Nevanlinna–Pick Type Interpolation in a Dual Algebra*  

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The duality approach of Sarason, Abrahamse, Ball, and others is extended to obtain a version of Nevanlinna–Pick interpolation for a weak star closed algebra of operators on Hilbert space. In this context factorization is the polar decomposition of a positive trace class operator. Several concrete Nevanlinna–Pick type theorems are seen to follow from the main result and various versions of the Wold decomposition.

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INTRODUCTION

The purpose of this paper is to describe a general version of Nevanlinna–Pick interpolation suggested by the duality methods of Sarason [9], Abrahamse [1], Ball [5], and others as well as the abstract approach of Agler [3]. The main result is a matrix valued version of Nevanlinna–Pick interpolation for a dual algebra of operators on Hilbert space. In the abstract setting, the duality results from viewing $\mathcal{L}(H)$, the bounded linear operators on the (complex and separable) Hilbert space $H$, as the dual of $\mathcal{T}(H)$, the trace class operators. Factorization in this context is then the polar decomposition combined with the spectral theorem for a positive trace class operator.

In the body of the paper we describe how to recover versions of Nevanlinna–Pick interpolation on the unit disc from the main result and the Wold decomposition and for a multiply connected domain from the main result and a type of Wold decomposition of Abrahamse and Douglas [2]. These results are similar to those of Abrahamse [1] and Ball [5]. Further, we describe how to obtain a version of Agler’s Abstract Interpolation Theorem [4]. In the remainder of the introduction, we state our main result.

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To state the result precisely, we need to introduce some notations and terminology. Let $\mathcal{T}^+(H)$ denote the positive (semidefinite) trace class operators on $H$. Given $Q \in \mathcal{T}^+(H)$, the sesquilinear form
\[
\langle X, Y \rangle_Q = \text{trace}(XQY^*)
\] (0.1)
defines a preHilbert space structure on $L(H)$. The resulting Hilbert space we denote $K(Q)$ with inner product $\langle \cdot, \cdot \rangle_Q$ and norm $\| \cdot \|_Q$. Let $\{X\}_Q$ denote the class of $X \in L(H)$ in $K(Q)$. Each $T \in \mathcal{T}(H)$ defines an operator on $K(Q)$, denoted $T_Q$, by the formula
\[
T_0[X]_Q = [TX]_Q, \quad X \in L(H).
\] (0.2)
Given $A$, a subalgebra of $L(H)$, let $A(Q)$ denote the Hilbert space obtained from the closure of $A$ (more precisely, the set $\{[X]_Q : X \in A\}$) in $K(Q)$. If $T \in A$, then the operator $T_Q$ leaves $A(Q)$ invariant.

Fix $I$, a weak* closed ideal of the subalgebra $A$. Let $N_0 = N_I$ denote the orthogonal complement of $I(Q)$ in $A(Q)$ and let $P_{N_0}$ denote the corresponding orthogonal projection. It is not hard to see that $N_0$ is an invariant subspace for $P_{N_0}$ whenever $T \in A$, where $P_{N_0}$ denotes the orthogonal projection of $K(Q)$ onto $A(Q)$.

We can now state our main result.

0.3. Theorem. Suppose $A \subset L(H)$ is a weak* closed subalgebra of $L(H)$ containing the identity and $\mathcal{I}$ is a weak* closed ideal of $A$. Given $\phi \in A$, there exists $\psi \in A$ such that $\phi - \psi \in \mathcal{I}$ and
\[
\|\psi\| = \inf\{\|\phi + a\| : a \in \mathcal{I}\}
\]
\[
= \sup\{\|P_{A(Q)}\phi + P_{N_0}\|_Q : Q \in \mathcal{T}^+(H)\}. \tag{0.4}
\]
We remark that a weak* closed subalgebra of $L(H)$ containing the identity is called a dual algebra. Also, the existence of $\psi$ such that the first equality in (0.4) holds follows since $\mathcal{I}$ is weak* closed; and hence the unit ball in $\mathcal{I}$ is weak* compact. The content of Theorem 0.3 is the second equality in (0.4).

The paper is organized as follows. Section one contains preliminaries on the duality between $L(H)$ and $\mathcal{T}(H)$. Theorem 0.3 is proved in Section two. In section three we consider the case that $A$ is an algebra of multiplication operators acting on a reproducing kernel Hilbert space and obtain a version of Agler's Abstract Interpolation Theorem. The Wold decomposition of Abrahamse and Douglas and (0.3) are combined in Section four to yield Nevanlinna–Pick interpolation on nice multiply connected domains. In section five the special case of a uniform algebra which is approximating in modulus is considered. The results in this section are
similar to those of Cole, Lewis and Wermer [11]. In Section six, a matrix valued version of some of the results in Section five are presented. Sections five and six together generalize the results of Section four and naturally apply to matrix valued interpolation for $H^\infty$ of the polydisc and other domains in $\mathbb{C}^n$.

1. Preliminaries, $\mathcal{L}(H)$ as a Dual Space

Throughout $H$ is a complex separable Hilbert space and $\mathcal{L}(H)$ denotes the bounded linear operators on $H$. In this section we recall some basic facts about the predual of $\mathcal{L}(H)$, the trace class operators, as well as fix notation.

For $A \in \mathcal{L}(H)$, the quantity

$$\|A\|_2^2 = \sum \|Ae_j\|^2 = \sum \|Af_j\|^2 = \sum_{i,j} |\langle Ae_j, f_j \rangle|^2$$

(1.1)

is independent of the choice of orthonormal basis $\{e_j\}$ and $\{f_j\}$ of $H$. $\mathcal{H}(H)$, the Hilbert-Schmidt class, consists of those $A \in \mathcal{L}(H)$ such that $\|A\|_2$ is finite. The trace class, $\mathcal{T}(H)$ can be defined by

$$\mathcal{T}(H) = \{ AB^*: A, B \in \mathcal{H}(H) \}. \tag{1.2}$$

For $A \in \mathcal{T}(H)$,

$$\text{trace}(A) = \sum \langle Ae_j, e_j \rangle \tag{1.3}$$

is independent of orthonormal basis $\{e_j\}$ of $H$ and is called the trace of $A$. If $A \in \mathcal{T}(H)$, then $|A| = (A^*A)^{1/2} \in \mathcal{T}(H)$ and the trace norm of $A$ is defined by

$$\|A\|_1 = \text{trace}(|A|). \tag{1.4}$$

Moreover,

$$|\text{trace}(A)| \leq \|A\|_1. \tag{1.5}$$

Using the trace, it is possible to define a natural inner product on $\mathcal{H}(H)$ by

$$\langle A, B \rangle = \text{trace}(AB^*). \tag{1.6}$$

$\mathcal{H}(H)$ is a Hilbert space and $\mathcal{T}(H)$ is a Banach space.
If $A \in \mathcal{F}(H)$ and $T \in \mathcal{L}(H)$, then $AT, TA \in \mathcal{F}(H)$, trace$(AT) = \text{trace}(TA)$ and
\[ \|AT\| \leq \|T\| \|A\|. \] (1.7)

In particular, for $T \in \mathcal{L}(H)$ fixed, (1.5) and (1.7) imply that
\[ L_T(A) = \text{trace}(AT) \] (1.8)
defines a bounded linear functional on $\mathcal{F}(H)$ with
\[ \|L_T\| = \|T\|. \] (1.9)

It turns out that the map
\[ \mathcal{L}(H) \ni T \mapsto L_T \in \mathcal{F}(H)^* \] (1.10)
is an isometric isomorphism of $\mathcal{L}(H)$ onto $\mathcal{F}(H)^*$. Define, for $\mathcal{S}$ a subspace of $\mathcal{F}(H)$,
\[ \mathcal{S}^\perp = \{ A \in \mathcal{F}(H) : \text{trace}(AS) = 0 \text{ for every } S \in \mathcal{S} \} \] (1.11)
and for $\mathcal{S}$ a subspace of $\mathcal{L}(H)$,
\[ \mathcal{S}^\perp = \{ T \in \mathcal{L}(H) : \text{trace}(ST) = 0 \text{ for every } S \in \mathcal{S} \}. \] (1.12)

$\mathcal{S}^\perp$ is (trace) norm closed in $\mathcal{F}(H)$ and $\mathcal{S}^\perp$ is weak* closed in $\mathcal{L}(H)$. Moreover, $(\mathcal{S}^\perp)^\perp$ is the weak* closure of $\mathcal{S}$ in $\mathcal{L}(H)$. In particular, if $\mathcal{S}$ is weak* closed, then $(\mathcal{S}^\perp)^\perp = \mathcal{S}$. We have the standard fact:

1.13. Proposition. If $\mathcal{S}$ is a weak* closed subspace of $\mathcal{L}(H)$, then the map
\[ \rho : (\mathcal{F}(H)/(\mathcal{S}^\perp)^\perp)^* \to \mathcal{S} \]
given by
\[ \rho(L) = L\pi, \]
where $\pi$ denotes the quotient map $\pi : \mathcal{F}(H) \to \mathcal{F}(H)/(\mathcal{S}^\perp)^\perp$, is an isometric isomorphism.

For $\mathcal{S}$ a weak* closed subspace of $\mathcal{L}(H)$, we adopt the notation
\[ \mathcal{S}_* = \mathcal{F}(H)/(\mathcal{S}^\perp), \] (1.14)
and call $\mathcal{S}_*$ the predual of $\mathcal{S}$ and say that $\mathcal{S}$ is a dual space.
In the sequel we will be interested in $M_N(\mathscr{S})$, the $N \times N$ matrices with entries from $\mathscr{S}$ for $N$ a positive integer and $\mathscr{S}$ a closed subalgebra of $\mathcal{L}(H)$. In this case $M_N(\mathscr{S})$ is a weak* closed subalgebra of $\mathcal{L}(H^{(N)})$, where $H^{(N)}$ denotes the orthogonal direct sum of $H$ with itself $N$ times. It follows that $M_N(\mathscr{S})$ has predual

$$M_N(\mathscr{S})^* = \mathcal{S}(H^{(N)})^{1\perp} M_N(\mathscr{S}).$$  \hspace{1cm} (1.15)$$

We close this section by recalling the spectral theorem for positive (semi-definite) trace class operators. Since trace class operators are compact, if $Q \in \mathcal{S}(H)$ is positive semidefinite then the spectral theorem for positive compact operators produces an orthogonal set $\{q_j\}$ of vectors such that

$$Q = \sum q_j q_j^*,$$  \hspace{1cm} (1.16)$$

where $x y^*$, for $x, y \in H$, denotes the rank one operator given by

$$x y^*(h) = (h, y)x$$  \hspace{1cm} (1.17)$$

and the sum converges in the norm topology on $\mathcal{L}(H)$.

2. Main Result

The purpose of this section is to give a proof of Theorem 0.3 from the introduction. Accordingly, a Hilbert space $H$, a dual algebra $\mathcal{A} \subset \mathcal{L}(H)$, and a weak* closed ideal $\mathcal{I}$ of $\mathcal{A}$ are fixed throughout the remainder of this section.

We recall some notation and terminology from the introduction. Let $\mathcal{S}^+(H)$ denote the positive trace class operators on $H$. Given $Q \in \mathcal{S}^+(H)$, the sesquilinear form

$$\langle X, Y \rangle_Q = \text{trace}(XY^*)$$  \hspace{1cm} (2.1)$$

defines a preHilbert space structure on $\mathcal{L}(H)$. The resulting Hilbert space we denote $K(Q)$ with inner product $\langle \cdot, \cdot \rangle_Q$ and norm $\|\cdot\|_Q$. Let $[X]_Q$ denote the class of $X \in \mathcal{S}(H)$ in $K(Q)$. And when the meaning is clear from the context, we will write $[X]$ or $X$ for $[X]_Q$. Each $T$ in $\mathcal{L}(H)$ defines a (bounded linear) operator on $K(Q)$ denoted $T_Q$, by

$$T_Q[X]_Q = [TX]_Q$$  \hspace{1cm} (2.2)$$

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as the following computation, for $X \in \mathcal{L}(H)$, shows

$$
\langle T_0[X], T_0[X] \rangle_{\mathcal{O}} = \text{trace}(TXQX^*T^*)
= \text{trace}(T^*TXQX^*)
\leq \|T^*T\| \|XQX^*\|_{\mathcal{O}}
\leq \|T\|^2 \langle [X], [X] \rangle_{\mathcal{O}},
$$

(2.3)

where $\|\cdot\|_{\mathcal{O}}$ denotes the Hilbert–Schmidt norm. In particular $\|T_0\| \leq \|T\|$. Given $\mathcal{F}$ is a subalgebra of $\mathcal{L}(H)$, let $\mathcal{F}(Q)$ denote the Hilbert space obtained from the closure of $\mathcal{F}$ (more precisely, the set $\{[X]_{\mathcal{O}} : X \in \mathcal{F}\}$) in $K(Q)$. Note, if $T \in \mathcal{F}$, then $\mathcal{F}(Q)$ is invariant for $T_0$.

