} : \frac{|z_1|^2}{\psi(z_1)} + \frac{|z_3|^2}{\phi(z_1)} < 1 \right\}. \tag{5}
\]

By the familiar criterion for Hartogs domains, \( \tilde{\Omega} \) is pseudoconvex if and only if \( \Omega \) is pseudoconvex and \( -\log \phi, -\log \psi \) are psh.

Our starting point is the following proposition:
Proposition 0. The Bergman kernel $\tilde{K}$ of $\tilde{\Omega}^{d_2, d_3}$ is given by

$$\tilde{K}(z; t) = \sum_{k, l = 0}^{\infty} \frac{(k + l + d_2 + d_3)!}{k! l! n^{d_2 + d_3}} K_{\psi^k \psi^l \phi^{d_2 + d_3}}(z_1, t_1) \langle z_2, t_2 \rangle^k \langle z_3, t_3 \rangle^l.$$  

The series converges uniformly on compact subsets of $\tilde{\Omega}$.

Proof. Arguing as in Ligocka [Lig, Proposition 0] shows that

$$\tilde{K}(z; t) = \sum_{\alpha, \beta} K_{\alpha \beta}(z_1, t_1) z_2^{\alpha_2} z_3^{\alpha_3},$$

where the summation is over all multiindices $\alpha \in \mathbb{N}^{d_2}, \beta \in \mathbb{N}^{d_3}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and

$$g_{\alpha \beta}(z_1) = \int_{|x|^2 \psi(z_1) + |x|^2 \psi(z_1) < 1} |x_2^\beta| |x_3^\alpha| d^2 x_2 d^2 x_3$$

$$= \frac{n^{d_2 + d_3} \beta!}{(|\alpha| + |\beta| + d_2 + d_3)!} \phi(z_1)^{|\alpha| + d_3} \psi(z_1)^{|\beta| + d_2}.$$

Since $\sum_{|\alpha| + |\beta|} x^\beta y^\alpha = \langle x, y \rangle^k / k!$, the required assertion follows. $lacksquare$

The construction similar to (5) was first used by Forelli and Rudin [For], [FR], [Rud]; for other applications, see [Lig], [KLR] and the references therein.

Let us recall the asymptotic formula for the Bergman kernel due to Fefferman [Fef] and Boutet de Monvel-Sjöstrand [BS]. Let $\Omega$ be a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and $-\phi$ a $C^\infty$ defining function for $\Omega$, i.e., $\Omega = \{z : \phi(z) > 0\}$, $\phi$ is $C^\infty$ in a neighborhood of $\partial \Omega$, $\nabla \phi \neq 0$ on $\partial \Omega$, and the Levi matrix $(-\partial^2 \phi / \partial z_\alpha \partial \bar{z}_\beta)$ is positive definite on the complex tangent space (the last condition is equivalent to the Monge–Ampère matrix in (8) below having $n$ positive and $1$ negative eigenvalue, for any $z \in \partial \Omega$). Then there exist functions $a(x, y), b(x, y), \phi(x, y) \in C^\infty(C^n \times C^n)$ such that

(a) $a(x, y), b(x, y), \phi(x, y)$ are almost-analytic in $x, y$ in the sense that $\partial \phi(x, y) / \partial \bar{x}$ and $\partial \phi(x, y) / \partial y$ have a zero of infinite order at $x = y$, and similarly for $a(x, y)$ and $b(x, y)$;

(b) $\phi(x, x) = \phi(x)$;

(c) for $x \in \partial \Omega$,

$$a(x, x) = \frac{n!}{\pi} J[\phi](x) > 0,$$  

where $J[\phi]$ is the Jacobian of $\phi$. 

(7)
where $J[\phi]$ is the Monge–Ampère determinant

$$J[\phi] = -\det \begin{pmatrix} -\phi_{\bar{z}_j} & -\phi_{\bar{z}_k} \\ -\phi_{z_j} & -\phi_{z_k} \end{pmatrix}$$

whose positivity follows from the strong pseudoconvexity of $\partial \Omega$;

(d) the Bergman kernel of $\Omega$ is given by the formula

$$K(x, y) = \frac{a(x, y)}{\phi(x, y)} + b(x, y) \log \phi(x, y)$$

for $(x, y) \in \Omega_\varepsilon = \{ |x - y| < \varepsilon, \text{dist}(x, \partial \Omega) < \varepsilon \}$, where $\varepsilon > 0$ is sufficiently small;

(e) outside any $\Omega_\varepsilon$ the Bergman kernel is $C^\infty$ up to the boundary of $\Omega$;

(f) if the boundary $\partial \Omega$ is even real-analytic, then the functions $a(x, y), b(x, y)$ and $\phi(x, y)$ can in fact be chosen to be holomorphic in $x, y$ in a neighborhood of the boundary diagonal $\{(x, x); x \in \partial \Omega \}$ in $\mathbb{C}^n$, and outside any $\Omega_\varepsilon$ the Bergman kernel is holomorphic in $x, y$ in a neighborhood of $\Omega \times \Omega$.

The original proofs in [Fef] and [BS] deal only with (a)(e); part (f) is due to Kashiwara [Ka] and Bell [Bel].

Observe that if $\phi'(x, y)$ is another function satisfying (a) and (b), then $h = (\phi'/\phi) - 1$ vanishes at $x = y$ to an infinite order; thus (9) remains in force with $a' = (1 + h)^{n+1}a + \phi'^{n+1}b \log(1 + h)$ in the place of $a$ and $b$. It follows that even for any function $\phi(x, y)$ satisfying (a) and (b) there exist $a(x, y), b(x, y)$ such that the conclusions (a)–(d) hold. This allows us to work with a convenient $\phi(x, y)$ in concrete situations later on: for instance, if $\phi(x, y)$ is of the form $|x_1|^2 + (a \text{ function of } x_2, \ldots, x_n)$, we can take $\phi(x, y) = x_1 \bar{y}_1 + (a \text{ function of } x_2, \ldots, x_n, y_2, \ldots, y_n)$.

We will find convenient the following “local” version of Fefferman’s expansion, which can be obtained from the original result by using the (probably well-known) Lemmas 2 and 3 below.

**Proposition 1.** Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ boundary, $-\phi$ a defining function for $\Omega$, and $x_0 \in \partial \Omega$ a strongly pseudoconvex boundary point. Then there exist a neighborhood $U$ of $x_0$ in $\mathbb{C}^n$ and functions $a(x, y), b(x, y), \phi(x, y) \in C^\infty(U \times U)$ such that

(a) $\phi(x, y)$ is almost-analytic in $x, y$ in the sense that $\bar{\partial} \phi(x, y)/\bar{\partial} x$ and $\bar{\partial} \phi(x, y)/\bar{\partial} y$ have a zero of infinite order at $x = y$, and similarly for $a(x, y)$ and $b(x, y)$;

(b) $\phi(x, x) = \phi(x)$;
Lemma 2. Let $\Omega_1 \subset \Omega$ be two pseudoconvex domains with $C^\infty$ boundaries and $U$ a neighborhood of a point $x_0 \in \partial \Omega$ such that $U \cap \partial \Omega_1 = U \cap \partial \Omega$ and the piece of common boundary $U \cap \partial \Omega$ is strongly pseudoconvex. Then the difference $K_\Omega(x, y) - K_{\Omega_1}(x, y)$ is $C^\infty$ on $(U \cap \Omega_1) \times \Omega_1$.

Lemma 3. Let $\Omega$ be a pseudoconvex domain (possibly unbounded) and $x_0 \in \partial \Omega$ a strongly pseudoconvex point of its boundary. Then there exists a strongly pseudoconvex domain $\Omega_1 \subset \Omega$ such that $\partial \Omega$ and $\partial \Omega_1$ coincide in a neighborhood of $x_0$.

Proof of Lemma 2. For $\Omega$ strongly pseudoconvex, this is the content of Lemma 1 on p. 6 in [Fef]. The local version given here follows in the same way by J. J. Kohn's local regularity theorems for the $\partial$-operator and subelliptic estimates at $x_0$ [Ko, Theorems 1.13 and 1.16].

