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Journal of Functional Analysis 263 (2012) 1408–1434

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Ellipticity in pseudodifferential algebras of Toeplitz type

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Received 23 November 2011; accepted 3 June 2012

Available online 12 June 2012

Communicated by C. De Lellis

Abstract

Let $L^\star$ be a filtered algebra of abstract pseudodifferential operators equipped with a notion of ellipticity, and $T^\star$ be a subalgebra of operators of the form P_1AP_0, where $P_0, P_1 \in L^\star$ are projections, i.e., $P_j^2 = P_j$. The elements of $L^\star$ act as linear continuous operators in a scale of abstract Sobolev spaces, those of $T^\star$ in the corresponding subspaces determined by the projections. We study how the ellipticity in $L^\star$ descends to $T^\star$, focusing on parametrix construction, equivalence with the Fredholm property, characterisation in terms of invertibility of principal symbols, and spectral invariance. Applications concern *SG*-pseudodifferential operators, operators on manifolds with conical singularities, and Boutet de Monvel's algebra for boundary value problems. In particular, we derive invertibility of the Stokes operator with Dirichlet boundary conditions in a subalgebra of Boutet de Monvel's algebra. We indicate how the concept generalizes to parameter-dependent operators.

Keywords: Pseudodifferential operators of Toeplitz type; Stokes operator

1. Introduction

One of the major ideas in the theory of pseudodifferential operators is to study existence and regularity of solutions to partial differential equations in terms of a parametrix construction within an algebra of pseudodifferential operators. For example, given a μ-th order differential operator A on a closed smooth manifold M whose homogeneous principal symbol is pointwise invertible on the unit co-sphere bundle of M, one can construct a pseudodifferential operator B of

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0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2012.06.003

order $-\mu$ such that both $AB-1$ and $BA-1$ are smoothing operators, i.e., are integral operators with a smooth integral kernel. Such an operator B is called a parametrix of A. Concerning the partial differential equation $Au=f$ this has two important consequences:

(1) *Fredholm property*: If $H^s(M)$ denotes the standard L_2-Sobolev space of regularity s, the operator $A: H^s(M)\to H^{s-\mu}(M)$ is a Fredholm operator, i.e., its kernel is finite-dimensional and its range is finite co-dimensional. Hence for any f satisfying a finite number of orthogonality conditions, the (affine) solution space is finite-dimensional.
(2) *Elliptic regularity*: If f has regularity $s-\mu$ then any solution u which has some a priori regularity t must have regularity s.

The first property holds since smoothing operators are compact operators in any $H^s(M)$, the second is due to the fact that $B: H^{s-\mu}(M)\to H^s(M)$ and due to the smoothing property of the remainders.

This concept – to embed differential operators in a class of pseudodifferential operators and to construct parametrices to elliptic elements to obtain the Fredholm property and elliptic regularity of solutions – has by now been realized for a huge variety of different kinds of differential operators. Just to name a very few, let us mention Boutet de Monvel's algebra for boundary value problems [1], Schulze's calculi for manifolds with conical singularities, edges or higher singularities [19,3], and Melrose's b-calculus [10] for manifolds with corners.

Boutet de Monvel's algebra for boundary value problems on a compact manifold M with smooth boundary consists of operators of the form

$$\mathcal{A}=\begin{pmatrix} A_+ + G & K\\ T & Q\end{pmatrix}: \begin{matrix} H^s(M,E_0)\\ \oplus\\ H^s(\partial M,J_0)\end{matrix} \to \begin{matrix} H^{s-\mu}(M,E_1)\\ \oplus\\ H^{s-\mu}(\partial M,J_1)\end{matrix} \tag{1}$$

(of arbitrary order $\mu\in\mathbb{R}$), where E_j and J_j are vector bundles over M and ∂M, respectively, which are allowed to be zero-dimensional. Here, A_+ is the "restriction" to M of a pseudodifferential operator A defined on the double of M, G is a so-called singular Green operator, K is a potential operator, T is a trace operator, and Q is a usual pseudodifferential operator on the boundary. H^s refers to the L_2-Sobolev spaces. For further information on this calculus we refer the reader to the existing literature, for instance [1,6,13,20], or [17] where a nice and short introduction can be found. For example the Dirichlet problem for the Laplacian is included in this set-up as

$$\begin{pmatrix}\Delta\\ T\end{pmatrix}: H^s(M)\to \begin{matrix} H^{s-2}(M)\\ \oplus\\ H^{s-2}(\partial M)\end{matrix}, \quad s>\frac{1}{2},$$