It is a simple matter to determine the adjoint of $T_0$. We have, for $X, Y \in \mathcal{L}(H)$

$$
\langle T_0[X], [Y] \rangle_{\mathcal{O}} = \text{trace}(TXQY^*)
= \text{trace}(XQ(T^*Y)^*)
= \langle [X], (T^*)^2[Y] \rangle_{\mathcal{O}}.
$$

(2.4)

Thus

$$(T_0)^* = (T^*)_{\mathcal{O}}.
$$

(2.5)

Let $\mathcal{N}_Q = \mathcal{N}_{\mathcal{F}, Q}$ denote the orthogonal complement of $\mathcal{F}(Q)$ in $\mathcal{A}(Q)$.

$$
\mathcal{N}_Q = \mathcal{F}(Q) \ominus \mathcal{A}(Q).
$$

(2.6)

Also, let $P_{\mathcal{A}(Q)}$ denote the orthogonal projection of $K(Q)$ onto $\mathcal{A}(Q)$.

2.7. Lemma. If $\phi \in \mathcal{A}$, then $\mathcal{N}_Q$ is invariant for $P_{\mathcal{A}(Q)}\phi^*_Q$. Further, if $\phi \in \mathcal{F}$, then $P_{\mathcal{A}(Q)}\phi^*_Q P_{\mathcal{N}_Q}$ is the zero operator.

Proof. Let $X \in \mathcal{N}_Q$ and $Y \in \mathcal{F}$ be given. Compute

$$
\langle \phi^*_Q X, [Y] \rangle_{\mathcal{O}} = \langle X, \phi[Y] \rangle_{\mathcal{O}} = 0,
$$

(2.8)

since $\phi Y \in I$. Observing that $\mathcal{F}$ is dense in $\mathcal{F}(Q)$ shows $\mathcal{N}_Q$ is invariant for $\phi^*_Q$. Now, if $\phi \in \mathcal{F}$ then for $X \in \mathcal{N}_Q$ and $Y \in \mathcal{F}$,

$$
\langle \phi^*_Q X, [Y] \rangle_{\mathcal{O}} = \langle X, [\phi Y] \rangle = 0,
$$

(2.9)

since $\phi Y \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $\mathcal{A}(Q)$, it follows that $\phi^*_Q X = 0$. This completes the proof.
We can now state our main result.

2.10. Theorem. Suppose $H$ is a Hilbert space, $\mathcal{A} \subset \mathcal{L}(H)$ is a dual algebra, and $\mathcal{I}$ is a weak* closed ideal of $\mathcal{A}$. If $\phi \in \mathcal{A}$, then there exists $\psi \in \mathcal{A}$ such that $\phi - \psi \in \mathcal{I}$ and

$$
\|\psi\| = \inf \{ \|\phi + a\| : a \in \mathcal{I} \} = \sup \{ \|P_{\mathcal{I}(Q)}\phi P_{\mathcal{I}(Q)}\| : Q \in \mathcal{F}^+(H) \}. 
$$

(2.11)

As remarked earlier, the second equality in (2.11) is the content of (2.10).

Proof. Recall from Section one that $\mathcal{A}$ has a predual $A_\pi = \mathcal{F}(H)\dual{\mathcal{A}}$.

Let $\pi$ denote the corresponding quotient map,

$$
\pi : \mathcal{F}(H) \to A_\pi.
$$

Define

$$
\mathcal{I} = \{ F : a(F) = 0 \text{ for every } a \in \mathcal{I} \}.
$$

(2.12)

The proof that the infimum is smaller than the supremum in (2.11) consists of applying the Hahn–Banach Theorem to $L : \mathcal{I} \to \mathbb{C}$ given by

$$
L(F) = \phi(F).
$$

(2.13)

Given $F \in \mathcal{I}$ and $\varepsilon > 0$, there exists $G \in \mathcal{F}(H)$ such that $\pi(G) = F \in A_\pi$, and $\|G\|_1 \leq \|F\| + \varepsilon$, since $\|F\| = \inf \{ \|G\| : \pi(G) = F \}$. The trace class operator $G^*$ can be factored by the polar decomposition as $G^* = VQ$, where $Q = (GG^*)^{1/2}$ and $V$ is a partial isometry with range the closure of the range of $G^*$ and kernel the kernel of $G^*$. Compute, for $a \in \mathcal{A}$,

$$
\langle[a], [V^*] \rangle_Q = \text{trace}(aQV^*) = \text{trace}(aG) = a(F).
$$

Thus, if $a \in \mathcal{I}$, then, since $F \in \mathcal{I}^\perp$, $\mathcal{I}$

$$
\langle[a], [V^*] \rangle_Q = 0.
$$

(2.14)

From (2.14) and the fact that $\mathcal{I}$ is dense in $\mathcal{I}(Q)$, it follows that

$$
P_{\mathcal{I}(Q)}[V^*] \in \mathcal{I}(Q) \cap \mathcal{I}(Q).
$$

(2.15)
Recall the definition of $N_Q$ from (2.6) and let $P_{\mathcal{N}_Q}$ denote the orthogonal projection of $A(Q)$ onto $\mathcal{N}_Q$. From (2.15) it follows that

$$P_{\mathcal{N}_Q}[V^*]_Q = P_{\mathcal{N}_Q}P_{\mathcal{N}_Q}[V^*]_Q$$

(2.16).

Using (2.16), compute

$$L(F) = \text{trace}(\phi G)$$

$$= \text{trace}(\phi QV^*)$$

$$= \langle [\phi]_Q, [V^*]_Q \rangle_Q$$

$$= \langle [\phi]_Q, P_{\mathcal{N}_Q}[V^*]_Q \rangle_Q$$

$$= \langle [\phi I]_Q, P_{\mathcal{N}_Q}P_{\mathcal{N}_Q}[V^*]_Q \rangle_Q$$

$$= \langle \phi [I]_Q, P_{\mathcal{N}_Q}P_{\mathcal{N}_Q}[V^*]_Q \rangle_Q$$

$$= \langle [I]_Q, P_{\mathcal{N}_Q}\phi [I]_Q P_{\mathcal{N}_Q}[V^*]_Q \rangle_Q.$$  (2.17)

where in the last equality we have used (2.7) and $[I]_Q \in P_{\mathcal{N}_Q}(Q)$. Hence,

$$|L(F)| \leq \|P_{\mathcal{N}_Q}\phi [I]_Q P_{\mathcal{N}_Q}[V^*]_Q \| \|V^*\|_Q \|I\|_Q.$$  (2.18)

Finally, compute

$$\|V^*\|^2_Q = \langle [V^*]_Q, [V^*]_Q \rangle_Q$$

$$= \text{trace}(V^*QV)$$

$$= \text{trace}(Q)$$

$$= \|Q\|_1 = \|G\|_1$$

$$\leq \|F\| + \varepsilon,$$  (2.19a)

where we have used $V$ is a partial isometry. A similar computation shows

$$\|I\|^2_Q \leq \|F\| + \varepsilon,$$  (2.19b)

whence, from (2.18) and (2.19) it follows that

$$|L(F)| \leq \sup \{ \|P_{\mathcal{N}_Q}\phi [I]_Q P_{\mathcal{N}_Q}[V^*]_Q \| : Q \in \mathcal{F}(H) \} \|F\|.$$  (2.20)

The Hahn–Banach Theorem extends $L$ to all of $\mathcal{A}_a$ such that

$$\|L\| \leq \sup \{ \|P_{\mathcal{N}_Q}\phi [I]_Q P_{\mathcal{N}_Q}[V^*]_Q \| : Q \in \mathcal{F}(H) \}$$

$$L(F) = \phi(F), \quad F \in \mathcal{A}_a.$$  (2.21)
Since the dual of \( A \) is \( A \), there exists \( \psi \in A \) such that \( \|\psi\| = \|L\| \) and
\[
L(F) = \psi(F), \quad F \in A, \quad \phi(F) = \psi(F), \quad F \in A^\perp. \tag{2.22}
\]
Now, (2.22) says \( \phi - \psi \in (A^\perp)^+ \). Hence \( \phi - \psi \in I \).
Summarizing, there exists \( \psi \in A \) such that
\[
\phi - \psi \in I \tag{2.23}
\]
and
\[
\|\psi\| \leq \sup \{ \|P_{A(Q)}\phi^*P_{\perp Q}\| : Q \in I(H) \}. \tag{2.24}
\]
To finish the proof it is enough to show that the infimum in (2.11) is larger than the supremum in (2.24). Given \( Q \in I^+(H) \) and \( \varphi \in A \), we have, from (2.3) \( \|\varphi_Q\| \leq \|\varphi\| \). If \( \phi - \varphi \in I \), it follows from (2.7) that
\[
P_{A(Q)}\varphi_P^*P_{\perp Q} = P_{A(Q)}\varphi_P^*P_{\perp Q}. \tag{2.25}
\]
Hence, for \( \phi - \varphi \in I \),
\[
\|\varphi\| \geq \|\varphi_Q\|
= \|\varphi_P\|
\geq \|P_{A(Q)}\varphi_P^*P_{\perp Q}\|
= \|P_{A(Q)}\varphi_P^*P_{\perp Q}\|. \tag{2.26}
\]
This concludes the proof of (2.10).

3. REPRODUCING KERNEL HILBERT SPACES

3.1. Definition. Let \( X \) be a set and \( B \) a Hilbert space. A Hilbert space \( H \) of \( B \)-valued functions on \( X \) is a reproducing kernel Hilbert space if

1. For each \( \gamma \in B \) the function \( f_\gamma = \gamma \) is in \( H \);
2. There exists \( C \) such that \( \|f_\gamma\| \leq C\|\gamma\| \) for all \( \gamma \in B \); and there exists
   \[
k : X \times X \to \mathcal{L}(B)
\]
such that

3. if \( x \notin \mathcal{B} \) and \( s \notin X \), then \( k(\cdot, s)x \in H \);

4. for each \( s \in X \), there exists a constant \( C_s \) such that for every \( x \in \mathcal{B} \), \( \|k(\cdot, s)x\| \leq C_s\|x\| \) and;

5. if \( f \in H \), \( x \in \mathcal{B} \), and \( s \in X \), then \( \langle f, k(\cdot, s)x \rangle = \langle f(s), x \rangle \).

Let \( H^{\infty} \) denote those \( \mathcal{L}(\mathcal{B}) \)-valued functions \( \phi \) on \( X \) such that \( \phi f \in H \) whenever \( f \in H \) and for which there exists a \( C \) such that

\[
\|\phi f\| \leq C \|f\|. \tag{3.2}
\]

In this case, let \( M_\phi \) denote the operator equal to multiplication by \( \phi \) on \( H \); and we view \( H^{\infty}(k) \) as an algebra of operators on \( H \).

3.3. Lemma. If \( \phi \in H^{\infty} \), then, for all \( x \in \mathcal{B} \) and \( s \in X \),

\[
M_\phi^* k(\cdot, s)x = k(\cdot, s) \phi(s)^*x. \tag{3.4}
\]

Proof. If \( f \in H \), \( x \in \mathcal{B} \), and \( s \in X \), then

\[
\begin{align*}
\langle f, k(\cdot, s) \phi(s)^*x \rangle &= \langle f(s), \phi(s)^*x \rangle \\
&= \langle \phi(s)f(s), x \rangle \\
&= \langle M_\phi f, k(\cdot, s)x \rangle \\
&= \langle f, M_\phi^* k(\cdot, s)x \rangle, \tag{3.5}
\end{align*}
\]

from which the lemma follows.

3.6. Lemma. \( H^{\infty}(k) \) is weak* closed. Given \( s_1, \ldots, s_n \in X \), let \( \mathcal{I} \subset H^{\infty} \) denote the ideal consisting of those \( \phi \in H^{\infty} \) such that \( \phi(s_j) = 0 \) for each \( j \). Then \( \mathcal{I} \) is also weak* closed.

Proof. Let

\[
\begin{align*}
H^{\infty}(k) = \{ X \in \mathcal{T}(H) : \text{trace}(X\phi) = 0 \text{ for every } \phi \in H^{\infty}(k) \}. \\
\end{align*}
\]

It is enough to show that

\[
H^{\infty}(k) = (\overline{H^{\infty}(k)})^\perp = \{ T \in \mathcal{L}(H) : \text{trace}(TX) = 0 \text{ for every } X \in \overline{H^{\infty}(k)} \}. \tag{3.7}
\]
For \( s \in X \), let \( H_s = \{ f \in H : f(s) = 0 \} \). If \( x \in B \), and \( f \in H_s \), then the rank one operator \( f(\cdot)(k(\cdot, s) x)^* \) is in \( \frac{1}{2} H^\infty(k) \), since, for any \( \phi \in H^\infty(k) \),

\[
\text{trace}(\phi f(k(\cdot, s) x)^*) = \langle \phi f, k(\cdot, s) x \rangle = \phi(s) f(s) = 0.
\]

(3.8)

Fix \( T \in (\frac{1}{2} H^\infty)^{1} \). Note that, from (3.8),

\[
0 = \text{trace}(T f(k(\cdot, s) x)^*) = \langle T f, k(\cdot, s) x \rangle = (T f)(s).
\]

(3.9)

For \( s \in X \) define \( \psi(s) \in \mathcal{L}(B) \) by defining, for \( \gamma \in B \),

\[
\psi(s) \gamma = (T f^)(s).
\]

(3.10)

It is evident that \( \psi(s) \) is linear. The estimate

\[
\| \psi(s) \gamma \|^2 = \| (T f)(s) \|^2 = \langle T f^, k(\cdot, s) T f(s) \rangle \\
\leq \| T f^ \| \| k(\cdot, s) T f(s) \| \\
\leq C_s \| T f^ \|^2 \\
\leq C_s \| T \|^2 \| f^ \|^2 \\
\leq C_s \| T \|^2 C^2 \| \gamma \|^2,
\]

(3.11)

implies

\[
\| \psi(s) \gamma \| \leq \| T \| \sqrt{C_s} \| \gamma \|.
\]

Thus, \( \psi(s) \) is bounded.