Proof of Lemma 3. Let $u$ be a defining function for $\Omega = \{ u < 0 \}$ strictly-psh in a neighborhood $B(x_0, \delta)$ of $x_0$ (see e.g. [Krn], Proposition 3.2.1). Choose a $C^\infty$ function $\theta: [0, 1] \to \mathbb{R}^+$ such that $\theta \equiv 0$ on $[0, 1/2]$, $\theta'' \geq 0$ on $[1/2, 1]$ and $\theta(1 - \cdot) = +\infty$. Set $\Omega_1 = \{ x : u(x) + \theta(|x-x_0|^2/\delta^2) < 0 \}$. Then $\Omega_1 \cap \Omega = B(x_0, \delta)$, $\partial \Omega_1$ coincides with $\partial \Omega$ in $B(x_0, \delta/2)$, and as $\theta'' \geq 0$, $\theta(|x-x_0|^2/\delta^2)$ is psh, so $\Omega_1$ is strongly pseudoconvex.

Remark. The conclusion of Lemma 2 fails if $U \cap \partial \Omega = U \cap \partial \Omega_1$ is only assumed to be weakly pseudoconvex: for instance, take $\Omega = \{ \max(|z_1|, |z_2|) < 1 \} \subset \mathbb{C}^2$, $\Omega_1 = \{ \max(|z_1|, 2|z_2|) < 1 \}$, and $x_0 = (1, 0)$. Similarly, the hypothesis that $\Omega$ be pseudoconvex cannot be dispensed with: an example is $\Omega_1 = \{ z \in \mathbb{C}^2 : |z| < 2 \}$, $\Omega = \Omega_1 \cup \{ |z_1| < 3, 1 < |z_2| < 3 \}$, $x_0 = (2, 0)$. The author does not know if $\Omega$ can be allowed to be pseudoconvex but unbounded. On the other hand, the hypotheses that $\Omega_1$ be smoothly bounded and pseudoconvex are not needed in the proof and can be omitted (but we won’t have any use for this refinement in the sequel).

We can now apply Proposition 0 to obtain a weighted analog of Fefferman’s expansion.

Theorem 6. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^N$ with smooth boundary, $-\phi$ and $-\psi$ two $C^\infty$ defining functions of $\Omega = \{ \psi > 0 \}$ such that $-\log \phi$ and $-\log \psi$ are psh, and $x_0 \in \partial \Omega$ a strongly pseudoconvex boundary point. Then for any $d_2, d_3 \in \mathbb{N}$ there exist a neighborhood $U$ of $x_0$ in $\mathbb{C}^N$ and functions $a(x, y), b(x, y), \phi(x, y), \psi(x, y) \in C^\infty(U \times \Omega)$ such that
(a) $\phi(x, y)$ is almost-analytic in $x, y$ in the sense that $\partial \phi(x, y)/\partial x$ and $\partial \phi(x, y)/\partial y$ have a zero of infinite order at $x = y$, and similarly for $a(x, y), b(x, y)$ and $\psi(x, y)$;

(b) $\phi(x, x) = \phi(x), \psi(x, x) = \psi(x)$;

(c) for $x \in \partial \Omega$,

$$a(x, x) = (N + d_2 + d_3)! \left( \frac{\phi}{\psi} (x) \right)^{d_2} J[\phi](x); \quad (10)$$

(d) the weighted Bergman kernel with respect to $\phi^d \psi^d$ is given by the formula

$$K_{\phi^d \psi^d}(x, y) = \frac{a(x, y)}{\phi(x, y)^{N + d_2 + d_3 + 1}} + b(x, y) \log \phi(x, y) \quad (11)$$

on $(U \cap \Omega) \times (U \cap \Omega)$;

(e) if $\Omega$ is strongly pseudoconvex, then $K_{\phi^d \psi^d}$ is in addition a $C^\infty$ function up to the boundary away from the boundary diagonal $\{(x, x) : x \in \partial \Omega\}$;

(f) finally, if $\Omega$ is strongly pseudoconvex with real-analytic boundary and the functions $\phi, \psi$ are real analytic and $-\log \phi, -\log \psi$ strictly psh, then the functions $a(x, y), b(x, y), \phi(x, y), \psi(x, y)$ can be chosen to be holomorphic in $x, y$ on $U \times U$.

Proof. The hypotheses assure that $\tilde{\Omega} = \Omega^{d_2 + d_3}$ is a pseudoconvex domain with $C^\infty$ boundary and that $u(z) = |z_1|^2 + g(z_1) |z_2|^2 - \phi(z_1), \quad g = \phi/\psi \in C^\infty(\tilde{\Omega})$, is a $C^\infty$ defining function for $\tilde{\Omega}$.

Evaluating the derivatives

$$\frac{\partial u}{\partial z_{ij}} = -\partial_{ij} \phi + |z_2|^2 \partial_{ij} g, \quad \frac{\partial u}{\partial z_{ij}} = \bar{z}_{ij}, \quad \frac{\partial u}{\partial z_{ij}} = \bar{z}_{ij},$$

we see that the complex tangent space at the point $z_0 = (x_0, 0, 0)$ consists of all vectors $(X_1, X_2, X_3) \in C^{N + d_2 + d_3}$ with $X_1$ lying in the tangent space at $x_0$ of $\partial \Omega$ and $X_2, X_3$ arbitrary, and similarly the Levi form at $z_0$ is given by

$$L_u(X_1, X_2, X_3) = |X_1|^2 + g(x_0) |X_2|^2 + L_u^G(X_1)$$

which is positive definite, since $L_u^G$ is and $g > 0$. Thus $z_0$ is a strongly pseudoconvex point of $\partial \Omega$ and we may apply Proposition 1 to the Bergman kernel $K$ of $\Omega$. On the other hand, by Proposition 0

$$\tilde{K}(z_1, 0, 0; t_1, 0, 0) = \frac{(d_2 + d_3)!}{\pi^{N - d_2}} K_{\phi^d \psi^d}(z_1, t_1). \quad (12)$$
Substituting from (9) we thus obtain (11), and similarly (7) and a short computation of the Monge–Ampère determinant

\[
J[-u](x_0, 0, 0) = -\det \begin{bmatrix}
-\phi & -\partial_{x_k}\phi & 0 & 0 \\
-\partial_{x_i}\phi & -\partial_{x_i}\partial_{x_k}\phi & 0 & 0 \\
0 & 0 & g\delta_{2i, 2k} & 0 \\
0 & 0 & 0 & \delta_{3j, 3k}
\end{bmatrix}
\]

(13)

\[= g(x_0)^{\partial J[\phi]}(x_0) \]

yield (10). This settles the claims (a)–(d).

If all of \(\partial \Omega\) is strongly pseudoconvex, then by the above all points of \(\partial \Omega \cap \{z_2 = z_3 = 0\}\) are strongly pseudoconvex, hence so are all points of \(\partial \Omega \cap \{|z_2|^2 + |z_3|^2 < \delta\}\) for a small \(\delta > 0\). Setting \(\Omega_1 = \{z : u(z) + \theta(|z_2|^2 + |z_3|^2)/\delta < 0\}\), where \(\theta\) is the function from the proof of Lemma 3, we obtain a smoothly bounded strongly pseudoconvex domain \(\Omega_1 \subset \Omega\) such that \(\Omega\) and \(\Omega_1\) have the same intersection with \(V = \{|z_2|^2 + |z_3|^2 < \delta/2\}\). By Lemma 2, \(K - K_{\partial \Omega}\) is \(C^\infty\) on \((V \times V) \cap (\Omega_1 \times \Omega_1)\), while by part (e) of Fefferman’s theorem \(K_{\partial \Omega}\) is \(C^\infty\) on \(\Omega_1 \times \Omega_1\) away from the boundary diagonal. Thus, in particular, \(K\) is \(C^\infty\) on \((\Omega_1 \times \Omega_1) \cap \{z_2 = z_3 = 0\}\) away from the boundary diagonal; which, in view of (12), proves (e).