where $Tu=S(u|_{\partial M})$ with an invertible pseudodifferential operator S of order $3/2$ on the boundary. The parametrix (which in fact is an inverse in this case) is then of the form $(\,A_+ + G \quad K\,)$, where $u=(A_+ + G)f$ solves $\Delta u=f$ in M and $u|_{\partial M}=0$, while $u=K\varphi$ is harmonic in M and $u|_{\partial M}=S^{-1}\varphi$. Ellipticity in Boutet de Monvel's algebra is determined by the invertibility of both the homogeneous principal symbol of A and the principal boundary symbol associated with $\mathcal{A}$. For differential problems this corresponds to Shapiro–Lopatinskij ellipticity of boundary value problems.

Many boundary value problems fit into the framework of Boutet de Monvel's algebra, but there are important exceptions. For example, Dirac operators in even dimension cannot be completed with a boundary condition to be an elliptic element in Boutet de Monvel's algebra. The Dirichlet condition $u \mapsto u|_{\partial M}$ in this case cannot be used directly, but instead the celebrated Atiyah–Patodi–Singer boundary condition $u \mapsto P(u|_{\partial M})$ where P is the positive spectral projection of the tangential operator associated with the Dirac operator. In [23] it is shown that in fact any elliptic differential operator can be completed to a Fredholm problem by such kind of boundary conditions involving projections. Motivated by this fact Schulze in [20] extended Boutet de Monvel's calculus in such a way that this type of boundary conditions is included. In this extended calculus the operators are of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} A_+ + G & K \\ T & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P_0 \end{pmatrix}, \tag{2}$$

where P_j are zero-order pseudodifferential operators on the boundary which are projections, i.e., $P_j^2 = P_j$. Defining the closed subspaces

$$H^s(\partial M, J_j, P_j) := P_j\bigl(H^s(\partial M, J_j)\bigr)$$

of $H^s(\partial M, J_j)$, operators of the form (2) are considered as a maps

$$\begin{matrix} H^s(M, E_0) \\ \oplus \\ H^s(\partial M, J_0, P_0) \end{matrix} \rightarrow \begin{matrix} H^{s-\mu}(M, E_1) \\ \oplus \\ H^{s-\mu}(\partial M, J_1, P_1) \end{matrix}. \tag{3}$$

A concept of ellipticity is developed and it is shown that elliptic operators have a parametrix of analogous structure, i.e., as in (2) but with P_0 and P_1 interchanged. For elliptic operators the map (3) is Fredholm, and one has elliptic regularity in the scale of projected subspaces. In the works [14,15] of Savin and Sternin index formulas for boundary value problems with conditions in so-called even or odd subspaces over the boundary are derived. In [21] and [22] Schulze and the author realized a similar calculus for boundary value problems without transmission property and operators on manifolds with edges, respectively.

In the present paper we study such kind of calculi involving projections from a general point of view. We consider a filtered algebra $L^\star$ of "abstract" pseudodifferential operators equipped with a notion of ellipticity that is equivalent to the existence of a parametrix modulo smoothing operators, cf. Section 2 for details. Let us denote by $T^\star$ the subalgebra of operators of the form $P_1 A P_0$, where A belongs to $L^\star$ and $P_0, P_1 \in L^\star$ are projections, i.e., $P_j^2 = P_j$. We call $T^\star$ a *Toeplitz subalgebra* in view of the structure of classical Toeplitz operators, which are of the form PMP where P is the orthogonal projection of $L_2(S^1)$ onto the Hardy space of functions that extend holomorphically to the unit-disc, and M is the operator of multiplication by a bounded function on the unit-circle. We shall assume that the elements of $L^\star$ act as linear continuous operators in certain scales of "abstract" Sobolev spaces. Applying the projections to these spaces yields a natural scale of closed subspaces in which the elements of the Toeplitz subalgebra act. Recall that the algebra of operators from (2) fits in this general framework. In Section 3 we show that ellipticity in $L^\star$ naturally induces a concept of ellipticity in the Toeplitz subalgebra $T^\star$. Assuming that $L^\star$ is closed under taking adjoints and that ellipticity is equivalent to the Fredholm mapping property, we show that also the induced ellipticity in $T^\star$ is equivalent to the Fredholm mapping property. Moreover, if ellipticity in $L^\star$ is characterized by the invertibility of certain "abstract" principal symbols, we discuss how the corresponding symbolic structure looks like in the Toeplitz subalgebra. Also we give a sufficient condition ensuring the spectral invariance of Toeplitz subalgebras. In Section 3.5 we indicate how our approach can be extended to parameter-dependent operators; however, the parameter-dependent case is not the main focus of this paper and we plan to investigate it further in a separate publication.