If \( f \in H \) and \( s \in X \), then \( f - f(s) \in H_s \). From (3.9) we have, for \( x \in B \),

\[
\langle (T f)(s), x \rangle = \langle (T f_{(s)}) (s), x \rangle = \langle (T f)(s), x \rangle.
\]

Hence, \( (T f)(s) = \psi(s) f(s) \). In other words, \( \psi \in H^\infty(k) \) and \( T = \psi \).
The weak* closure of $\mathcal{I}$ is $(\mathcal{I}^*)^\perp$. Note $(\mathcal{I}^*)^\perp \subset H^\omega(k)$ and let $\phi \in (\mathcal{I}^*)^\perp$ be given. Since $y(k(\cdot, s), x)^* \in \mathcal{I}^\perp$ for every $j$ and $x, y \in \mathcal{I}$,

$$
0 = \text{trace}(\phi y(k(\cdot) x)^*) \\
= \langle \phi y, k(\cdot, s) x \rangle \\
= \langle \phi(s_j) y, x \rangle.
$$

Thus, $\phi \in \mathcal{I}$.]

Canonical examples of (0.3) arise as follows. Let $H$ denote a reproducing kernel Hilbert space of $B$ valued functions on a set $X$ with reproducing kernel $k$. If $\mathcal{A}$ is a dual subalgebra of $H^\omega(k)$ and points $x_1, \ldots, x_n$ in $X$ are fixed, then $\mathcal{I} = \{ \phi \in \mathcal{A} : \phi(x_j) = 0 \}$ is a weak* closed ideal of $\mathcal{A}$. Given $w_1, \ldots, w_m \in \mathcal{L}(\mathcal{A})$, if there exists $\phi \in \mathcal{A}$ such that $\phi(x_j) = w_j$, then (0.3) gives the minimum norm of an element $\psi \in \mathcal{A}$ such that $\psi(x_j) = w_j$. Of course, we can replace $\mathcal{A}$ and $\mathcal{I}$ by $\{ a^* : a \in \mathcal{A} \}$ and $\{ a^* : a \in \mathcal{I} \}$ respectively. This situation is described in more detail in the following example.

3.12. EXAMPLE. Let $X$ denote the finite set $\{1, 2, \ldots, N\}$. Suppose, $H_j$ is a countable collection of reproducing kernel Hilbert spaces of $\mathbb{C}^p$ valued functions on $X$ with reproducing kernels $k_j$. Let $D$ denote the set of $\mathbb{C}^p$ valued functions on $X$. Assume that each $\phi \in D$ defines a bounded operator $M_{\phi}$ on $H_j$ by

$$
M_{\phi}h = \phi h.
$$

(3.12.1)

Further, assume each $\phi \in D$ defines a bounded operator, denoted $M_{\phi}$ on $\mathcal{H} = \bigoplus H_j$ by the formula,

$$
M_{\phi} \oplus h_j = \bigoplus M_{\phi} h_j.
$$

(3.12.2)

In particular, we are assuming,

$$
\|M_{\phi}\| = \sup \{ \|M_{\phi}\| : j \} < \infty,
$$

(3.12.3)

where $\|M_{\phi}\|$ denotes the norm of $M_{\phi}$ as an operator on $H_j$. Fix $Y \subset X$ and let $\mathcal{A} = \{ M_{\phi}^*: a \in D \}$ and $\mathcal{I} = \{ M_{\phi}^*: a \in D, a(t) = 0 \text{ for } t \in Y \}$.

We conclude this section with an elaboration on the previous example. It describes how to recover a version of Agler’s Abstract Interpolation Theorem from (0.3).

3.13. EXAMPLE. Again, let $X$ denote the finite set $\{1, 2, \ldots, N\}$, but now let $D$ denote the set of $\mathbb{C}$ valued functions on $X$. For the remainder of this
example, by a reproducing kernel Hilbert space $H$ we mean $\mathbb{C}$ valued and such that each $a \in \mathcal{D}$ defines a bounded operator on $H$ as in (3.12.1).

The following lemmas are included for completeness.

3.13.1. Lemma. Suppose $H$ is a reproducing kernel Hilbert space on $X$ with reproducing kernel $k$. The set $\{k(\cdot, t): t \in X\}$ is a basis for $H$. Given $\phi \in \mathcal{D}$, let $M_\phi$ denote the operator on $H$ given by

$$M_\phi f(s) = \phi(s) f(s).$$

Then

$$M_\phi^* k(\cdot, t) = \overline{\phi(t)} k(\cdot, t).$$

3.13.4. Lemma. If $H$ is a reproducing kernel Hilbert space on $X$ with reproducing kernel $k$ and $\phi \in \mathcal{D}$, then

$$\|M_\phi\| = \inf \{ \rho \geq 0: ((\rho^2 - \phi(i) \overline{\phi(j)}) k(j, i))_{i,j=1}^N \geq 0 \},$$

where $M_\phi$ is multiplication by $\phi$ on $H$.

3.13.6. Lemma. Suppose $H_1$ and $H_2$ are reproducing kernel Hilbert spaces on $X$ with reproducing kernels $k_1$ and $k_2$ respectively. If there exists $c$ such that

$$k_1(s, t) = c(s) k_2(s, t) \overline{c(t)}, \quad s, t \in X,$$

then the map $V: H_2 \rightarrow H_1$ defined by

$$V k_2(\cdot, t) = \overline{c(t)} k_1(\cdot, t),$$

is an isometry and, for $\phi \in \mathcal{D},$

$$M_\phi^* V = VM_\phi^*.$$  

Moreover, if $c(s) = 0$ implies $k_2(s, s) = 0$, then $V$ is onto.

3.13.10. Definition. Let $\Gamma$ denote an index set and suppose for each $\alpha \in \Gamma$, there is an associated reproducing kernel Hilbert space $H_\alpha$ on the set $X$ with corresponding reproducing kernel $k_\alpha$. The collection $\mathcal{F} = \{(H_\alpha, k_\alpha): \alpha \in \Gamma\}$ is a family if it satisfies the following axioms

1. For each $\alpha \in \Gamma$ and $s \in X$, $k_\alpha(s, s) = 1$ or 0;
2. $\{(k_\alpha(i, j)): \alpha \in \Gamma\}$ is a compact subset of the $N \times N$ matrices;
(3) for each $\phi \in \mathcal{D}$,
\[ \sup \{ \| M_\phi \|_\alpha : \alpha \in \Gamma \} < \infty, \]
where $\| M_\phi \|_\alpha$ denotes the norm of $M_\phi$ multiplication by $\phi$ on $H_\alpha$;

(4) given sequences $\{ \alpha_j \}$ from $\Gamma$ and $\{ c_j \}$ from $\mathbb{C}^N$ such that $c_j(s) = 0$ whenever $k_\alpha(s, s) = 0$ and, for $s \in X$,
\[ \sum_j |c_j(s)|^2 = 1, \]
it follows that
\[ k(s, t) = \sum_j c_j(s) k_\alpha(s, t) c_j(t) \]
is in $\mathcal{F}$.

3.13.11. Definition. Let $\mathcal{F} = \{(H_\alpha, k_\alpha) : \alpha \in \Gamma\}$ be a family of reproducing kernel Hilbert spaces on $X$. Given a subset of $X$ let $[Y]_\alpha$ denote the span of the set $\{ k_\alpha(s, s) : s \in Y \}$ in $H_\alpha$. $\mathcal{F}$ is an NP family if, given disjoint subsets $Y$ and $Z$ of $X$ and $\phi \in \mathcal{D}$,
\[ \sup \{ \| M_\phi \|_\alpha : \alpha \in \Gamma \} \]
\[ \geq \sup \{ \| P_{Y \cup Z} \| \| M_\phi \|_\alpha : \alpha \in \Gamma \}, \]
where $P_\alpha$ denotes the orthogonal projection of $H_\alpha$ onto the subspace $M_\alpha$ of $H_\alpha$.

3.13.13. Theorem (Agler). Let $\mathcal{F}$ be an NP family of reproducing kernel Hilbert spaces on $X$. Given a subset of $X$ and $\phi \in \mathcal{D}$ there exists $\psi \in \mathcal{D}$ such that $\rho(s) = \phi(s)$ for $s \in Y$ and
\[ \sup \{ \| M_\phi \|_\alpha : \alpha \in \Gamma \} = \sup \{ \| M_\phi \|_\alpha : \alpha \in \Gamma \}. \]

Proof. Since the collection $\mathcal{H} = \{ k_\alpha : \alpha \in \Gamma \}$ is a compact subset of the $N \times N$ matrices, by (3.13.10)(2), we can find a countable subset $\alpha_j$ such that $\{ k_\alpha \}$ is dense in $\mathcal{H}$. Let $\mathcal{H} = \bigoplus H_\alpha$. Each $\phi \in \mathcal{D}$ defines a bounded operator $M_\phi$ on $\mathcal{H}$ as in (3.12.1) from the hypothesis (3.13.10)(3). With $Y$ fixed, let $\mathcal{A} = \{ \mathcal{M}_\phi^* : \phi \in \mathcal{D} \}$ and $\mathcal{I} = \{ \mathcal{M}_\phi^* : \phi \in \mathcal{D}, \phi(s) = 0 \text{ for } s \in Y \}$. From (0.3), there exists $\mathcal{M}_\phi^* \in \mathcal{A}$ such that $\mathcal{M}_\phi^* - \mathcal{M}_\phi^* \in \mathcal{I}$ and
\[ \| \mathcal{M}_\phi^* \| = \inf \{ \| \mathcal{M}_\phi^* + \mathcal{M}_\phi^* \| : \mathcal{M}_\phi^* \in \mathcal{I} \}
\[ = \sup \{ \| P_{\mathcal{A} + Q} \| \| \mathcal{M}_\phi^* \| : Q \in \mathcal{F}^+(\mathcal{H}) \}, \]
(3.13.15)
where, for simplicity, $P_Q$ denotes the orthogonal projection onto $\mathcal{N}_Q = \mathcal{N}(Q) \subseteq \mathcal{F}(Q)$. Since $(\mathcal{M}_*^\varphi)_Q = (\mathcal{M}_\varphi)_Q$ and $P_{\mathcal{N}(Q)}(\mathcal{M}_*^\varphi)_Q P_Q = P_{\mathcal{N}(Q)}(\mathcal{M}_\varphi)_Q P_Q$, we have $(P_{\mathcal{N}(Q)}(\mathcal{M}_*^\varphi)_Q P_Q)^* = P_{\mathcal{N}(Q)}(\mathcal{M}_\varphi)_Q P_Q$ Thus,

$$\|\mathcal{M}_*^\varphi\| = \sup\{\|P Q(\mathcal{M}_*^\varphi)_Q P_Q\| : Q \in \mathcal{T}^+(\mathcal{H})\}. \quad (3.13.16)$$

From (3.13.6), we need to consider the expressions $\|P Q(\mathcal{M}_*^\varphi)_Q P_Q\|$ for $Q$ a positive trace class operator on $\mathcal{H}$. We have $Q = \sum q_n q_n^*$ for some $q_n = \oplus q_{n,j} \in \mathcal{H}$ (with $q_{n,j} \in H_{n,j}$). For $\phi, \psi \in \mathcal{D}$,

$$\langle [\mathcal{M}_*^\varphi]_Q [\mathcal{M}_\varphi]_Q \rangle_Q = \sum_n \langle \mathcal{M}_*^\varphi q_n, \mathcal{M}_\varphi q_n \rangle = \sum_j \sum_n \langle M_{n,j} q_{n,j}, M_{n,j} q_{n,j} \rangle_{H_{n,j}}. \quad (3.13.17)$$

Writing $q_{n,j} = \sum_{t \in X} c_{n,j}(t) k(s, t)$, assuming $c_{n,j}(t) = 0$ if $k_{n,j}(t, s) = 0$, and using (3.13.3), the above computation can be continued.

$$\langle [\mathcal{M}_*^\varphi]_Q [\mathcal{M}_\varphi]_Q \rangle_Q = \sum_j \sum_n \phi(t) c_{n,j}(s) \psi(s) c_{n,j}(t) k_{n,j}(s, t). \quad (3.13.18)$$

Let

$$k(s, t) = \sum_j \sum_n c_{n,j}(s) c_{n,j}(t) k_{n,j}(s, t), \quad (3.13.19)$$

and let $H$ denote the Hilbert space of $\mathbb{C}$ valued functions with reproducing kernel $k$ determined by

$$\langle k(\cdot, t), k(\cdot, s) \rangle = k(s, t). \quad (3.13.20)$$

From (3.13.18), (3.13.19), and (3.13.20), it follows that

$$\langle [\mathcal{M}_*^\varphi]_Q [\mathcal{M}_\varphi]_Q \rangle_Q = \left\langle \sum_t \phi(t) k(\cdot, t), \sum_s \psi(s) k(\cdot, s) \right\rangle. \quad (3.13.21)$$