Finally, if \(\Omega\) is strongly pseudoconvex and \(-\log \phi, -\log \psi\) are strictly psh, then \(\Omega\) is strongly pseudoconvex. Thus under the hypotheses of part (f), \(\Omega\) is a strongly pseudoconvex domain with real-analytic boundary, and so the required assertion follows from (12) and the part (f) of Fefferman’s theorem.

Observe that whenever \(\phi \neq 0\) we have by an elementary matrix manipulation

\[
J[\phi] = -\det \begin{bmatrix}
-\phi & 0 \\
0 & -\partial^2\phi/\partial z_j \partial z_k + \partial \phi/\partial z_j \partial z_k
\end{bmatrix} = \phi^{N+1} \det \begin{bmatrix}
\partial^2\phi/\partial z_j \partial z_k
\end{bmatrix} \log \frac{1}{\phi}.
\]

(14)

The leading term (10) in (11) can thus be rewritten (in a slightly more symmetric way) as

\[
K_{\phi, \psi}(x, x) \sim \frac{(N + d_2 + d_3)!}{R^N (d_2 + d_3)!} \frac{\det (\partial \phi \log 1/\phi)}{\phi^{d_2} \psi^{d_3}} \text{ as } x \to x_0.
\]

(15)
Corollary 5. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^N$ with smooth boundary and $-\phi$ a defining function for $\Omega = \{ \phi > 0 \}$ such that $-\log \phi$ is strictly psh. Then for any strongly pseudoconvex point $x_0$ of $\partial \Omega$ and any positive function $g$ in $C^\infty(\Omega)$,

(a) the function $-\log \phi - (1/d) \log g$ is strictly psh if $d$ is sufficiently large;

(b) or any such $d$,

$$\phi^d(x) g(x) \cdot K_{g^d}(x, x) \sim \frac{(N+d)!}{d! \pi^N} \det \left( \partial \bar{\partial} \log \frac{1}{\phi} \right)$$

as $x \to x_0$.

Proof. The second part follows from the first upon taking $d_1 = 0$, $d_2 = d$, $\psi = \phi^{1/d}$ in Theorem 4. To prove the first part, observe that

$$
\begin{bmatrix}
-\phi & -\phi_k \\
-\phi_j & -\phi_k - (\phi/d)(\log g)_{j,k}
\end{bmatrix}
= \phi \begin{bmatrix}
1 & 0 \\
\phi_j/\phi & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & (-\log \phi - (1/d) \log g)_{j,k}
\end{bmatrix}
\begin{bmatrix}
1 & \phi_k/\phi \\
0 & 1
\end{bmatrix},
$$

where for brevity we have used the subscripts $j, k$ to denote differentiations by $z_j, z_k$. Thus $-\log \phi - (1/d) \log g$ is strictly psh at $z \in \Omega$ if and only if the square matrix

$$
\begin{bmatrix}
-\phi & -\phi_k \\
-\phi_j & -\phi_k - (\log g)_{j,k}
\end{bmatrix} - \frac{1}{d} \begin{bmatrix}
0 & 0 \\
0 & (\log g)_{j,k} \phi
\end{bmatrix}
= A(z) = -\frac{1}{d} B(z)
$$

has 1 negative and $N$ positive eigenvalues. However, in view of the hypotheses on $\phi$ and $g$, both matrix-valued functions $A(z)$ and $B(z)$ are $C^\infty$ on $\Omega$ and $A(z)$ has the required signature $(N, 0, 1)$ for each $z \in \Omega$; thus $\sup_{z \in \Omega} \| B(z) \| = b < \infty$ and $\mathcal{M} = \{ A(z), z \in \Omega \}$ is a compact subset of the open set $\mathcal{M}$ of all Hermitian matrices of signature $(N, 0, 1)$. Taking $d$ so large that $b/d < \text{dist}(\mathcal{M}, \mathcal{N}, \mathcal{M})$, the assertion follows. \[\square\]

Corollary 6. Let $\Omega$ and $\phi$ be as in Corollary 5 and in addition let $\Omega$ be strongly pseudoconvex. Then for any point $x_0 \in \partial \Omega$ and $d$ sufficiently large,

$$\phi(x)^d \cdot K_{g^d}(x, x) \to \frac{(d-1)!}{(d-N-1)! \pi^N}$$

as $x \to x_0$. 

A FORELLI-RUDIN CONSTRUCTION
Proof. By (14) $J(\phi) > 0$ on $\Omega$, so we may apply the last corollary with $g = J(\phi)$.

Remarks. (1) For $d = 0$, a more general variant of (16) was obtained by Hörmander ([Hö], Theorem 3.5.1): if $\Omega \subset \mathbb{C}^N$ is bounded and pseudoconvex, $x_0 \in \partial \Omega$ is a strongly pseudoconvex boundary point and $g \in C(\Omega)$, $g > 0$, then

$$g(x) K_g(x, x) \sim \frac{N!}{\pi} \det \left( \partial \overline{\partial} \log \frac{1}{\phi} \right)$$

(17)
as $x \to x_0$, where $-\phi$ is any defining function for $\Omega$.

(2) If $g$ is a positive continuous function on $\Omega$ and $\tilde{K}_g$ stands for the Bergman kernel of $\mathcal{O}^{d_2, d_3}$ with respect to the weight $g(z_1)(z = (z_1, z_2, z_3)) \in \mathcal{O}^{d_2, d_3}$, then the following generalization of the formula (6) holds:

$$\tilde{K}_g(z; t) = \sum_{k, \ell = 0}^\infty \frac{(k + l + d_2 + d_3)!}{k! l! d_2^k d_3^\ell} K_g^{k, l, \ell, \alpha, \beta, \gamma}(z_1, t_1) \langle z_2, t_2 \rangle^\ell \langle z_3, t_3 \rangle^k.$$

(3) Using the last two remarks and a modification of the argument in the proof of Theorem 4, it is possible to give an alternative proof of the growth estimates in Corollaries 5(b) and 6 (however, the more refined formula (11) cannot be obtained in this way). In particular, Corollary 5(b) holds, in fact, even for all $d \in \mathbb{N}$, and Corollary 6 for all integers $d \geq N + 1$; and in Corollary 5(b) it suffices that $-\log \phi$ be psh (i.e. not necessarily strictly psh).

(4) The weights $\phi^d \det(-\partial \overline{\partial} \log \phi)$ in Corollary 6 turn up in the applications to the Berezin quantization: in that situation $-\log \phi$ is automatically strictly psh since it is a potential for the Kähler metric $g_{\phi} = \partial \overline{\partial}(-\log \phi) / \partial z_j \overline{\partial} z_k$ on $\Omega$, and $\det(g_{\phi}) = \det(-\partial \overline{\partial} \log \phi)$ is the corresponding Riemannian volume element.

(5) Setting $\rho = \phi^d g$, the formula (16) can be written either as

$$\pi^N \rho(x) K_\rho(x, x) \sim C_\rho \det \left( \partial \overline{\partial} \log \frac{1}{\rho} \right),$$

(18)
or as

$$\pi^N \rho(x) K_\rho(x, x) \sim D_\rho \det \left( \partial \overline{\partial} \log \frac{1}{\rho} \right),$$

(19)
which are thus valid, with $C_\rho = (N + d)! / d!$ and $D_\rho = C_\rho / d$, on a strongly pseudoconvex domain $\Omega$ for $\rho$ such that $-\log \rho$ is psh and $\rho$ has a zero
of order precisely \(d\) on \(\partial \Omega\) \((d=1, 2, 3, \ldots)\). The first of these formulas remains valid also for \(d=0\) (cf. (17)). The second, on the other hand, remains valid when \(N=1\) also for some functions \(\rho\) with (formally) \(d=\infty\): for instance, if \(\Omega\) is the unit disc \(D\) and \(\rho(z) = \exp(-c/(1-|z|^2))\), \(c>0\), then it is not difficult to compute that (19) is fulfilled with \(D=1\). Some related results for functions \(\rho = \rho(|z|)\) on \(D\) that decay very fast as \(|z| \to 1\) were established by T. R. Kriete \([\text{Kri}]\). In the context of \(\Omega\) as the complex plane \(\mathbb{C}\) and functions \(\rho\) depending only on the modulus \(|z|\) and decaying fast at \(\infty\), a very thorough analysis of the asymptotics of \(K_\rho\) has recently been done by F. Holland and R. Rochberg \([\text{HR}]\).