In the last three sections of this article we discuss various concrete examples that are all covered by our approach. The first concerns classical *SG*-pseudodifferential operators on $\mathbb{R}^n$, cf. [12,2,3]; roughly speaking these are pseudodifferential operators whose symbols have asymptotic expansions into homogeneous components both with respect to the variable and corresponding co-variable, cf. Section 4 for the details (by passing to the radial compactification of $\mathbb{R}^n$, i.e., the closed n-dimensional unit ball, the class of *SG*-pseudodifferential operators described here corresponds to the class of Melrose's scattering operators on the unit ball, cf. [11]). Actually this calculus can also be adapted to cover more general non-compact manifolds, cf. [16]; for simplicity of presentation we restrict ourselves to the Euclidean space. In Section 5 we consider Schulze's cone algebra [19] of pseudodifferential operators on manifolds with conical singularities (note that this calculus is closely related with Melrose b-calculus [10], cf. [9] for a comparison). The final Section 6 is dedicated to Boutet de Monvel's algebra. We show that the above described extended Boutet de Monvel calculus is obtained as a particular case of our general approach; in fact we can allow arbitrary projections from Boutet de Monvel's algebra and not only those of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & P \end{smallmatrix}\right)$ with pseudodifferential projections P on the boundary. This allows us to give an application concerning the Stokes operator, i.e., the Laplacian considered on divergence free (solenoidal) vector fields which plays an important role in the analysis of the Navier–Stokes equations. More precisely, if M is a smoothly bounded compact domain in $\mathbb{R}^n$ and P denotes the Helmholtz projection for M then the Stokes operator with Dirichlet boundary conditions is

$$P\Delta : H^s_\sigma\big(M, \mathbb{C}^n\big) \cap \{u \mid u|_{\partial M} = 0\} \to H^{s-2}_\sigma\big(M, \mathbb{C}^n\big), \quad s > 3/2, \tag{4}$$

where $H^s_\sigma(M, \mathbb{C}^n)$ denotes the image of $H^s(M, \mathbb{C}^n)$ under the Helmholtz projection. In [4] Giga has shown that this operator in case $s = 2$ (and even in the more general setting of L_p-Sobolev spaces) is the generator of an analytic semigroup and that its resolvent has a certain pseudodifferential structure. Grubb and Solonnikov [8,6] studied the resolvent in terms of a parameter-dependent version of Boutet de Monvel's algebra. In our context of Toeplitz subalgebras, we can view the Stokes operator as an element in a Toeplitz subalgebra of Boutet de Monvel's algebra. Taking for granted that (4) is an isomorphism for $s = 2$ (in fact, it is a self-adjoint positive operator) we will derive in Section 6.2 that it is an isomorphism for any $s > 3/2$ and we shall show that the inverse is of the form $P(A_+ + G)P$, where $A_+ + G$ belongs to Boutet de Monvel's algebra. Though this result might not be new, it follows without effort as a straightforward corollary from our general approach.

2. The general set-up

We introduce algebras of "abstract" pseudodifferential operators, incorporating some standard features frequently met in pseudodifferential analysis: asymptotic summation, parametrix and Fredholm property. Also we introduce the notion of subalgebra of Toeplitz type. The notations of this section will be used throughout the paper.

2.1. Algebras of abstract pseudodifferential operators

Let $\mathbb{N}_0$ denote the non-negative integers and consider a set of operators

$$L^\star = \bigcup_{-\mu\in\mathbb{N}_0} \bigcup_{\mathbf{g}\in\mathbf{G}} L^\mu(\mathbf{g}), \tag{5}$$

where μ represents the "order of operators" while $\mathbf{G}$ is a set of "admissible" pairs of data $\mathbf{g} = (g_0, g_1)$; at this stage $\mathbf{G}$ should be considered as data specifying the class of considered operators, and to which will be assigned a precise meaning depending on the concrete application. For a first example see Example 2.1 below.