That is, we have a unitary map $U: \mathcal{A}(Q) \to H$ given by

$$U[\mathcal{M}_*^\varphi]_Q = \sum_s \psi(s) k(\cdot, s) \quad (3.13.22)$$

such that, for $a \in \mathcal{D}$,

$$U(\mathcal{M}_*^\varphi)_Q = M_{*}^a U. \quad (3.13.23)$$
It follows that $U\mathcal{I}(Q)$ is the span of $\{k(\cdot, s): s \notin Y\}$. Thus, if we let $P$ denote the orthogonal projection of $H$ onto the orthogonal complement of $U\mathcal{I}(Q)$ in $H$, then, for each $a \in \mathcal{D},$

$$P_Q(\mathcal{M}_a^*)_Q P_Q = PM_a^*P \text{ unitarily equivalent.} \quad (3.13.24)$$

To finish the proof, for $s \in X$, let $r(s) = \sum |c_{n_j}(s)|^2$ and let $d(s) = r(s)^{-1/2}$ if $r(s) > 0$, and $d(s) = 0$ if $r(s) = 0$. In the notation of (3.13.19), let $c'_{n_j}(s) = d(s) c_{n_j}(s)$. It follows that

$$k'(s, t) = \sum_{j,n} c'_{n_j}(s) c'_{n_j}(t) k_n(s, t)$$

$$= \sum_{j,n} d(s) c_{n_j}(s) d(t) c_{n_j}(t) k_n(s, t)$$

$$= d(s) d(t) k(s, t). \quad (3.13.25)$$

From (3.13.10)(4) $k'$ is in $\mathcal{F}$. Hence there exists $\alpha_0$ such that $k' = k_{\alpha_0}$. Let $H_{\alpha_0}$ denote the corresponding Hilbert space as in (3.13.20). Notice that $d(s) = 0$ implies $k(s, s) = 0$. Hence, from (3.13.6), it follows that $W: H_{\alpha_0} \to H$ given by

$$Wk_{\alpha_0}(\cdot, t) = d(t) k(\cdot, t) \quad (3.13.26)$$

is a unitary operator and, for $a \in \mathcal{D},$

$$M_a^* W = WM_a^* \quad (3.13.27)$$

From (3.13.26) it follows that $W^*$ maps $U\mathcal{I}(Q)$, the span of $\{k(\cdot, t): t \notin Y\}$, onto $[Y]_{\alpha_0}$, the span of $\{k_{\alpha_0}(\cdot, t): t \notin Y\}$. In the notation of (3.13.11) we have, for $a \in \mathcal{D},$

$$PM_a^* P = P_{[X]_{\alpha_0} \oplus [X \setminus Y]_{\alpha_0}} M_a^* P_{[X]_{\alpha_0} \oplus [X \setminus Y]_{\alpha_0}}. \quad (3.13.28)$$

Using (3.13.24), (3.13.28) and (3.13.12) it follows that

$$\|P_Q(\mathcal{M}_a^*)_Q P_Q\| = \|PM_a^* P\|$$

$$= \|P_{[X]_{\alpha_0} \oplus [X \setminus Y]_{\alpha_0}} M_a^* P_{[X]_{\alpha_0} \oplus [X \setminus Y]_{\alpha_0}}\|$$

$$\leq \sup \{\|M_a^* P_{[Y]_x}\|: x \in \Gamma\}. \quad (3.13.29)$$

To complete the proof, notice, using $\mathcal{M}_a^* = \bigoplus M_{a_\alpha}^*$ acting on $\mathcal{H} = \bigoplus H_{\alpha_0}$, $\{k_{\alpha_0}\}$ is dense in $\{k_{\alpha}: \alpha \in \Gamma\}$, $[Y]_x$ is invariant for $M_{a_\alpha}^*$, $p(s) = \phi(s)$ for $s \in Y$, (3.13.29), and (3.13.15)
\[
\|M^*_{\mu}\| = \sup \{ \|M^*_{\mu}\|_{\alpha} : f \}
\]
\[
= \sup \{ \|M^*_{\mu}\|\alpha : \alpha \in \Gamma \}
\]
\[
\geq \sup \{ \|M^*_{\mu}P(\nu)\|\alpha : \alpha \in \Gamma \}
\]
\[
= \sup \{ \|A^*_{\mu}P(\nu)\|\alpha : \alpha \in \Gamma \}
\]
\[
\geq \sup \{ \|P_{Q}(M^*_{\mu})P_{Q}\| : Q \in \mathcal{T}^+(\mathcal{H}) \}
\]
\[
= \|A^*_{\mu}\|.
\]
(3.13.30)

Whence the inequalities in (3.13.30) are equalities. This proves the Theorem.

4. INTERPOLATION ON MULTIPLY CONNECTED DOMAINS

A version of Nevanlinna–Pick Interpolation on the unit disc, or more generally a nice multiply connected domain in \( \mathbb{C} \) can be obtained from (0.3) and the Wold decomposition for the disc, and (0.3) and the Abrahamse Douglas version of the Wold decomposition for the more general domains. The results are similar to those obtained by Abrahamse [1] and Ball [5]. The essential difference is that the factorization of an \( L^2 \) function as a product of \( H^1 \) and conjugate \( H^\infty \) (with some twisting) is replaced by the intimately related Wold decomposition. To obtain a concrete theorem from (0.3) in these cases it is necessary to view a trace class operator on a vector valued \( L^2 \) space as an integral operator. This section begins with a discussion of this construction for \( H^2 \) of a multiply connected domain. Next a version of Nevanlinna–Pick interpolation is obtained; and then refined using the Wold decomposition. In subsequent sections many of the results in this section are generalized to a uniform algebra which has a good representation as an algebra of operators on an \( L^2 \) space.

Let \( \Omega \) denote a nice multiply connected domain in \( \mathbb{C} \), where by nice we mean the \( \Omega \) is open connected bounded and its boundary \( \partial \Omega \) consists of \( g+1 \) disjoint simple closed analytic arcs. Let \( A(\Omega) \) denote those functions which are continuous on the closure of \( \Omega \) and analytic in \( \Omega \). Let \( ds \) denote arclength measure on \( \partial \Omega \) and \( H^2(\partial \Omega) \) the closure of \( A(\Omega) \) in \( L^2(ds) \). \( H^2(\partial \Omega) \) has a reproducing kernel which we will denote, without reference to \( \Omega \), by \( k \); i.e., for each \( z \in \Omega \), \( k = k(\cdot, z) \in H^2(\partial \Omega) \) and, for every \( f \in H^2(\partial \Omega) \),

\[
f(z) = \langle f, k(\cdot, z) \rangle = \int_{\partial \Omega} f(k(\cdot, z)) \ ds. \tag{4.1}\]

Let \( H^2(\Omega) \) denote the closed linear span of the set \( \{ k(\cdot, z) : z \in \Omega \} \). It is well known, but not immediate, that the map defined by (4.1) gives a unitary equivalence between \( H^2(\partial \Omega) \) and \( H^2(\Omega) \).

For a positive integer \( l \) let \( H^2_l(\Omega) \) denote the direct sum of \( H^2(\Omega) \) with itself \( l \) times. \( H^2_l(\Omega) \) has reproducing kernel \( k^l \) the \( l \times l \) matrix valued function with \( k \) in each diagonal position and 0 elsewhere. Let \( H^2(0) \) denote \( H^2(0) \) and \( L^\infty(\text{ds}) \) and let \( H^2_{\text{ess}}(\Omega) \) denote the \( l \times l \) matrices with entries from \( H^2(\Omega) \). For notational ease, let \( A = H^2_{\text{ess}}(\Omega) \). View \( A \) as an algebra of multiplication operators on \( H^2_l(\Omega) \) by associating the operator \( M_f \) on \( H^2_l(\Omega) \) defined by \( M_f = f \cdot A \). In this way \( A \) is a subalgebra of \( L^\infty(H^2_l(\Omega)) \).

The following lemma identifies the norm of an element \( f \in A \) and shows that \( A \) can be identified with \( H^2(0) \) defined in section three. Proofs are included for completeness.

**4.2. Lemma.** Let \( \phi \) be an \( l \times l \) matrix with entries from \( H^2(\Omega) \) be given. If \( r < \text{ess} \sup \{ \| \phi(\zeta) \| : \zeta \in \Omega \} \), then there exists a bounded function \( f \in H^2_l(\Omega) \) such that \( \| f \| = 1 \) and \( \| \phi f \| \geq r \). If \( \phi \in A \), then

\[
\| M_\phi \| = \text{ess} \sup \{ \| \phi(\zeta) \| : \zeta \in \partial \Omega \}. \tag{4.2.1}
\]

Moreover, \( A = H^\infty(0) \).

**Proof.** We merely outline a proof. For notational ease, denote \( \text{ess} \sup \{ \phi(\zeta) : \zeta \in \partial \Omega \} \) by \( \| \phi \|_\infty \). Fix \( r, \delta > 0 \) such that \( r < r + \delta < \| \phi \|_\infty \). The set

\[
E = \{ \zeta : \| \phi(\zeta) \| > r + \delta \} \tag{4.2.2}
\]

has positive measure. Given \( \varepsilon > 0 \), there exists an \( N \) and unit vectors \( v_j \in \mathbb{C}^l \), \( j = 1, \ldots, N \), such that for every unit vector \( v \in \mathbb{C}^l \) there exists a \( j \) such that

\[
\| v_j - v \| \leq \varepsilon. \tag{4.2.3}
\]

Let

\[
E_j = \{ \zeta : \| \phi(\zeta) v_j \| \geq r \}. \tag{4.2.4}
\]

For almost every \( \zeta \in E \) there exist a unit vector \( v_\zeta \) such that \( \| \phi(\zeta) v_\zeta \| \geq r + \delta \). There exists \( j_\zeta \) such that \( \| v_j - v_\zeta \| \leq \varepsilon \). Hence, for almost every \( \zeta \),

\[
\| \phi(\zeta) v_\zeta \| \geq r + \delta - \varepsilon \| \phi \|_\infty. \tag{4.2.5}
\]

With \( \varepsilon = \delta/\| \phi \|_\infty \), it follows that for almost every \( \zeta \in E \) there exists \( j_\zeta \) such that \( \zeta \in E_{j_\zeta} \). Hence there exists a \( j_\zeta \) such that \( E_{j_\zeta} \) has positive measure. Fix such a \( j \) and let \( w = v_j \) and \( F = E_{j_\zeta} \).
Given \( \varepsilon > 0 \), there exists a closed set \( C \) and an open set \( O \) such that \( C \subset F \subset O \) and the measure of \( O \setminus C \) is less than \( \varepsilon \). Since \( A(\Omega) \) is approximating in modulus, the argument of (5.13) produces \( f \in A(\Omega) \) such that \( |f|^2 < 1 + \varepsilon \) on \( \partial \Omega \), \( ||f|^2 - 1| < \varepsilon \) on \( C \) and \( |f|^2 < \varepsilon \) on the complement of \( O \). We have
\[
\int_{\partial \Omega} |f|^2 \, ds \leq \int_C (1 + \varepsilon) \, ds + \int_{\partial \Omega \setminus C} (1 + \varepsilon) \, ds + \int_{\partial \Omega \setminus O} \varepsilon \, ds \\
\leq (1 + \varepsilon) \int_C \, ds + 2 \varepsilon + \varepsilon^2. \tag{4.2.6}
\]

We can now estimate using (4.2.6)
\[
\|\phi f\| = \int \|\phi f\|^2 |f|^2 \, ds \\
\geq \int_C \|\phi f\|^2 (1 - \varepsilon) \, ds \\
\geq \int_C r(1 - \varepsilon) \, ds \\
\geq \frac{(\|f\|^2 - 2 \varepsilon - \varepsilon^2)}{1 + \varepsilon} (1 - \varepsilon) r. \tag{4.2.7}
\]

Hence
\[
\frac{\|\phi f\|}{\|f\|} \geq \frac{(\|f\|^2 - 2 \varepsilon - \varepsilon^2)}{\|f\|^2(1 + \varepsilon)} (1 - \varepsilon) r. \tag{4.2.8}
\]

Choosing \( \varepsilon \) small proves the first part of the Lemma.

If \( \phi \in \mathcal{A} \), then \( M_\phi \) is a bounded operator on \( H^\gamma(\Omega) \). Hence, given \( r < \|\phi\|_\infty \) and a unit vector \( f \in H^\gamma(\Omega) \) such that \( \|\phi f\| \geq r \), obtains
\[
\|M_\phi \| \geq \|M_\phi f\| \geq r. \tag{4.2.9}
\]

Thus, \( \|M_\phi \| \geq \|\phi\|_\infty \). The other inequality is easily verified. This proves the second part of the Lemma.

If \( \phi \in H^{\omega}(k_j) \), then it follows, by considering the constant (vector valued) functions in \( H^\gamma(\Omega) \), that \( \phi \in H^\gamma(\Omega) \). \( \phi \in H^{\omega}(k_j) \) means, by definition, \( M_\phi \) is a bounded operator on \( H^\gamma(\Omega) \). Hence, arguing as above, \( \|M_\phi \| \geq \|\phi\|_\infty \). Whence, \( \phi \in \mathcal{A} \).

An additional construction is needed before we can state the central result of this section. Given \( Q \in \mathcal{F}^\gamma(H^\gamma(\Omega)) \), we wish to view \( Q \) as an
integral kernel. Since $Q$ is a positive trace class operator, there exists $q_\rho \in H_\Sigma^2(\Omega)$ such that $Q = \sum q_\rho \rho^*$. Identify $Q$ with the $l \times l$ matrix valued function on $\partial \Omega$

$$Q(\zeta) = \sum q_\rho(\zeta) \rho(\zeta)^*. \tag{4.3a}$$

The sum in (4.3) converges (coordinate-wise) in $H_\Sigma^2(\Omega)$ and $Q(\zeta)$ is, for almost every $\zeta$, a positive (semidefinite) matrix.