## 2. WEIGHTED BERGMAN KERNELS

We now use Fefferman’s asymptotic expansion together with Proposition 0 to determine the asymptotics of \(K_{\rho \phi_m}(z, z)\) as \(z\) is fixed, \(M\) is fixed and \(m \to \infty\).

Denote

\[ u_k(\tau) = \begin{cases} (1-\tau)^k & (k < 0) \\ (1-\tau)^k \log \frac{1}{1-\tau} & (k \geq 0). \end{cases} \]

**Lemma 7.** Let \(f\) be a function holomorphic in the unit disc \(D\) such that

\[ f(\tau) = \sum_{j=-n-1}^{m} a_j u_j(\tau) + g(\tau), \]

where \(g \in C^m(D)\). Then the Taylor coefficients of \(f\) satisfy

\[ f_k = \sum_{j=0}^{n} \binom{j+k}{k} a_{j-1} + \sum_{j=0}^{m} \binom{-1}{j} \binom{k}{j+1}^{-1} a_j + O(k^{-m}) \quad \text{as} \quad k \to \infty. \]

**Proof.** This is immediate from the formulas

\[
(1-\tau)^{-j-1} = \sum_{k=0}^{\infty} \binom{k+j}{k} \tau^k \quad (j \geq 0),
\]

\[
(1-\tau)^j \log \frac{1}{1-\tau} = \tau^{-1} \log \tau - \tau^{-1} - \tau^{-j}
\]

\[= + \sum_{j=k+1}^{\infty} \frac{(-1)^j j!}{k(k-1)\cdots(k-j)} \tau^k \quad (j \geq 0), \quad (20)\]
and the Cauchy estimates
\[ |(k + 1) \cdots (k + m) g_{k + m}| = \frac{1}{2\pi} \int_0^{2\pi} g^{(m)}(e^{it}) e^{-it\phi} dt \leq \|g^{(m)}\|_{\infty}. \]

**Theorem 8.** Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^N \) with smooth boundary, \( -\phi, -\psi \) two \( C^\infty \) defining functions for \( \Omega = \{ \phi > 0 \} \) such that \(-\log \phi, -\log \psi \) are psh, and let \( x_0 \in \Omega \) be a point at which \(-\log \phi \) is strictly psh. Fix an integer \( M \geq 0 \). Then as \( k \to \infty \), there is an asymptotic expansion
\[ K_{\phi, \psi, k}(x_0, x_0) = \frac{k^N}{\pi^N \phi(x_0)^k} \psi(x_0)^M \left[ \sum_{j=0}^{\infty} b_j(x_0)/k^j \right] \]
where the coefficients \( b_j(x_0) \) depend only on the jets of \( \phi, \psi \) at \( x_0 \). In particular,
\[ b_0 = \det \left[ \partial^d \log \frac{1}{\phi} \right]. \]

**Proof.** Let \( d \) be any positive integer and consider the domain \( \hat{\Omega} = \Omega^{M,d} \). By Proposition 0,
\[ K(x_0, 0, z_3; x_0, 0, t_3) = \sum_{k=0}^{\infty} \frac{(k + d + M)!}{\pi^{M+N} k!} K_{\phi_k, \psi_k}(x_0, x_0) \left( z_3, t_3 \right)^k. \]

As in the proof of Theorem 4, \( \phi/\psi = g \) extends to a positive \( C^\infty \) function on \( \Omega \), and \( u(z) = |z_1|^2 + g(z_1) |z_2|^2 - \phi(z_1) \) is a \( C^\infty \) defining function for \( \hat{\Omega} \); the hypotheses assure that \( \hat{\Omega} \) is pseudoconvex and that all \( z \in \partial \hat{\Omega} \) with \( z_1 = x_0 \) and \( z_2 = 0 \) are its strongly pseudoconvex boundary points. By continuity, there is \( \delta > 0 \) such that all points of \( \partial \hat{\Omega} \) with \( |z_1 - x_0|^2 + |z_2|^2 < \delta \) are strongly pseudoconvex. We may assume that \( \delta < \text{dist}(x_0, \partial \Omega) \). Choose a \( C^\infty \) function \( \theta \) on \( [0, 1) \) such that \( \theta^{\prime} \geq 0 \), \( \theta \equiv 0 \) on \( [0, 2/3] \) and \( \theta(t) = -\log(1 - t) \) on \( (3/4, 1) \), and define \( u'(z) = |z_1|^2 + g(z_1) |z_2|^2 - \exp[-\theta((|z_1 - x_0|^2 + |z_2|^2)/\delta)] \phi(z_1) \) and \( \hat{\Omega}' = \{ u' < 0 \} \). A similar argument as in the proof of Lemma 3 shows that \( \hat{\Omega}' \) is a strongly pseudoconvex domain \( \subset \hat{\Omega} \) and that \( \hat{\Omega} \) and \( \hat{\Omega}' \) coincide in a neighborhood of \( \Pi = \{ z : |z_1 - x_0|^2 + |z_2|^2 < \delta/2 \} \). Thus the conclusions (a)–(c) of Fefferman’s theorem are applicable to \( \hat{\Omega}' \), and also by Lemma 2 \( K_{\hat{\Omega}' - \hat{\Omega}} = K \) is \( C^\infty \) on \( \Pi \cap \hat{\Omega} \times \Pi \cap \hat{\Omega} \). It follows that \( \hat{K} \) is \( C^\infty \) on \( \Pi \cap \hat{\Omega} \times \Pi \cap \hat{\Omega} \) minus the boundary diagonal \( S = \{ (z, z) : z \in \Pi \cap \hat{\Omega} \} \), while near \( S \) it is of the form
\[ \hat{K}(z, t) = \frac{a(z, t)}{[-u(z, t)]^{N+2d-M+1} + b(z, t) \log[-u(z, t)]}. \]
with some almost-analytic \( C^\infty \) functions \( a, b \) and \( u \) satisfying \( u(z, z) = u(z) \) and
\[
a(z, z) = \frac{(N + d + M)!}{\pi^{N + d + M}} J[ -u ](z), \quad z \in H \cap \partial \mathcal{D}.
\]
In particular, this applies to points of the form \((z, t) = (P_x, P_t)\) with \( P_x = (x_0, 0, \tau \sqrt{\phi(x_0)} e_d), \) where \( \tau \in D \) and \( e_d \) is the vector \((0, 0, \ldots, 0, 1) \in \mathbb{C}^d. \) Observe that, in view of the remark preceding Proposition 1, we may assume that
\[
u(x_0, z_2, z_3; x_0, t_2, t_3) = \langle z_3, t_3 \rangle + g(x_0) \langle z_2, t_2 \rangle - \phi(x_0).
\]
Consequently, the function of one complex variable
\[
f(\tau) = \tilde{K}(P_x, P_t), \quad \tau \in D,
\]
is \( C^\infty \) on \( D \backslash \{1\}, \) while near \( \tau = 1 \) it is of the form
\[
f(\tau) = \frac{G(\tau)}{(1 - \tau)^{N + d + M + 1}} + H(\tau) \log(1 - \tau),
\] (25)
where
\[
G(\tau) = \frac{a(P_x, P_t)}{\phi(x_0)^{N + d + M}} + (1 - \tau)^{N + d + M + 1} b(P_x, P_t) \log \phi(x_0)^{N + d + M + 1}
\]
and \( H(\tau) = b(P_x, P_t) \) are \( C^\infty \) functions on \( \mathbb{C} \) which are “almost-analytic” at \( \tau = 1 \) in the sense that \( \partial G \) and \( \partial H \) have a zero of infinite order at that point. The last condition implies that as \( \tau \to 1, \)
\[
\frac{G(\tau) - G(1)}{\tau - 1} \to \partial G(1).
\]
Proceeding inductively, it transpires that we have an asymptotic expansion
\[
G(\tau) = G(1) - \partial G(1) \cdot (1 - \tau) + \frac{1}{2!} \partial^2 G(1) \cdot (1 - \tau)^2 + \cdots \quad \text{as } \tau \to 1,
\]
and similarly for \( H. \) Thus Lemma 7 can be applied to \( f(\tau), \) with \( n = N + d + M, \) any integer \( \geq 0, \) and
\[
a_{l = N - d - M - 1} = \frac{(-1)^l}{l!} \partial^l G(1), \quad l = 0, \ldots, N + d + M,
\]
\[
a_l = \frac{(-1)^{l+1}}{l!} \partial^l H(1), \quad l \geq 0.
\]
As the Taylor coefficients of \( f \) are, in view of (23), given by
\[
f_k = \frac{(k + d + M)!}{\pi^{d+M} k!} K_{\phi, \psi u}(x_0, x_0) \phi(x_0)^k,
\]
we thus conclude that
\[
\frac{(k + d + M)!}{\pi^{d+M} k!} K_{\phi, \psi u}(x_0, x_0) \phi(x_0)^k
\]
\[
= \sum_{j=0}^{N+d+M} \binom{k+j}{j} \frac{(-1)^{N+d+M-j}}{(N+d+M-j)!} \partial^{N+d+M-j} G(1)
\]
\[
- \sum_{j=0}^{N+1} \frac{1}{(j+1)!} \binom{k}{j+1}^{-1} \partial^j H(1) + O(k^{-m})
\]
as \( k \to \infty \), for any \( m \geq 0 \). As \( (k+j)!/k! \sim (k+d)^j \) for each fixed \( j \), we get
\[
K_{\phi, \psi u}(x_0, x_0) = \frac{1}{\phi(x_0)^k} \left[ \sum_{l=-M-d-m+1}^{N} (k+d)^l B_l + O(k^{-d-M-m}) \right]
\]
with some numbers \( B_k \) depending only on the derivatives of \( H \) and the first \( N+d+M \) derivatives of \( G \) at \( \tau = 1 \), i.e. only on the jet of the function \( b \) and the \( (N+d+M) \)-jet of the function \( a \) in (24) at \( (P_1, P_1) \), which are known to depend only on the jet of the boundary \( \partial \Omega \) at the point \( P_1 \) (see e.g. [BFG], p. 312), i.e. only on the jets of \( \phi \) and \( \psi \) at \( x_0 \). In particular, the leading coefficient (at \( (k+d)^N \)) is
\[
\frac{B_0}{\phi(x_0)^k} = \frac{\pi^{d+M} G(1)}{\phi(x_0)^k (N+d+M)!} = \frac{\pi^{d+M} d(P_1, P_1)}{(N+d+M)! \phi(x_0)^{N+d+k+M+1}}
\]
\[
= J[-u](x_0, \sqrt{\phi(x_0)} e_d) \phi(x_0)^{N+k+d+M+1} / \pi^N.
\]
Standard matrix manipulations show that
\[
J[-u](z) = \frac{\phi^{N+d+M+1}}{\psi^M} \det \left[ \left( 1 - \frac{|z|^2}{\psi^2} \right) \cdot \partial \partial \log \frac{1}{\phi} + \frac{|z|^2}{\psi^2} \cdot \partial \partial \log \frac{1}{\psi} \right],
\]
so
\[
\frac{B_0}{\phi^k} = \frac{1}{\pi^{d+M}} \det \left[ \partial \partial \log \frac{1}{\phi} \right].
\]
Writing \( k \) in the place of \( k + d \) we thus get

\[
K_{\phi \psi}(x_0, x_0) = \frac{\pi^{-N}}{\phi(x_0)^M \psi(x_0)^M} \sum_{l=-d-M-m+1}^N k^l b_l(x_0) + O(k^{-d-M-m}),
\]

where \( b_l = \pi^N \phi^d \psi^M b_l \) depends only on the jets of \( \phi, \psi \) at \( x_0 \) and \( b_0 \) is given by (22). Since \( d \) and \( m \) can be arbitrary positive integers, the assertion of the theorem follows.

**Corollary 9.** Let \( \Omega \) be a strongly pseudoconvex domain in \( \mathbb{C}^N \) with smooth boundary and \(-\phi \) a \( C^\infty \) defining function for \( \Omega \) such that \(-\log \phi \) is strictly psh. Then

\[
\lim_{k \to \infty} (\pi/k)^N \phi(x)^k K_{\phi \psi \det[-\bar{\partial} \log \phi]}(x, x) = 1 \quad \forall x \in \Omega. \tag{26}
\]

**Proof.** As in Corollary 6, observe that \( 0 < \phi^{N+1} \det[-\bar{\partial} \log \phi] = J[\phi] \in C^\infty(\bar{\Omega}) \), pick \( M \in \mathbb{N} \) so large that \(-\log \phi - (1/M) \log J[\phi] \) is strictly psh (Corollary 5(a)), and apply the last theorem with \( \psi = J[\phi]^{1/M} \phi \).

**Remarks.** (1) The hypothesis that \(-\log \phi \) be strictly psh at \( x_0 \) cannot be dispensed with: take \( \Omega = D, M = 0, \phi(x) = 1 - |x|^{2m} \) where \( m \) is an integer \( > 1 \). Then \(-\log \phi \) is psh but not strictly psh at \( x_0 = 0 \). An explicit formula for \( \bar{\partial} = \bar{K}_{\partial} \); \( \partial = \{(x, z) \in C \times C^d : |z|^2 + |x|^{2m} < 1\} \) has been calculated by D’Angelo [dA],

\[
\bar{K}(x, z; x, z) = \sum_{j=0}^2 c_j \frac{(1 - |z|^2)^{-2 + j/m}}{(1 - |z|^2)^j/m - |x|^{2m} + |x|^2}.
\]

for some constants \( c_0, c_1, c_2 \). In particular, taking \( x = 0 \) and using Proposition 0 one can see that

\[
\phi^k(0) K_{\phi \psi}(0, 0) = \sum_{j=0}^2 \frac{c_j}{k+1} \left( k + 1 + \frac{1}{m} \right) c \cdot k^{1/m},
\]

so there are fractional powers of \( k \) entering into (21).

If \(-\log \phi \) is not psh, Theorem 8 fails even more drastically; see Remark 4 below.
Similarly, for \( \Omega, \phi \) as above and \( \chi = -\partial\partial \log \phi \) one computes that

\[
[\pi K_{\phi,x}(0,0)]^{-1} = \frac{1}{\pi} \int_{\Omega} \phi(x)^k \chi(x) \, dx = \frac{m}{k - 1},
\]

so

\[
\lim_{k \to \infty} \frac{\pi \phi(0)^k K_{\phi,x}(0,0)}{k} = \frac{1}{m}
\]

instead of 1 as in (26).

Finally, taking \( \Omega = \mathbb{C}, \phi(x) = 1 - \exp(-|x|^2) \) and \( \chi = -\partial\partial \log \phi \), a computation like (27) shows that as \( k \to \infty \),

\[
\frac{\pi \phi(0)^k K_{\phi,x}(0,0)}{k} \sim \frac{1}{\log k} \to 0.
\]

This suggests that perhaps the value of the limit (26) may reflect the type of the point \((x_0, \sqrt{\phi(x_0)}) \in \partial \Omega \subset \mathbb{C}^2\).