Let $H_\Sigma^\infty(\Omega)$ denote the direct sum of $H_\Sigma^\infty(\Omega)$ with itself $l$ times. Let $\mathcal{P}_j^+$ denote those positive $l \times l$ matrix valued functions $Q$ on $\partial \Omega$ for which there exists a constant $C_Q$ such that

$$|f^*Qf| ds \leq C_Q |f^*f| ds \tag{4.3b}$$

for every $f \in H_\Sigma^\infty(\Omega)$, where $Q^*$ is the pointwise transpose of $Q$. Given $Q \in \mathcal{P}_j^+$, let $H_\Sigma^\infty(\Omega)$ denote the Hilbert space of functions obtained by closing up $H_\Sigma^\infty(\Omega)$ in the inner product

$$\langle f, g \rangle = \int g^*Qf| ds. \tag{4.3c}$$

Let $H_\Sigma^\infty(\Omega)$ denote the orthogonal direct sum of $H_\Sigma^\infty(\Omega)$ with itself $l$ times. Given $Q \in \mathcal{P}_j^+$ and $\phi \in \mathcal{A}$, we define an operator on $H_\Sigma^\infty(\Omega)$ as follows. For $f = (f_j)_{j=1}^l \in H_\Sigma^\infty(\Omega)$, let

$$|M_{\phi, Q}f| = \left( \sum_s |\phi_{s,j} f_s| \right)_{j=1}^l. \tag{4.4a}$$

To see that $M_{\phi, Q}$ is in fact a bounded operator, note that, from (4.4a), for any $f \in H_\Sigma^\infty(\Omega)$,

$$\|M_{\phi, Q}f\|_{H_\Sigma^\infty(\Omega)} \leq \sum_s \|\phi_{s,j}\|_{\infty} \|f_s\|
\leq l^2 \max_s \|\phi_{s,j}\|_{\infty} \|f\|_{H_\Sigma^\infty(\Omega)}. \tag{4.4b}$$

Given $f \in H_\Sigma^\infty(\Omega)$, let $f_j$ denote the transpose of the $j$th row of $f$, so that $f_j$ is a column vector. Let

$$\tilde{f} = \bigoplus_{j=1}^l f_j, \tag{4.5}$$

thought of as a column vector (so an element of $H_\Sigma^\infty(\Omega)$). Given $Q \in \mathcal{O}^+(H_\Sigma^\infty(\Omega))$ the following lemma shows $Q \in \mathcal{O}^+$ as well. Consider $[f]_Q \rightarrow \tilde{f}$ is a map from $\mathcal{A}$ into $H_\Sigma^\infty(\Omega)$ as a densely defined map on $\mathcal{A}(\Omega)$. In particular the lemma says that $f \mapsto \tilde{f}$ is a unitary map which intertwines
Given $x_1, \ldots, x_n \in \Omega$, let $H^2(\Omega)$ denote the closure in $H^2_0(\Omega)$ of those functions in $H^2_0(\Omega)$ which vanish at each $x_j$.

4.6. Lemma. In the above notation,

$$[f]_Q \to \hat{f}$$

is a unitary map (and so extends to all of $\mathcal{A}(Q)$) onto $H^2(Q)$ such that

$$(M_\phi)_Q [f]_Q = M_\phi, Q \hat{f}$$

for all $\phi \in \mathcal{A}$. In particular $Q \in \mathcal{P}^+$ and $M_\phi, Q$ is a bounded operator. Moreover, given $x_1, \ldots, x_n \in \Omega$, let $\mathcal{I}$ denote the ideal of those $\phi \in \mathcal{A}$ such that $\phi(x_j) = 0$ for each $j$. Then (4.6.1) maps $\mathcal{I}(Q)$ onto $H^2_0(Q)$.

Proof. Given $f \in \mathcal{A}$ let $f_j$ denote the transpose of its $j$th row, $f^t$ its $t$th column, and $Q_{s,t}$ the $(s, t)$ entry of the matrix function $Q$ and write $Q = \sum q_{s,t} q_{s,t}^*$ as in (4.3) and let $q_{s,t}$ denote the $j$th entry of $q_{s,t}$. For $f, g \in \mathcal{A}$ we have

$$\langle [f]_Q, [g]_Q \rangle_Q = \text{trace}(f Q g^*) = \sum_j \sum_{s,t} Q_{s,t} f_j g^*_j q_{s,t}^*$$

Thus, writing $\hat{f}$ as in (4.5) it follows that the map $\mathcal{A}(Q) \to H^2(Q)$ densely defined by on $\mathcal{A}$ by $f \to \hat{f}$ is unitary. Thus, we have

$$\|f\|_2 \|Q\|_1 \geq \text{trace}(f Q g^*) = \|f\|_Q^2 = \|f\|_{H^2_0(Q)}^2,$$

which shows $Q \in \mathcal{P}^+$. Moreover, for $\phi, f \in \mathcal{A}$, the formula (4.6.2) is readily verified; and consequently, $M_\phi, Q$ is a bounded operator.

Given $f \in \mathcal{A}$, $f \in \mathcal{I}$ if and only if $f \in H^2_0(\Omega)$. This completes the proof.

We can now state our main result of this section.
4.7. Theorem. Let points $x_1, \ldots, x_n \in \Omega$ and $l \times l$ matrices $w_1, \ldots, w_n$ be given. Let $\phi$ denote any function in $\mathcal{A}$ such that $\phi(x_j) = w_j$. There exists $\psi \in \mathcal{A}$ such that $\psi(x_j) = w_j$ and

$$\left\| \psi \right\| = \inf \left\{ \left\| \phi \right\| : \phi \in \mathcal{A}, \phi(x_j) = w_j \right\} = \sup \left\{ \left\| M_{\psi, Q}^* P_{V_0} \right\| : Q \in \mathcal{F}_I^+ \right\}, \tag{4.7.1}$$

where $P_{V_0}$ denotes the orthogonal projection of $H^2_0(Q)$ onto $H^2_0(Q) \oplus H^2_0(Q)$. In particular, there exist $\psi \in \mathcal{A}$ such that $\psi(x_j) = w_j$ and $\left\| \psi \right\| \leq 1$ if and only if the supremum in (4.7.1) is at most one.

Proof. Recall from (4.2) that $\mathcal{A}$ is a weak* closed subalgebra of $\mathcal{L}(H^2_0(Q))$ which contains the identity. Let $\mathcal{I}$ denote the elements of $\mathcal{A}$ which vanish at each $x_j$. From section three $\mathcal{I}$ is weak* closed in $\mathcal{A}$. Let $\phi$ denote any element of $\mathcal{A}$ which satisfies $\phi(x_j) = w_j$. An application of (0.3) produces a $\psi \in \mathcal{A}$ such that $\psi - \phi \in \mathcal{I}$ and

$$\left\| \psi \right\| = \inf \left\{ \left\| \phi - \psi \right\| : \phi \in \mathcal{I} \right\} = \sup \left\{ \left\| P_{\mathcal{I}}(M_{\psi}^*) Q P_{V_0} \right\| : Q \in \mathcal{F}^+(H^2_0(Q)) \right\}. \tag{4.7.2}$$

Given $Q \in \mathcal{F}^+(H^2_0(Q))$, from Lemma 4.6 there exists a unitary map $W : \mathcal{A}(Q) \to H^2_0(Q)$ which carries $\mathcal{I}$ (onto $H^2_0(Q)$, which satisfies, for each $\phi \in \mathcal{A}$,

$$W(M_{\psi}) = M_{\psi} Q W. \tag{4.7.3}$$

This means that, in (4.7.1), the supremum is at least as big as the infimum.

Given $Q \in \mathcal{F}_I^+$ and $\phi \in \mathcal{I}$, $M_{\psi, Q}$ maps the dense subset $H^2_0(Q)$ of $H^2_0(Q)$ into $(H^2_0(Q))$. Hence,

$$M_{\psi, Q} P_{V_0} = 0. \tag{4.7.4}$$

Thus,

$$\left\| M_{\psi, Q}^* P_{V_0} \right\| = \left\| M_{\psi, Q} P_{V_0} \right\|. \tag{4.7.5}$$

Since

$$\left\| M_{\psi, Q}^* P_{V_0} \right\| \leq \left\| M_{\psi, Q} \right\|, \tag{4.7.6}$$

to finish the proof it suffices to show that $\left\| M_{\psi, Q} \right\| \leq \left\| \psi \right\|$. To this end, recall that $\left\| \psi \right\| = \left\| \psi \right\|_{\infty}$. Hence, for $f = (f_j)_{j=1}^l \in H^2_0(Q)$,
\[ \| M_{\psi, Qf} \|^2 = \left( \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \psi_{s,t} f_s \right) \]
\[ = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \psi_{s,t}^* Q(\psi_{s,t} f_s) \, ds \]
\[ = \int \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \psi_{s,t}^* Qf_s \, ds \]
\[ = \int \sum_{s=0}^{\infty} \psi_{s,t}^* Qf_s \, ds, \quad (4.7.7) \]

where \( \psi = \psi^* \psi \) pointwise. Consequently,

\[ \| \psi \|^2 \| f \|^2 - \| M_{\psi, Qf} \|^2 \]
\[ = \int \left( \| \psi \| I - \psi \right) \otimes Qf_s \, ds, \]
\[ = \int \left( \| \psi \| I - \psi \right) \otimes Q \left( \begin{array}{c} f_1 \\ \vdots \\ f_n \end{array} \right) \left( \begin{array}{c} f_1^* \\ \vdots \\ f_n^* \end{array} \right) \, ds, \quad (4.7.8) \]

where \( \delta_{s,t} \) is one if \( s = t \) and zero otherwise, and \( I \) denotes the identity matrix. Since \( \| \psi \|^2 \) is pointwise at most \( \| \psi \| \), \( \| \psi \| I - \psi \) is pointwise positive. Since \( Q \) is also pointwise positive, it follows that the integrand above (4.7.8) is pointwise positive. The result follows.

Theorem 4.7 can be refined by use of the Wold decomposition. We first take up the case where \( \Omega = D \), the unit disc, and merely indicate the modifications needed for the general case.

For simplicity, we write \( H^2 \) for \( H^2(D) \) etc. Let \( S \) denote the operator of multiplication by \( \zeta \) on \( H^2 \). \( S \) is known as the unilateral shift. A version of the Wold decomposition suitable for our purposes is the following.

4.8. Proposition. If \( V \), an operator on a Hilbert space \( H \), is an isometry and there exists vectors \( \gamma_1, \ldots, \gamma_l \) such that the span of the set \( \{ V_t \gamma_j : j \geq 0, 1 \leq t \leq l \} \) is dense in \( H \), then there exists \( m \leq l \), a Hilbert space \( K \), a unitary operator \( W \) on \( K \) and a unitary operator \( U : H \to H^2_m \otimes K \) such that

\[ UV = (S^{(m)} \otimes W) U, \quad (4.8.1) \]

where \( S^{(m)} \) denote the direct sum of \( S \) with itself \( m \) times.

We note that the \( m \) in the Wold decomposition is the dimension of the orthogonal difference \( H \otimes VH \).
Recall, given \( \phi \in H^\infty \), \( M_{\phi, Q} \) denotes the operator of multiplication by \( \phi \) on \( H^2(Q) \) defined, densely on \( H^2 \), by
\[
\phi f = (\phi f)_i.
\]

4.9. Lemma. Let \( Q \in \mathcal{F} \) be given. In the notation of (4.8), there exists \( m \leq l \), and a unitary map \( U : H^2(Q) \to H^2_m \) such that, for all \( \phi \in H^\infty \),
\[
UM_{\phi, Q} = \phi(S)^{(m)} U.
\]
Moreover, given \( x_1, \ldots, x_n \in \mathbb{D} \), \( U \) maps \( H^2(Q) \) onto \( (H^2_m)_x \), where \( H^2(Q)_x \) is the closure of those function in \( H^2 \) which vanish at each \( x_j \) in \( H^2(Q) \).

Proof. Recall. Let \( V \) denote the operator of multiplication by \( \xi \) on \( H^2(Q); V = M_{\xi, Q} \). Then \( V \) is an isometry and the vectors \( \{ e_j : 1 \leq j \leq l \} \) are cyclic for \( V \), where \( e_j \) denotes the function in \( H^2 \) which is 1 in the \( j \)-th entry and 0 elsewhere. From the Wold decomposition, there exists an \( m \leq l^2 \), a Hilbert space \( K \) a unitary \( W \) on \( K \), and a unitary \( U : H^2(Q) \to H^2_m \otimes K \) such that \( UV = (S^{(m)} \otimes K) U \). It follows that for any \( \phi \in H^\infty \),
\[
U \phi(V) = (\phi(S)^{(m)} \otimes \phi(W)) U.
\]

We now wish to show that there is no unitary summand \( W \). In general, an isometry \( T \) which has no reducing subspace—a subspace which is invariant for both \( T \) and \( T^* \)—is called a pure isometry. Thus we are to show that \( V \) is a pure isometry. The first step is to see that \( V \) is pure in the case that \( Q = qq^* \), for some vector \( q = (q_j)_{j=1}^l \in H^2 \), so that \( Q(\xi) = (q_i(\xi) q_j(\xi))_{i,j=1}^{l} \), where as before \( Q' \) is the pointwise transpose of \( Q(\xi) \). In this case, given \( f = (f_j)_{j=1}^l \) and \( g = (g_j)_{j=1}^l \) in \( H^\infty \),
\[
\langle f, g \rangle_Q = \int \sum_{i,j} g_i^* q_j f_j q_i \, ds = \int \sum_{i,j} g_i q_j, f_j q_i \, ds = \left( \sum_{j} f_j q_j \, ds, \sum_{j} g_j q_j \right)_{H^2}.
\]
Hence the map
\[
X_q : H^2(qq^*) \to H^2
\]
densely defined on $H_i^\infty$ by
\[ X_i f = \sum q_j f_j \] (4.9.4)
is an isometry which intertwines $V$ with the shift $S$. Thus $V$ is unitarily equivalent, via $X_i$, to $S$ restricted to the shift invariant subspace $X_i H^2(qq^*)$ of $H^2$. Since $S$ is a pure isometry it follows that $V$ is a pure isometry.

For the general case, we argue as follows. There exist vectors $q_n \in H^2_i$ such that $Q = \sum q_n q_n^*$. The map
\[ X: H^2(Q) \to \bigoplus_{n=1}^{\infty} H^2, \] (4.9.5)
defined for $f \in H^2_i$ by $Xf = \bigoplus Xq_i f$ is an isometry which intertwines $V$ and $S$. The operator $\bigoplus S$ is a pure isometry, since it is the direct sum of the shift. Arguing as above, it follows that $V$ is a pure isometry.

Thus we have that the isometry $U$ maps $H^2(Q)$ onto $H^2_m$ and, for every $\phi \in H^\infty$
\[ U\phi(V) = \phi(S)^{(m)} U. \] (4.9.6)
Hence
\[ U\phi(V) e_j = \phi(S)^{(m)} U e_j. \] (4.9.7)
Let $H^\infty_x$ denote those $f \in H^\infty$ which vanish at each $x_j$. By choosing $\phi \in H^\infty_x$, it follows from (4.9.7) that $U$ maps $H^\infty_x$ into $(H^2_m)_x$, functions in $H^2_x$ which vanish at each $x_j$. Thus $U$ maps $H^2(Q)_x$ into $(H^2_m)_x$. By considering $U^* = U^{-1}$ instead of $U$ and arguing as above we see that $U^*$ maps $(H^2_m)_x$ into $H^2(Q)_x$. The moreover follows.

The following Theorem is a version of matrix valued interpolation for the unit disc.

4.10. Theorem. Let distinct points $x_1, \ldots, x_n \in \mathbb{D}$ and $l \times l$ matrices $w_1, \ldots, w_n$ such that $\|w_j\| < 1$ be given. There exists $\psi \in \mathcal{A}$ such that $\psi(x_j) = w_j$ and $\|\psi\| \leq 1$ if and only if the Pick matrix
\[ ((I_i - w_i w_i^*) k(x_i, x_j)) \] (4.10.1)
is positive semidefinite, where $I_i$ is the $l \times l$ identity.
Proof. Recall $\mathcal{A} = H^2_{\mathbb{T}}$ and $\mathcal{I}$ denotes those functions in $A$ which vanish at each $x_j$. Let $\phi \in \mathcal{A}$ denote any function such that $\phi(x_j) = w_j$. From Theorem 4.7, there exists $\psi$ such $\psi(x_j) = w_j$ and

$$\|\psi\| = \sup\{\|P_{\mathcal{H}_Q}(M_{\phi}^*)_{\mathcal{Q}} P_{\mathcal{I}}\| : Q \in \mathcal{F}^*(H^2)\},$$

(10.4.2)

where $P_{\mathcal{I}}$ is the orthogonal projection of $H^2(Q)$ onto $\mathcal{N}_Q = H^2(Q) \cap H^2(Q)_Q$. It turns out that there is essentially only one term in the supremum above and determining its norm produces the Pick matrix of (4.10.1).

From Lemma 4.9 there exists an $m \leq l$, and a unitary map $U$ from $H^2(Q)$ onto $H^2_{\mathbb{T}}$ such that, in the notation of (4.9),

$$U|_{M_{\phi}} = \varphi(S)^{(m)} U$$

(10.3.3)

for $\phi \in \mathcal{A}$. Under $U$, $H^2(Q)_Q$ is mapped onto $(H^2_{m})_Q$. Let $U_i$ denote the direct sum of $U$ with itself $l$ times. Then $U_i$ is a unitary map from $H^2(Q)$ onto $(H^2_{m})$, such that

$$U_i|_{M_{\phi}} = M_{\phi}^{l_i} U_i$$

(10.4.4)

for $\phi \in \mathcal{A}$. Under $U_i$, $H^2(Q)_Q$ is mapped onto $(H^2_{m})_Q$. Hence

$$P_{H^2(Q)} M_{\phi}^{l_i} P_{\mathcal{I}} = M_{\phi}^{l_i} P_{\mathcal{I}}$$

unitarily equivalent, (10.5.5)

where $\mathcal{N} = (H^2_{m}) \cap ((H^2_{m})_Q)$ and $P_{\mathcal{I}}$ is the corresponding orthogonal projection. Notice that $\mathcal{N}$ can also be written as

$$(H^2_{m}) \cap ((H^2_{m})_Q) = \oplus (H^2_{m}) \cap ((H^2_{m})_Q).$$

(10.6.6)

Let $P$ denote the orthogonal projection of $H^2_{\mathbb{T}}$ onto $H^2 \cap (H^2_{m})_{x}$. With respect to the decomposition (10.6.6) $P_{\mathcal{I}}$ is the direct sum of $m$ copies of $P$ and for $\phi \in \mathcal{A}$ we have the unitary equivalence,

$$M_{\phi}^{l_i} P_{\mathcal{I}} = \bigoplus_{i} M_{\phi}^{l_i} P_i$$

(10.7.7)

where $M_{\phi}$ is the operator of multiplication by $\phi$ on $H^2_{\mathbb{T}}$. Hence, from (10.7.7), $\|M_{\phi}^{l_i} P_{\mathcal{I}}\| = \|M_{\phi}^{l_i} P_i\|$. Let $k$ denote the Szego kernel for the unit disc. To finish the proof observe that, for any vectors $v_1, \ldots, v_m$ in $C^l$,

$$\left| \sum_{i} M_{\phi}^{l_i} k(\cdot, x_i) v_i \right|^2 = \left| \sum_{i} k(\cdot, x_i) \phi(x_i)^* v_i \right|^2$$

$$= \sum_{i} v_i^* \phi(x_i) \phi(x_i)^* v_i k(x_i, x_i).$$

(10.8.8)
Similarly,
\[ \left\| \sum k(\cdot, x_j) v_j \right\|^2 = \sum v^*_j v_j k(x_i, x_j). \]  
(4.10.9)

Since vectors of the form \( k(\cdot, x_j) v_j \) are dense in \( H^2_l \otimes (H^2_l)^\ast \), it follows that \( \|M_{\phi} P\| \leq 1 \) if and only if
\[ \sum v^*_j ((I - \phi(x_i) \phi(x_j)^\ast) k(x_i, x_j)) v_j \geq 0 \]  
(4.10.10)
for all choices of vectors \( v \); if and only if
\[ ((I - \phi(x_i) \phi(x_j)^\ast) k(x_i, x_j)) \]  
(4.10.11)
is positive semidefinite. This completes the proof.

Now let \( \Omega \) denote a nice multiply connected domain. The bundle shifts of rank \( l \) are the pure \( l \)-cyclic subnormal operators whose minimal normal extension has spectrum in the boundary of \( \Omega \) [2]. These bundle shifts are parameterized, up to unitary equivalence, by representations of the fundamental group of \( \Omega \) into the space of \( l \_l \) unitary matrices modulo conjugation by a fixed unitary. For \( Q \in \mathcal{P}_l^+ \), the operator equal to multiplication by \( \zeta \) on \( H^2(Q) \) is a bundle shift of rank \( m \leq l \). The argument used for the disc above (4.10) shows that the supremum in (4.7) can be taken over the (compact) collection of bundle shifts. We note that the assignment of a bundle shift to multiplication by \( \zeta \) on \( H^2 \) is a factorization of \( Q \).

5. Interpolation in Uniform Algebras

This section describes a version of Nevanlinna–Pick interpolation in a uniform algebra which is approximating in modulus and which has a good representation as a collection of operators on a reproducing kernel Hilbert space, where good is made precise below. This generalizes the results of the previous section. In particular, trace class operators again give rise to integral operators. For simplicity, we focus on the scalar valued case \( (l = 1) \), and discuss the matrix valued case only briefly in the next section. The results in this section reveal two aspects of (0.3). First, it may be that the algebra \( A \) can be represented as an algebra of operators on several different Hilbert spaces. Each choice of Hilbert space obtains, at least superficially if not substantively, a different theorem. Second, in concrete examples, the proof of (0.3) often requires only an appropriately chosen subcollection of positive trace class operators, instead of every trace class operator.
operator. As an example, the results apply to $H^\infty$ of the polydisc represented as an algebra of multiplication operators on $H^2(D')$. This is discussed at the end of this section. The results here are similar to those of Cole, Lewis and Wermer [11].

Let $X$ denote a compact Hausdorff space. A uniform algebra $\mathfrak{A}$ is a uniformly closed subalgebra of $C(X)$ which contains the constant functions and separates points of $X$. The Silov boundary of $\mathfrak{A}$, denoted $\partial \mathfrak{A}$, is the smallest closed subset of $X$ such that

$$\max \{|f(x)| : x \in X\} = \max \{|f(x)| : x \in \partial \mathfrak{A}\},$$

(5.1)

for every $f \in \mathfrak{A}$.

Given $m$, a (positive, regular, Borel) measure on the uniform algebra $\mathfrak{A} \subset C(X)$, let $\mathfrak{A}^2(m)$ denote the closure of $\mathfrak{A}$ in $L^2(m)$. If, for $x \in X$, the map, defined densely on $\mathfrak{A}$, by

$$\mathfrak{A}^2(m) \ni f \to f(x)$$

(5.2)

is bounded on $\mathfrak{A}^2(m)$, $x$ is said to be an $m$-bounded point evaluation and we identify $x$ with the map (5.2). If $x$ is an $m$-bounded point evaluation then there exists a unique $k(\cdot, x) \in \mathfrak{A}^2(m)$ such that for every $f \in \mathfrak{A}^2(m)$

$$f(x) = \int f(k(\cdot, x)) \, dm = \langle f, k(\cdot, x) \rangle.$$  

(5.3)

Let $\mathfrak{A}^\infty(m)$ denote the algebra $\mathfrak{A}$ viewed as a subspace of $L^\infty(m)$. Then

$$\mathfrak{A}^\infty(m) = \{ f \in L^1(m) : \int af \, dm = 0 \text{ for every } a \in \mathfrak{A}^\infty(m) \}. $$

(5.4)

5.5. HYPOTHESIS. Suppose $\mathfrak{A} \subset C(X)$ is a uniform algebra and $m$ is a measure on $\partial \mathfrak{A}$. Here are two conditions we will, for the most part, assume $m$ satisfies.

1. $C(\partial \mathfrak{A}) \cap \mathfrak{A}^\infty(m)$ is dense in $\mathfrak{A}^\infty(m)$ (in the $L^1(m)$ norm); and
2. the linear span of the functions $k_m(\cdot, x)$, for $m$-bounded point evaluations $x$, is dense in $\mathfrak{A}^2(m)$.

Given $\mathfrak{A} \subset C(X)$ a uniform algebra and $m$ a measure on $\partial \mathfrak{A}$, let $X_m$ denote the set of $m$-bounded point evaluations. Let $k'_m : X_m \times X_m \to \mathbb{C}$ given by

$$k'_m(y, x) = k_m(y, x).$$

(5.6)
Let $H'$ denote the reproducing kernel Hilbert space of functions on $X_m$ with reproducing kernel $k'_m$ determined by

$$
\langle k'_m(\cdot, y), k'_m(\cdot, x) \rangle = k'_m(x, y).
$$

(5.7)

That is $H'$ is the Hilbert space obtained by closing up finite linear combinations of $\{k'_m(\cdot, x) : x \in X_m\}$ in the norm induced by (5.7). There is a natural mapping of $\mathfrak{U}(m)$ onto $H'$ which associates the function $\phi$ on $X_m$ defined by

$$
\hat{\phi}(x) = \langle \phi, k(\cdot, x) \rangle.
$$

(5.8)

to $\phi \in \mathfrak{U}(m)$. If condition (5.5)(2) is satisfied then this mapping defines a unitary equivalence between $\mathfrak{U}(m)$ and $H'$. Recall, from (3.2), the definition of $H^{\infty}(k'_m)$ as the bounded multiplication operators on $H'$.