(3) For \( M = 0 \), (23) simplifies to

\[
\tilde{K}(z_1, z_3; t_1, t_3) = \sum_{k=0}^{\infty} \frac{(k+d)!}{k! \pi^d} K_{\phi,x}(z_1, t_1; z_3, t_3)^k.
\]

On the other hand, for \( \Omega \subset \mathbb{C} \) and \( d = 1 \), so that \( \tilde{\Omega} \) is a Reinhardt domain in \( \mathbb{C} \), the first coefficients of the asymptotic expansion (24) of \( \tilde{K} \) were identified explicitly by Nakazawa [Na1], [Na2]. Using his formulas we can thus obtain explicit expressions for the first coefficients of the asymptotic expansion (21), as we did for \( b_0 \). Since the derivation involves only routine calculations, we only state the final result without proof.

**Theorem 10.** Let \( \Omega \) be a smoothly bounded domain in \( \mathbb{C} \) and \(-\phi\) a \( C^\infty \)-defining function for \( \Omega \) such that \(-\log \phi \) is subharmonic on \( \Omega \) and strictly subharmonic at \( x \in \Omega \). Then as \( k \to \infty \),

\[
K_{\phi,x}(x, x) = \frac{kb_0(x)}{\pi \phi(x)^k} \left[ 1 + \frac{\beta_1(x)}{k} + \frac{\beta_2(x)}{k^2} + \frac{\beta_3(x)}{k^3} + O(k^{-4}) \right],
\]

where

\[
b_0 = \partial\partial \log \frac{1}{\phi}, \quad \beta_1 = \frac{1}{2b_0} \partial\partial \log b_0, \quad \beta_2 = \frac{1}{3b_0} \partial\partial \beta_1, \quad \beta_3 = \frac{1}{4b_0} \partial\partial \beta_2 - \frac{1}{12b_0} \partial\partial (\beta_1^2).
\]
An immediate consequence of Theorem 8 is that for \( \Omega \subset \mathbb{C}^N \) pseudoconvex with smooth boundary and \( -\phi, -\psi \) two \( C^\infty \) defining functions for \( \Omega \) such that \( -\log \phi \) and \( -\log \psi \) are psh,

\[
\lim_{k \to \infty} K_{\phi,\psi}^N(x, x)^{1/k} = 1/\phi(x)
\]

(28)

at any point \( x \in \Omega \) where \( -\log \phi \) is strictly psh. This is an improvement upon some results in [E3]; it is unknown to the present author whether (28) holds at points where \( -\log \phi \) is psh but not strictly psh. For an arbitrary positive lower-semicontinuous function \( \phi \) on \( \Omega \) (not necessarily a defining function or such that \( -\log \phi \) is psh), it follows from (23), the formula for the radius of convergence, and the fact that the domain of convergence of a power series is always a log-convex complete Reinhardt domain, that

\[
\limsup_{k \to \infty} K_{\phi,\psi}^N(x, x)^{1/k} = 1/\phi^*(x)
\]

where \( \log(1/\phi^*) \) is the greatest psh minorant of \( \log(1/\phi) \); in particular, (28) cannot hold if \( -\log \phi \) is not psh. See [E3] and [Bre] (also Section 2.4 in Sadullaev [Sad]) for related matters.

3. THE BEREZIN TRANSFORM

In this section we will deal with the strongly pseudoconvex and real analytic situation only. We begin by showing that in that case Theorem 8 can be appreciably sharpened.

**Theorem 11.** Let \( \Omega \) be a strongly pseudoconvex domain in \( \mathbb{C}^N \) with real-analytic boundary and \( -\phi, -\psi \) two \( C^\omega \) defining functions for \( \Omega \) such that \( -\log \phi, -\log \psi \) are strictly psh. Let \( \phi(x, y), \psi(x, y) \) denote the functions holomorphic in \( x, y \) in a neighborhood of the diagonal such that \( \phi(x, x) = \phi(x), \psi(x, x) = \psi(x) \) (the existence of \( \phi(x, y), \psi(x, y) \) is a consequence of real analyticity). Fix an integer \( M \geq 0 \). Then for \( (x, y) \) near the diagonal, there is an asymptotic expansion

\[
K_{\phi,\psi}^N(x, y) = \frac{k^N}{2^{N\phi(x, y)\psi(x, y)}M} \sum_{j=0}^{\infty} b_j(x, y) k^{-j} \quad \text{as} \quad k \to \infty
\]

(29)
uniformly on compact subsets, where

\[ h_\theta(x, y) = \det \left[ \frac{\partial^2}{\partial x_j \partial y_k} \log \frac{1}{\phi(x, y)} \right] \tag{30} \]

and the coefficients \( h_i \) are holomorphic in \( x, y \) and depend only on the jets of \( \phi, \psi \) at \( (x, y) \).

**Proof.** As in the proof of Theorem 8, consider the domain \( \Omega = \tilde{\Omega}^{M, \phi} \) where \( d \) is any positive integer. The hypotheses guarantee that \( \Omega \) is strongly pseudoconvex with real analytic boundary, \( g = \phi/\psi \) is a positive \( C^{\mu} \) function in a neighborhood of \( \Omega \), \( u(z) = |z_3|^2 + g(z_1) |z_2|^2 - \phi(z_1) \) is a \( C^{\mu} \) defining function for \( \Omega \), and by Proposition 0,

\[ \tilde{K}(x, 0, z; y, 0, t) = \sum_{k=0}^{\infty} \frac{(k + d + M)!}{k! \pi^{d+M}} K_{\phi, \psi}^{a, b, u}(x, y) \langle z_3, t_3 \rangle^k. \tag{31} \]

By Fefferman’s theorem, \( \tilde{K}(z, t) \) is a \( C^{\mu} \) function on \( \overline{\Omega} \times \overline{\Omega} \) minus the boundary diagonal \( \{(z, z) \in \Omega, \psi \in \phi \} \), while in a neighborhood \( \mathcal{U} \) of the boundary diagonal in \( C^{N+d+M} \times C^{N+d+M} \) it is of the form (9) on \( \mathcal{U} \cap (\Omega \times \Omega) \) with functions \( a(z, t), b(z, t), u(z, t) \) holomorphic in \( z, t \) on \( \mathcal{U} \). By the uniqueness theorem for holomorphic functions, we must in particular have

\[ u(z, t) = \langle z_3, t_3 \rangle + g(z_1, t_1) \langle z_2, t_2 \rangle - \phi(z_1, t_1) \]

on \( \mathcal{U} \), where \( g(x, y) = \phi(x, y) / \psi(x, y) \). Let us now specialize to \( z, t \in \tilde{\Omega} \) with \( z_3 = t_2 = 0 \), i.e. to points of the form \( (x, 0, z_3) \) with \( (x, z_3) \in D := \{ x \in \Omega, |z_3|^2 < \phi(x) \} \subset C^{N+d} \). It follows that the function

\[ g(x, z_3; y, t_3) = \tilde{K}(x, 0, z_3; y, 0, t_3) \]

is \( C^{\mu} \) on \( \overline{D \times D} \) minus the boundary diagonal \( A = \{(x, z_3; y, t_3) : x = y, z_3 = t_3, |z_3| = \sqrt{\phi(x)} \} \), while near \( A \) it is of the form

\[ g(x, z_3; y, t_3) = \frac{a'(x, z_3; y, t_3)}{[\phi(x, y) - \langle z_3, t_3 \rangle]^{N+d+M+1}}, \]

\[ + b'(x, z_3; y, t_3) \log [\phi(x, y) - \langle z_3, t_3 \rangle], \]

where \( a', b' \)—the restrictions of \( a, b \) to \( z_2 = t_2 = 0 \)—are holomorphic in \( x, y, z_3, t_3 \) in a neighborhood \( \mathcal{V} \) of \( A \). In view of (31), the left-hand side depends only on the scalar product \( \langle z_3, t_3 \rangle \), again by the uniqueness
theorem for holomorphic functions, the same has to be true for the functions $a'$ and $b'$: $a'(x, z_3; y, t_3) = a''(x, y, \langle z_3, t_3 \rangle)$ and similarly for $b'$. Switching to the variable $\tau = \langle z_3, t_3 \rangle / \phi(x, y)$ we thus see that