$\mathfrak{A}$ is approximating in modulus, if, for every nonnegative continuous function $g$ on $\partial \mathfrak{A}$ and $\varepsilon > 0$ there exists $f \in \mathfrak{A}$ such that

$$
||f(x)| - g(x)|| < \varepsilon \text{ for every } x \in \partial \mathfrak{A}.
$$

5.9. Proposition. Let $\mathfrak{A} \subset C(X)$ be a uniform algebra which is approximating in modulus and suppose $m$ is a measure on $\partial \mathfrak{A}$ which satisfies both conditions of (5.5). Let $\mathcal{A}$ denote the weak* closure of $\mathfrak{U}(m)$ in $L^{\infty}(m)$. Then

1. $\mathcal{A} = \mathfrak{U}(m) \cap L^{\infty}(m)$;
2. if $\phi \in \mathcal{A}$, then $\hat{\phi} \in H^{\infty}(k'_m)$;
3. the mapping

$$
\mathcal{A} \ni \phi \mapsto \hat{\phi} \in H^{\infty}(k'_m)
$$

(5.9.1)

is an isometric algebra homomorphism of $\mathcal{A}$ onto $H^{\infty}(k'_m)$.

Before stating the principle result of this section, we need to introduce some additional notations. Let $\mathfrak{A} \subset C(X)$ be a uniform algebra and let $m$ be a measure on $\partial \mathfrak{A}$. Let $X_m$ denote the $m$-bounded point evaluations. For each $a \in \mathfrak{A}$ such that $|a| > 0$ on $\partial \mathfrak{A}$, let $\mathfrak{U}(a)$ denote the Hilbert space obtained by closing up $\mathfrak{A}$ in the norm induced from the inner product

$$
\langle f, g \rangle_a = \int f \bar{g} |a|^2 \, dm.
$$

(5.10)
Evidently, $\mathcal{U}^2(m)$ and $\mathcal{U}^2(a)$ are equal as sets and, for each $x \in X_m$, there exists $k_a(\cdot, x) \in \mathcal{U}^2(a)$ such that
\[
\langle f, k_a(\cdot, x) \rangle_a = f(x)
\] (5.11)
for all $f \in \mathcal{U}^2(a)$. We can now state the principle result of this section.

5.12. Theorem. Suppose $\mathcal{U} \subset C(X)$ is a uniform algebra which is approximating in modulus and $m$ is a measure on $\partial \mathcal{U}$ which satisfies both conditions of (5.5). Let $X_m$ denote the set of $m$-bounded point evaluations and let $H'$ and $K_m(x)$ denote the corresponding Hilbert space and reproducing kernel as in (5.7). Suppose $x_1, \ldots, x_n \in X_m$ and that the kernels $k(\cdot, x_j) \in \mathcal{U}$. Let $k_a(x)$ denote the kernels from (5.11). If $\phi \in A$, then there exists $\rho \in H^\infty(K_m(x))$ such that $|\rho| \leq 1$ and $\rho(x_j) = \phi(x_j)$ if and only if the Pick matrices
\[
((1 - w_j w_j^*) k_a(x_i, x_j))_{i,j=1}^n
\] (5.12.1)
are positive semidefinite for every $a \in \mathcal{U}$ such that $|a| > 0$ on $\partial \mathcal{U}$.

5.12.2. Remark. As discussed in detail below, the result applies to the polydisc algebra represented as multiplication operators on $H^2$ of the polydisc. In this case $\mathcal{U}$ is $H^\infty$ of the polydisc. The Theorem can also be applied to multiply connected domains in $\mathbf{C}$ to recover scalar valued interpolation.

For notational ease, let $\|\phi\|_{m, \infty}$ denote the $L^\infty(m)$ norm of $\phi \in L^\infty(m)$; and let $\|\phi\|_2$ denote the norm of $\phi$ in $L^2(m)$. We begin with the following lemma.

5.13. Lemma. Let $\phi \in \mathcal{U}^2(m)$ be given. If $r < \|\phi\|_{m, \infty}$ then there exists $a \in \mathcal{U}$ such that $\|a\|_2 = 1$ and $\|a \phi\|_2 \geq r$. Here we don’t need to assume $m$ satisfies (5.5)(1); but we do assume that $\mathcal{U}$ and $m$ otherwise are as in (5.12).

Proof. Pick $\delta > 0$ such that $r < r + \delta < \|\phi\|_{m, \infty}$. Let
\[
E = \{ z \in \partial \mathcal{U} : |\phi(z)\| \geq r + \delta \}. 
\] (5.13.1)
Then $m(E) > 0$. Since $m$ is regular, there exists a closed set $C$ and an open set $O$ such that $C \subset E \subset O$ and $m(O \setminus C) < \varepsilon$. From the Tietze Extension Theorem there exists $f : \partial \mathcal{U} \to [0, 1]$ such that $f = 0$ on $K$ and $f = 1$ on $C$. Since $\mathcal{U}$ is approximating in modulus, there exists $a \in \mathcal{U}$ such that $|a|^2 - f^2 < \varepsilon$. We have
\[ |a|^2 > 1 - \varepsilon^2 \quad \text{on } C \]
\[ |a|^2 < 1 + \varepsilon^2 \quad \text{on } O \]
\[ |a|^2 < \varepsilon^2 \quad \text{on } \partial \mathbb{H} \setminus O. \quad (5.13.2) \]

Hence
\[
\|a\|^2 \leq \int_O + \int_{\partial \mathbb{H} \setminus O} |a|^2 \, dm \\
\leq (1 + \varepsilon^2)(m(C) + m(O \setminus C)) + \varepsilon^2 m(\partial \mathbb{H} \setminus O) \\
\leq (1 + \varepsilon^2)m(C) + \varepsilon((1 + \varepsilon) + \varepsilon m(\partial \mathbb{H})). \quad (5.13.3)
\]

We have
\[
\int |\phi| \frac{|a|^2}{\|a\|^2} \, dm \geq \frac{1}{\|a\|^2} \int_C |\phi|^2 |a|^2 \, dm \\
\geq \frac{1}{\|a\|^2} (1 - \varepsilon^2)(r + \delta)m(C) \\
\geq \frac{(1 - \varepsilon^2)(r + \delta)m(C)}{(1 + \varepsilon^2)m(C) + \varepsilon((1 + \varepsilon) + \varepsilon m(\partial \mathbb{H}))}. \quad (5.13.4)
\]

The result follows by choosing \( \varepsilon > 0 \) small enough.

5.14. Proof of Proposition 5.9. Begin by noting that
\[
\mathcal{A} = (L^1(m)/\mathcal{A}^{\infty}(m))^* \\
= (\mathcal{A}^{\infty}(m))^\perp \\
= (\mathcal{A}^*)^\perp. \quad (5.14.1)
\]

Let \( \phi \in \mathcal{A} \) and \( \psi \in \mathcal{A}^{\infty}(m) \) be given. To show that \( \phi \psi \in \mathcal{A} \) it is enough to show that
\[
\int \phi \psi f \, dm = 0, \quad (5.14.2)
\]
for every \( f \in \mathcal{A}^{\infty}(m), f \in \mathcal{A}^{\infty}(m) \) if and only if
\[
\int f \, dm = 0 \quad (5.14.3)
\]
for every $a \in \mathcal{A}(m)$. Hence, if $f \in \mathcal{A}(m)$, then, since $\mathcal{A}(m)$ is an algebra, it follows that for every $a \in \mathcal{A}(m)$

$$\int f \psi a \, dm = 0.$$  \hfill (5.14.4)

It follows that $f \psi \in \mathcal{A}(m)$. By combining (5.14.2) and (5.14.4) and using the fact that $\psi \in \mathcal{A}$, it follows that $f \psi \in \mathcal{A}$.

Fix $\phi \in \mathcal{A}$. Since $\phi \in L^\infty(m)$, it defines a bounded operator on $L^2(m)$ by $f \mapsto \phi f$. To see that this operator leaves $\mathcal{F}(m)$ invariant fix $a \in \mathcal{A}$. From above $\phi a \in \mathcal{A}$; to see it is in fact in $\mathcal{A}(m)$, let $b \in L^2(m) \subset \mathcal{F}(m)$ be given. It is immediate that $\phi b \in \mathcal{A}(m)$; i.e., $\phi a$ is orthogonal to $b$. Hence $\phi a \in \mathcal{A}(m)$. In particular $+$, choosing $a = 1$ shows $\phi \in \mathcal{A}(m)$. This shows $\mathcal{A} \subset \mathcal{F}(m) \cap L^\infty(m)$.

Now suppose $\phi \in \mathcal{F}(m) \cap L^\infty(m)$. To see that $\phi$ is in fact in $\mathcal{A}$, we need to show that

$$\int \phi f \, dm = 0$$  \hfill (5.14.5)

for every $f \in \mathcal{A}(m) \subset L^1(m)$. From (5.5)(1), it is enough to show that (5.14.5) holds for every bounded $f \in \mathcal{A}(m)$. But such an $f$ satisfies $f \in L^2(m) \subset \mathcal{F}(m)$, from which it now follows that $\phi \in \mathcal{A}$. We conclude $\mathcal{A} = \mathcal{F}(m) \cap L^\infty(m)$.

Fix $\phi \in \mathcal{A}$ and let $M_\phi$ denote the operator on $H^2(m)$ given by $M_\phi f = \phi f$. It is evident that the norm of $M_\phi$ is at most the norm of $\phi \in \mathcal{A}$. On the other hand, Lemma 5.13 implies that the norm of $M_\phi$ is at least the norm of $\phi \in \mathcal{A}$. Let $M_\phi$ denote the operator of multiplication by $\phi$ on $H$. $M_\phi$ is unitarily equivalent to $M_\phi$ as seen by the computation, for $\psi \in \mathcal{A}$,

$$(\phi \psi)(x) = \langle \phi \psi, k(\cdot, x) \rangle$$  

$$= \langle M_\phi \psi, k(\cdot, x) \rangle$$  

$$= \langle \phi, M_\phi^* k(\cdot, x) \rangle$$  

$$= \hat{\psi}(x) \langle \phi, k(\cdot, x) \rangle$$  

$$= \hat{\psi}(x) \phi(x).$$  \hfill (5.14.6)

It follows that

$$\|\phi\|_{m, \infty} = \|\hat{\phi}\|_{H^\infty(\mathcal{A})},$$  \hfill (5.14.7)
To see that \( \phi \rightarrow \hat{\phi} \) is onto \( H^\infty(k'_m) \), let \( \psi \in H^\infty(k'_m) \) be given. There exists \( \varphi \in \mathfrak{H}(m) \) such that \( \hat{\phi} = \psi \). Moreover, for each \( a \in \mathfrak{A} \),
\[
\|\hat{\phi}\|_{H^\infty(k'_m)} \|a\|_{H^\infty} \geq \|\hat{a}\|_{H^\infty}.
\]
Thus, from Lemma 5.13,
\[
\|\phi\|_{H^\infty(k'_m)} \geq \|\hat{\varphi}\|_{\mathfrak{H}(m)}. \quad (5.14.8)
\]
Hence \( \varphi \in \mathfrak{A} = L^\infty(m) \cap \mathfrak{H}(m) \).

5.15. Proof of Theorem 5.12. The proof consists using Proposition 5.9 to view \( H^\infty(k'_m) \) as a subalgebra of \( L^\infty(m) \) rather than of \( \mathfrak{A}(\mathfrak{H}(m)) \) and applying the duality arguments of section two. At a crucial stage, we'll use the approximating in modulus hypothesis to factor a positive \( C(\mathfrak{H}) \) function which corresponds to the positive trace class operator \( Q \) in the polar factorization of the trace class operator \( G \) as \( G = VQ \) in section two.

Let \( I \) denote those \( a \in H^\infty(k'_m) \) such that \( a(x_j) = 0 \) for \( j = 1, \ldots, n \). Identify \( I \) with the corresponding ideal in \( \mathfrak{A} \). Since \( L^\infty(m) \) is the dual of \( L^1(m) \), we have, in the notation of (0.3)
\[
\mathfrak{A} = L^1(m)/\mathfrak{A}
\]
where
\[
\mathfrak{A} = \{ g \in L^1(m) : \langle a, g \rangle = 0 \text{ for every } a \in \mathfrak{A} \}.
\]
Observe that \( \mathfrak{A} = L^\infty(k'_m) \) as in (5.14.1); and hence the hypothesis (5.6)(2) implies that \( C(\mathfrak{H}) \cap \mathfrak{A} \) is dense in \( \mathfrak{A} \). We now follow the proof of (0.3) applied to \( \mathfrak{A} \), \( \mathfrak{I} \), and any \( \phi \in \mathfrak{A} \) such that \( \hat{\phi}(x_j) = w_j \), making appropriate approximations along the way.