$$g(x, z_3; y, t_3) = F(x, y, \tau)$$

for a function $F$ holomorphic in $x, y, \tau$ on the domain $D_1 = \{(x, y, \tau) : x = y, \tau = 1\}$, while near $A_1$ it is of the form

$$F(x, y, \tau) = \frac{G(x, y, \tau)}{(1 - \tau)^{N+d+M+1}} + H(x, y, \tau) \log(1 - \tau), \quad (32)$$

where $H(x, y, \tau) = b''(x, y, \tau \phi(x, y))$ and

$$G(x, y, \tau) = \frac{a''(x, y, \tau \phi(x, y)) \phi(x, y)^{N+d+M+1}}{(1 - \tau)^{N+d+M+1}} + (1 - \tau)^{N+d+M+1} b''(x, y, \tau \phi(x, y)) \log\phi(x, y)^{N+d+M+1}$$

are holomorphic functions of $x, y, \tau$ in a neighborhood of $A_1$ in $C^{N+d+M+1}$. Thus for each such $x, y$ we have by Taylor's formula

$$G(x, y, \tau) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\partial^j G}{\partial \tau^j}(x, y, 1) (1 - \tau)^j \quad (33)$$

for $\tau$ near 1, and similarly for $H$. Diminishing $\mathcal{N}$ if necessary, we can achieve that $F$ be actually holomorphic in $x, y, \tau$ in a neighborhood of $\mathcal{N} \times D$ away from $A_1$. From (32) we therefore obtain, for any integer $m \geq 0$,

$$F(x, y, \tau) = \sum_{j=0}^{N+d+M} G_j(x, y) (1 - \tau)^{-j-1} + \sum_{j=0}^{m} H_j(x, y)(1 - \tau)^j \log(1 - \tau) + R_m(x, y, \tau),$$
where the function $R_m(x, y, \cdot)$ is holomorphic on $D$ and $C^m$ on $D$ for each fixed $x, y,$ and $G_j(x, y)$ and $H_j(x, y)$ are holomorphic in $x, \bar{y}$ in a neighborhood of the diagonal of $\Omega$. On the other hand, by (31)

$$F(x, y, \tau) = \sum_{k=0}^{\infty} \frac{(k+d+M)!}{k! \pi^{d+M}} K_{\phi \cdot t, \psi}(x, y) \phi(x, y)^k \cdot \tau^k,$$

and an application of Lemma 7 in the same way as before proves the theorem. The formula (30) follows by the same argument as in the proof of Theorem 8, or by using (22) and noting that $b_0(x, y)$ is holomorphic in $x$ and $\bar{y}$. Finally, the uniformity of (29) on compact subsets follows from the uniformity on compact subsets of the Taylor expansion (33).

Recall that for any weight function $\rho$ such that $K_\rho$ is not identically zero, the Berezin transform $B_\rho$ on $\Omega$ is defined by

$$B_\rho f(y) = \int_\Omega f(x) \frac{|K_\rho(x, y)|^2}{K_\rho(y, y)} \rho(x) \, dx$$

for all $y$ for which $K_\rho(y, y) \neq 0$. In particular, by Theorem 8, in the context of the following theorem $B_{\phi \cdot t, \psi} f(y)$ will always be well-defined as soon as $k$ is sufficiently large, for each $y \in \Omega$. Then we have the following result.

**Theorem 12.** Let $\Omega$ be a strongly pseudoconvex domain in $\mathbb{C}^N$ with real-analytic boundary and $-\phi, -\psi$ two $C^\infty$ defining functions for $\Omega$ such that $-\log \phi, -\log \psi$ are strictly psh. Fix an integer $M \geq 0$. Then for any $f \in L^\infty(\Omega)$ which is $C^\infty$ in a neighborhood of a point $y \in \Omega$, there is an asymptotic expansion

$$B_{\phi \cdot t, \psi} f(y) = \sum_{j=0}^{\infty} Q_j f(y) \cdot k^{-j},$$

where $Q_j$ are linear differential operators whose coefficients involve only the derivatives of $\phi, \psi$ at $y$ and $Q_0$ is the identity operator.

**Proof.** Let $U$ be a neighborhood of $y$ such that the asymptotic expansion (29) holds for $x$ in a neighborhood of $U$, and split the integration in (34) into integration over $U$ and over $\Omega \setminus U$. Let $\tilde{\Omega}$ be the domain from the proof of Theorem 11, $e_d$ the vector $(0, 0, \ldots, 0, 1) \in \mathbb{C}^d$, and consider the function

$$f(x, s) = \tilde{K}(x, 0, se_d, y, 0, \sqrt{\phi(y)} e_d).$$
By part (d) of Fefferman’s theorem, \( f \) is a \( C^\infty \) function on the set 
\[ W = \{(x, s) : x \in \Omega \setminus U, |s|^2 \leq \phi(x)\}. \]
Thus for any integer \( j \geq 0 \),
\[ \sup_{(x, s) \in W} \left| \frac{\partial f}{\partial s^j} \right| = c_j < +\infty. \]

On the other hand, in view of (31),
\[ f(x, s) = \sum_{k=0}^{\infty} \frac{(k + d + M)!}{k! \pi^{d+M}} K_{\varphi + \epsilon, \mu}(x, y) \phi(y)^{k+1/2} x^k. \] (36)
Applying Cauchy estimates to the function \( s \mapsto \partial f(x, s)/\partial s^j \), holomorphic in the disc \([|s| < \sqrt{\phi(x)}]\), we thus obtain
\[ |K_{\varphi + \epsilon, \mu}(x, y) \phi(y)^{k+1/2} x^k| \leq \frac{(k-j)! \pi^{d+M}}{(k+d+M)!} c_j, \]
for all \( x \in \Omega \setminus U, k \geq j \); that is,
\[ |K_{\varphi + \epsilon, \mu}(x, y) \phi(x)^k \phi(y)^k| \leq c_j k^{-2(d+M+j)} \quad \forall x \in \Omega \setminus U, \ k \geq d + j, \]
and, upon invoking (29) with \( x = y \),
\[ \frac{|K_{\varphi + \epsilon, \mu}(x, y)|^2}{K_{\varphi + \epsilon, \mu}(y, y)} \phi(x)^k \phi(y)^k \leq \frac{c_j^2 \phi(x)^M}{k^{N+2d+2d+2M}} \]
for all \( x \in \Omega \setminus U \) and \( k \geq j + d \). It follows that the integral over \( \Omega \setminus U \) is \( O(k^{-j}) \) for any \( j \).

It remains to deal with the integral over \( U \). In that case, by (29) and (21), we have an asymptotic expansion
\[ \frac{|K_{\varphi + \epsilon, \mu}(x, y)|^2}{K_{\varphi + \epsilon, \mu}(y, y)} \phi(x)^k \phi(y)^M = \frac{k^N \phi(x)^k \phi(y)^k \phi(x)^M \phi(y)^M |b_0(x, y)|^2}{\pi^N |\phi(x, y)|^{2M}} |\psi(x, y)|^{2M} \sum_{j=0}^{\infty} \beta_j(x, y) k^{-j} \]
as \( k \to \infty \), where \( \beta_0 = 1 \) and the expansion is uniform in \( x \) by virtue of the choice of \( U \). The last fact makes it legitimate to interchange the integration and summation signs and conclude that as \( k \to \infty \) the integral over \( U \) has an asymptotic expansion
\[ \left( \frac{k}{\pi} \right)^N \sum_{j=0}^{\infty} k^{-j} \int_U f(x) \beta_j(x, y) \frac{\phi(x)^M \phi(y)^M |b_0(x, y)|^2}{|\phi(x, y)|^{2M} |\phi(y)|^{2M}} \frac{\phi(x) \phi(y)}{|\phi(x, y)|^2} dx. \] (37)
Finally, recall the familiar formula for the asymptotics of Laplace integrals: if $D$ is a bounded region in $\mathbb{R}^n$, $F$ a complex-valued and $S$ a real-valued functions in $C^\infty(D)$, and $S$ peaks at a single point $x_0 \in D$, then as $\lambda \to +\infty$

$$\int_D F(x) e^{i S(x)} \, dx = \left( \frac{2\pi}{\lambda} \right)^{n/2} e^{i S(x_0)} \sqrt{|\text{Hess } S(x_0)|} \sum_{j=0}^{\infty} a_j \lambda^{-j}, \quad (38)$$

where the coefficients $a_j$ depend only on the derivatives of $F$ and $S$ at $x_0$, $a_0 = F(x_0)$, and

$$\text{Hess } S(x_0) = \det \left[ \frac{\partial^2 S}{\partial x_j \partial x_k}(x_0) \right].$$