Let \( \pi \) denote the quotient map
\[
\pi : L^1(m) \rightarrow \mathfrak{A}
\]
and let
\[
\mathfrak{I} = \{ F \in \mathfrak{A} : a(F) = 0 \text{ for every } a \in \mathfrak{A} \}.
\]
Let \( L \) denote the linear map
\[
L : \mathfrak{I} \rightarrow \mathbb{C}
\]
\[
L(F) = \phi(F).
\]
We wish to apply the Hahn–Banach Theorem to \( L \).
If \( F \in \mathcal{A} \), then there exists constants \( c_j \) such that, for \( \psi \in \mathcal{A} \),

\[
\psi(F) = \sum c_j \hat{\psi}(x_j)
\]

\[
= \sum \left< \psi, \sum c_j k(\cdot, x_j) \right>.
\]

(5.15.6)

where \( k(\cdot, x_j) \) is the reproducing kernel for \( \mathcal{A}^2(m) \). Hence, if \( G \in L^1(m) \) and \( \pi(G) = F \), then there exists \( G_0 \in \mathcal{A} \) such that

\[
G = \sum c_j k(\cdot, x_j) + G_0.
\]

(5.15.7)

Fix \( F \in \mathcal{A} \) and \( \varepsilon > 0 \). There exists \( G \in L^1(m) \) such that \( \pi(G) = F \) and \( \|G\|_1 < \|F\| + \varepsilon \). Writing \( G \) as in (5.15.7), we can choose \( H_0 \in C(\partial \mathcal{A}) \cap \mathcal{A} \) such that,

\[
H = \sum c_j k(\cdot, x_j) + H_0 \in C(\partial \mathcal{A})
\]

(5.15.8)

satisfies both

\[
\|H\|_1 \leq \|F\| + 2\varepsilon
\]

\[
\pi(H) = F.
\]

(5.15.9)

Finally, since \( \mathcal{A} \) is approximating in modulus, there exists \( a \in \mathcal{A} \) such that

\[
|H| + \varepsilon > |a|^2 > |H| + \frac{\varepsilon}{2}
\]

(5.15.10)

on \( \partial \mathcal{A} \).

Let \( V \) denote the function \( V = H/|a|^2 \). If \( \psi \in \mathcal{A} \), then, since \( H \in \mathcal{A} \),

\[
\langle \psi, V \rangle_a = \int \psi \hat{V} |a|^2 \, dm
\]

\[
= \langle \psi, H \rangle
\]

\[
= \psi(H) = 0.
\]

(5.15.11)

Thus, \( V \) is orthogonal to \( \mathcal{A}^2(a)_x \). Let

\[
\mathcal{N}_{\varepsilon, a} = \mathcal{A}^2(a) \ominus \mathcal{A}^2(a)_x
\]

(5.15.12)
and let $P_{\nu_{x,a}}$ denote the orthogonal projection of $\mathcal{A}(a)$ onto $\mathcal{N}_{x,a}$. We have $P_{\nu_{x,a}} V = P_{\nu_{x,a}} V$. Thus

$$
\phi(F) = \langle \phi, H \rangle \\
= \langle \phi, V |a|^2 \rangle \\
= \langle \phi, P_{\nu_{x,a}} V \rangle_a \\
= \langle \phi, P_{\nu_{x,a}} V \rangle_a \\
= \langle 1, M^*_a P_{\nu_{x,a}} V \rangle_a \\
= \langle 1, P_{\nu_{x,a}} M^*_a P_{\nu_{x,a}} V \rangle_a.
$$

(5.15.13)

Hence

$$
|\phi(F)| \leq \|P_{\nu_{x,a}} M^*_a P_{\nu_{x,a}} \| \|V\|_a.
$$

(5.15.14)

Estimate, using $|V| \leq 1$ and (5.15.10) and (5.15.9)

$$
\|V\|_a^2 = \int V \bar{V} |a|^2 \, dm \\
\leq \int |a|^2 \, dm \\
\leq \|H\|_1 + \varepsilon \int dm \\
\leq \|F\| + \varepsilon \left(2 + \int dm \right).
$$

(5.15.15)

Similarly, estimate

$$
\|1\|_a^2 \leq \|F\| + \varepsilon \left(2 + \int dm \right).
$$

(5.15.16)

Combining (5.15.14), (5.15.15) and (5.15.16) and letting $\varepsilon$ tend to 0 obtains

$$
|\phi(F)| \leq \|F\| \sup \{ \|P_{\nu_{x,a}} M^*_a P_{\nu_{x,a}} \| : a \in \mathcal{A} \}
$$

and $|a| > 0$ on $\partial \mathcal{A}$. (5.15.17)

The remainder of the proof now proceeds as in (0.3), but with the supremum in (5.15.17). We obtain $\rho \in \mathcal{A}$ such that

$$
\hat{\rho}(x_j) = \hat{\phi}(x_j) = w_j
$$

(5.15.18)
and the norm of $\rho$ is given by the supremum above, and this is the smallest possible norm given (5.15.18).

The supremum in (5.15.17) must be connected with the right hand side of (5.12.1) to finish the proof. Now, in the notation of (5.11), the vectors $\{k_n(\cdot, x_j): j = 1, \ldots, n\}$ span the space $\mathcal{N}_{e,d}$ defined in (5.15.12). Moreover, these are eigenvectors for $M^*_a$, the adjoint of multiplication by $\phi$ on $\mathcal{H}^2(a)$, with

$$M^*_a k_n(\cdot, x_j) = \hat{\phi}(x_j) k(\cdot, x_j).$$  

Hence, the norm of $P_{\mathcal{H}^2(a)} M^*_a P_{\mathcal{H}^2(a)} \otimes \mathcal{H}^2(a)$ is less than one if and only if the corresponding Pick matrix in (5.12.1) is positive semidefinite. Thus, if all of the Pick matrices of (5.12.1) are positive semidefinite, then $\|\rho\| \leq 1$. The converse is similar.

We end this section by applying (5.12) to the polydisc algebra. Let $\mathbb{D}^l$ denote $l$ copies of the unit disc and let $A = A(\mathbb{D}^l)$ denote the algebra of functions which are continuous on the closure of $\mathbb{D}^l$ and analytic in $\mathbb{D}^l$ considered as a uniform subalgebra of continuous functions on the closure of $\mathbb{D}^l$. The Silov boundary of $A$ is the $l$ fold torus,

$$T^l = \{(t_1, \ldots, t_l) \in \mathbb{C}^l: |t_j| = 1 \text{ for each } j\}.$$

Let $H^2 = H^2(\mathbb{D}^l)$ denote the Hardy space of functions analytic in the interior of $\mathbb{D}^l$ in the inner product

$$\int_{T^l} f g \, ds,$$  

where $ds = (1/(2\pi l)) d\theta_1 \, d\theta_2 \ldots d\theta_l$ (Haar measure) on $T^l$. $H^2$ has reproducing kernel $k$ simply the $l$ fold (tensor) product of the usual Szego kernel:

$$k(\zeta, z) = \Pi \frac{1}{1 - \zeta_j \bar{z}_j}.$$  

Thus, for each $\zeta = (\zeta_1, \ldots, \zeta_l)$ with $|\zeta_j| < 1$, $k(\zeta, z) \in A$. Let $\mathbb{Z}^l$ denote $l$ tuples of integers and let $\mathbb{Z}^l_+$ denote those tuples all of whose terms are non-negative. Let $Y^n$ denote the complement of $\mathbb{Z}^l_+$ in $\mathbb{Z}^l$. Let $\mathscr{Y}$ denote the $L^1(ds)$ closure of the span of the set

$$\{z_1^{x_1} \ldots z_l^{x_l}: x = (x_1, \ldots, x_l) \in Y^n\}.$$  

The subspace of $L^\infty(ds)$ given by

$$\mathscr{Y}^\perp = \left\{ f \in L^\infty(ds): \int f g ds = 0 \text{ for every } g \in \mathscr{Y} \right\}$$  

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is $H^\infty(\mathbb{D})$, since it consists precisely of those $f \in L^\infty(ds)$ whose Fourier coefficients

$$f_x = \int f\overline{x_1^1 \overline{x_2^2} \cdots \overline{x_l^l}} ds$$

vanish whenever $x \in \mathcal{Y}$. Since $\mathcal{Y}$ is closed,

$$\mathcal{Y} = \overline{\mathcal{Y}}$$

$$= H^\infty(\mathbb{D})$$

$$= A^\infty(ds). \quad (5.20)$$

Thus, $C(T^1) \cap A^\infty(ds)$ is dense in $A^\infty(ds)$. A discussion of the approximating in modulus property of $A$ can be found in [7]. Thus, (5.12) can be applied to $A$ and $H^2(\mathbb{D})$. Moreover, in this case the weak* closure of $A$ is $H^\infty(\mathbb{D})$ and the norm of an element of $H^\infty$ as an operator on $H^2$ is its $H^\infty$ norm.

We remark that (5.12) implies that (scalar valued) Nevanlinna–Pick interpolation on a polydisc depends only upon the principle ideals in the polydisc algebra. This is the same as for a multiply connected domain. In both cases these principle ideals correspond to the cyclic Silov modulus for the corresponding uniform algebras. Multiplicity is needed to treat matrix valued interpolation.

6. Matrix Interpolation in a Uniform Algebra

In this section we briefly discuss matrix valued interpolation for a uniform algebra. Let $\mathfrak{A} = C(X)$ denote a uniform algebra with Silov boundary $\partial \mathfrak{A}$. Throughout, we assume that $\mathfrak{A}$ is approximating in modulus and $m$ is an NP measure on $\mathfrak{A}$. For notational ease, let $k = k_m$ denote the corresponding kernel of (5.7).

Fix $l$, a positive integer. Let $\mathfrak{A}^l(l)$ denote the orthogonal direct sum of $\mathfrak{A}^l(m)$ with itself $l$ times. $\mathfrak{A}^l(l)$ has reproducing kernel $k_l$, the matrix valued function with $k$ in each diagonal entry and 0 elsewhere. Let $H^\infty(l)$ denote the $l \times l$ matrices with entries from $H^\infty$. Let $\mathfrak{A}$ denote $H^\infty(l)$ viewed as an algebra of operators on $\mathfrak{A}^l(l)$ by assigning the operator $M$, equal to multiplication by $\phi$, to $\phi \in \mathfrak{A}$. Using Lemma 5.13 and arguing as in Lemma 4.2, it follows that $\mathfrak{A} = H^\infty(l_1)$ and the norm of $\phi \in \mathfrak{A}$ is the same as the norm of $\phi \in L^\infty(l_1)$.

Given $Q \in \mathcal{F}^\times(\mathfrak{A}^l(m))$ there exists vectors $q_\phi \in \mathfrak{A}^l(m)$ such that $Q = \sum q_\phi q_\phi^*$. Let $Q$ also denote the matrix valued function $Q(x) = \sum q_\phi(x) q_\phi^*(x)$. 

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This sum converges, coordinate-wise, in $\mathfrak{A}^2(m)$ and, for almost every $\zeta$, $Q(\zeta)$ is a positive matrix.

Let $H^+_{\gamma}(k)$ denote the direct sum of $H^{\gamma}(k)$ with itself $l$ times. Given $f \in H^+_{\gamma}(k)$ we can view $f$ as an element of $\mathcal{A}$ by embedding it as a row vector in the first row. Hence, arguing as in (4.5)-(4.6), if $Q \in \mathcal{T}+(\mathfrak{A}^2(m))$, then

$$
\|[f]\|_2^2 = \sum_p \langle f q_n, f q_n \rangle = \sum q_n * f * f q_n dm = \int f * Q * f dm \leq C \int f * f dm,
$$

where $C = \|Q\|_1$, the trace norm of $Q$, and $Q(\zeta)$ is the pointwise transpose of $Q(\zeta)$.

Now let $\mathcal{P}^+_{\gamma}$ denote those positive matrix valued functions on $\partial \mathfrak{A}$ such that there exists a $C$ such that (6.1) holds for every $f \in H^+_{\gamma}(k)$. Given $Q \in \mathcal{P}^+_{\gamma}$, let $H^+_2(Q)$ denote the closure of $H^+_{\gamma}(k)$ in the inner product,

$$
\langle f, g \rangle_Q = \int g * Q f dm.
$$

Given $\phi \in \mathcal{A}$, let $\mathfrak{M}_{\phi, Q}$ denote the operator on $H^+_2(Q)$ defined by

$$
\mathfrak{M}_{\phi, Q} f = \left( \sum \phi_j * f_j \right)_{j=1}^l,
$$

where $f = (f_j) \in H^+_2(Q)$. As in Section four, $\mathfrak{M}_{\phi, Q}$ defines a bounded operator.

From (6.1) it follows that if $Q \in \mathcal{T}+(\mathfrak{A}^2(m))$, then $Q(\zeta) \in \mathcal{P}^+_{\gamma}$. Moreover, the map $f \to \hat{f}$ defined as in (4.5) determines a unitary map from $\mathfrak{A}^2(\zeta)$ onto $H^+_2(Q)$ which intertwines $(M_\phi)_Q$ and $\mathfrak{M}_{\phi, Q}$. Given $m$-bounded point evaluations $x_1, ..., x_n \in X$, let $(\mathfrak{A}^2(\zeta))_x$ and $(H^+_2(Q))_x$ denote the closure of the functions in $H^+_{\gamma}(k)$ which vanish at each $x_j$ in $\mathfrak{A}^2(m)$ and $H^+_2(\zeta)$ respectively. The map $\hat{f}$ sends $(\mathfrak{A}^2(\zeta))_x$ onto $(H^+_2(Q))_x$. We can now state the main result of this section.
6.4. THEOREM. Given m-bounded point evaluations $x_1, \ldots, x_n \in X$ and $\phi \in \mathcal{A}$, there exists $\psi \in \mathcal{A}$ such that $\psi(x_j) = \phi(x_j)$ for $1 \leq j \leq n$ and

$$
\|\psi\| = \inf \{ \|\phi\| : \phi \in \mathcal{A}, \phi(x_j) = w_j \}
$$

$$
= \sup \{ \|M_{x_j}^* Q P_{x_j}\| : Q \in \mathcal{P}^+ \},
$$

where $P_{x_j}$ denotes the orthogonal projection of $H_2^2(Q)$ onto $H_2^2(Q) \ominus (H_2^2(Q))_f$.

The proof is the same as that of (4.7).

REFERENCES