(See [Fed], Theorem II.4.1, or [BH], Section 8.3.) Observe that owing to the strict plurisubharmonicity of $-\log \phi$, the function $x \mapsto \phi(x) \phi(y)/|\phi(x, y)|^2$ has a strict local maximum at $x = y$. Diminishing $U$ if necessary, we may thus assume that the function

$$S(x) = \log \frac{\phi(x) \phi(y)}{|\phi(x, y)|^2} \quad (39)$$

peaks only at $x = y$ on $U$; shrinking $U$ further if needed we may likewise assume that $f$ is $C^\infty$ on $U$. Consequently, the formula (38) can be applied to the integrals in (37); and since a short computation reveals that

$$\text{Hess } S(y) = 4b_0^2 (\det[-\partial^2 \log \phi])^2 = 4b_0^2 (y)^2,$$

the assertion of the theorem follows. \[\Box\]

Remarks. (1) The coefficients $a_j$ in (38) can in principle be evaluated explicitly (see [Fed], Proposition II.4.1, [BH], pp. 338–339), but even in the simplest case $j = 1$ the calculations are quite formidable; in our context (i.e. for $D$ a domain in $\mathbb{C}^N \cong \mathbb{R}^{2N}$ and $S$ of the form (39)) they have been carried out by Berezin [Ber], Appendix 1 with the following result:

$$a_1 = \sigma(x_0) F(x_0) + b_0(x_0) \tilde{J}(F/b_0)(x_0),$$

where $\tilde{J}$ is the differential operator

$$\tilde{J} = \sum_{j,k} g^\theta \frac{\partial^2}{\partial x_j \partial x_k}, \quad (g^\theta) := \text{the inverse matrix } \left( \frac{-\partial^2 \log \phi}{\partial x_j \partial x_k} \right), \quad (40)$$

and

$$\sigma(x) = \frac{1}{2} \tilde{J} \log b_0(x),$$
with \( b_0 \) given by (22). (In fact Berezin erroneously has 3/2 instead of 1/2 in the last formula, see [E4]; this mistake has been copied in Lemma 2.2 in [E2].) Feeding this into (37) it follows that

\[
Q_1 f(y) = \left[ \beta_1(y, y) - \sigma(y) - \tilde{A} \log \frac{1}{\psi_1(y)^{1/2}} \right] f(y) + \tilde{A} f(y). \tag{41}
\]

However, it is immediate from (34) that \( B_\rho f = f \) for any \( \rho \) if \( f \) is a bounded holomorphic function; hence, all the operators \( Q_j \) must annihilate bounded holomorphic functions. In particular, the expression in the square brackets in (41) must vanish, and we see that:

**Theorem 13.** \( Q_1 = \tilde{A} \).

Taking again \( \psi = \phi J[\phi]^{1/M} \) with \( M \) large enough as in the proof of Corollary 9 and using the machinery from the first section of [E2], we thus arrive at the following important corollary.

**Corollary 14.** Let \( \Omega \) be a strongly pseudoconvex domain in \( \mathbb{C}^N \) with real-analytic boundary, \( -\phi \) a \( C^\infty \) defining function for \( \Omega \) such that \( -\log \phi \) is strictly psh, and \( (g_{j\bar{k}}) \) the Kähler metric on \( \Omega \) defined by the potential \( -\log \phi \). Then the Berezin quantization can be carried out on the Kähler manifold \((\Omega, g_{j\bar{k}})\).

(2) For \( \psi^M = -\det(\partial \bar{\partial} \log \phi) \), a formula for \( Q_2 \) in (35) has also been computed by the present author [E4]; the result is

\[
Q_2 = \frac{1}{2} \tilde{A}^2 + \frac{1}{2} \sum_{j,k} R^k \frac{\partial^2}{\partial z_j \partial \bar{z}_k},
\]

where

\[
R^k = \sum_{l,m} g^{j\bar{k}} \frac{\partial^2 \log h_0(x)}{\partial z_l \partial \bar{z}_m}.
\]

In this situation, the various quantities above have very natural interpretations in terms of the metric \( g_{j\bar{k}} \): \( g^k \) is just the contravariant metric tensor, \( h_0(x) \, dx \) is the Riemannian volume element, \( \tilde{A} \) is the Laplace–Beltrami operator, \( 2\sigma \) the scalar curvature, and \( R^k \) are the contravariant components of the Ricci tensor.

(3) The vanishing of the square bracket in (41) immediately gives a formula for the coefficient \( b_1 \) in the asymptotic expansion (21):

\[
b_1(y) = b_0(y) \beta_1(y, y) = b_0(y) \left[ \sigma(y) + \tilde{A} \log \frac{1}{\psi(y)^{1/2}} \right].
\]
Taking in particular $M = 0$ we thus obtain

$$K_\phi(x, x) = \frac{k^N b_0(x)}{\pi^N \phi(x)^k} \left[ 1 + \frac{1}{2k} \bar{A} \log b_0(x) + O(k^{-2}) \right]. \quad (42)$$

Of course for $\Omega$ a domain in $\mathbb{C}^n$, $\bar{A} = (1/b_0) \bar{\partial} \partial$, so (42) agrees with the corresponding formula in Theorem 10 obtained on the basis of Nakazawa’s calculations. However, it is also possible to proceed in the other direction: by using Proposition 0 and the formulas (20) we obtain from the asymptotic expansion (21) the coefficients of the Fefferman’s expansion (on the diagonal) for the Hartogs domain $\tilde{\Omega}$. In particular, (42) thus yields the following extension of Nakazawa’s result (for the first-order coefficient) to a class of Hartogs domains in $\mathbb{C}^n$.

**Theorem 15.** Let $\Omega$ be a strongly pseudoconvex domain in $\mathbb{C}^n$ with real-analytic boundary, $-\phi$ a $C^\infty$ defining function for $\Omega$ such that $-\log \phi$ is strictly psh, and $\tilde{\Omega} \subset \Omega \times \mathbb{C}^d$ the Hartogs domain corresponding to the defining function $u(x, t) = |t|^2 - \phi(x)$, $x \in \Omega$, $t \in \mathbb{C}^d$. Then the Bergman kernel $K$ of $\tilde{\Omega}$ satisfies

$$K(x, t; x, t) = \frac{(N + d)!}{\pi^{N+d}} J[\phi] \left[ \frac{1}{(-u)^{N+d+1}} + \bar{A} \log J[\phi] \right] \left[ \frac{1}{2(N + d) \phi(x)} \frac{1}{(-u)^{N+d}} \right]$$

$$+ O\left( \frac{1}{(-u)^{N+d}} \right)$$

as $(x, t)$ approaches $\partial \tilde{\Omega} \setminus \{ t = 0 \}$. Here

$$J[\phi](x) = J[-u](x, t) = \phi(x)^{N+1} \det [-\bar{\partial} \log \phi]$$

is the Monge–Ampère determinant, and $\bar{A}$ is given by (40).

In principle, the higher order coefficients in the asymptotic expansion of $K$ could be obtained in the same way (but with much more work).

**REFERENCES**