We assume that with any single datum g there is associated a scale of Hilbert spaces

$$H^s(g), \quad s \in \mathbb{N}_0, \tag{6}$$

and that each element $A \in L^\mu(\mathbf{g})$, $\mathbf{g} = (g_0, g_1)$, induces continuous linear operators

$$A : H^s(g_0) \to H^{s-\mu}(g_1)$$

for any $s \geqslant 0$. Furthermore we ask that

$$L^\mu(\mathbf{g}) \subset L^\nu(\mathbf{g}), \quad \mu \leqslant \nu,$$

and that for two data $\mathbf{g} = (g_0, g_1)$ and $\mathbf{g}' = (g_1, g_2)$ composition of operators induces mappings

$$L^\mu\big(\mathbf{g}'\big) \times L^\nu(\mathbf{g}) \to L^{\mu+\nu}\big(\mathbf{g}' \circ \mathbf{g}\big), \quad \mathbf{g}' \circ \mathbf{g} := (g_0, g_2). \tag{7}$$

Due to this composition property we also speak – by abuse of language – of the "algebras" $L^\star$ or $L^\mu(\mathbf{g})$. The classes of "smoothing" operators are defined as

$$L^{-\infty}(\mathbf{g}) := \bigcap_{-\mu\in\mathbb{N}_0} L^\mu(\mathbf{g}).$$

Example 2.1. Let M be a smooth compact (Riemannian) manifold. A "datum" is any pair $g = (M, E)$ where E is a smooth (Hermitian) vector bundle over M. We let

$$H^s(g) = H^s(M, E)$$

denote the standard L_2-Sobolev spaces of sections into E of regularity s. For $\mathbf{g} = (g_0, g_1)$ with $g_j = (M, E_j)$ let

$$L^\mu(\mathbf{g}) = L^\mu_{\mathrm{cl}}(M; E_0, E_1)$$

be the space of classical pseudodifferential operators, mapping sections into E_0 to sections into E_1 (with "classical" we mean that the local pseudodifferential symbols have complete asymptotic expansions into homogeneous components; see Section 4 for details).

A standard feature in pseudodifferential analysis is the possibility of asymptotic summation. We incorporate this by the following definition.

Definition 2.2. We call the algebra $L^\star$ *asymptotically complete* if for any admissible $\mathbf{g}$ and any sequence of operators $A_j \in L^{-j}(\mathbf{g})$ there exists an $A \in L^0(\mathbf{g})$ such that $A - \sum_{j=0}^{N-1} A_j \in L^{-N}(\mathbf{g})$ for any positive integer N.

Next we come to ellipticity and parametrices.

Definition 2.3. An operator $A \in L^0(\mathbf{g})$, $\mathbf{g} = (g_0, g_1)$, is called elliptic if there exists an operator $B \in L^0(\mathbf{g}^{(-1)})$ with $\mathbf{g}^{(-1)} := (g_1, g_0)$ such that

$$BA - 1 \in L^{-\infty}(\mathbf{g}_0), \qquad AB - 1 \in L^{-\infty}(\mathbf{g}_1),$$

where $\mathbf{g}_0 = (g_0, g_0)$, $\mathbf{g}_1 = (g_1, g_1)$. Any such B is called a parametrix of A.

Obviously parametrices are uniquely determined modulo smoothing operators. As a consequence of a Neumann series argument the existence of a parametrix in case of asymptotic completeness is equivalent to the existence of left- and right-inverses modulo operators of order -1: The operator $A \in L^\mu(\mathbf{g})$ is elliptic if there exist $B_0, B_1 \in L^{-\mu}(\mathbf{g}^{(-1)})$ such that $B_0 A - 1 \in L^{-1}(\mathbf{g}_0)$ and $AB_1 - 1 \in L^{-1}(\mathbf{g}_1)$. For details see the proof of Proposition 3.2, below.

The fact that parametrices are inverses modulo smoothing operators implies elliptic regularity of associated equations: If $A \in L^0(\mathbf{g})$ is elliptic, $f \in H^s(g_1)$, and $u \in H^0(g_0)$ then $Au = f$ implies $u \in H^s(g_0)$. In applications one is also interested in the fact that the smoothing remainders yield compact operators in the associated spaces, since this implies that elliptic operators are Fredholm operators. Even more, one also wants that the Fredholm property of an operator implies its ellipticity, meaning that the notion of ellipticity is actually optimal.

Definition 2.4. We say that $L^\star$ has the *Fredholm property* if, for any admissible $\mathbf{g}$, the following hold:

(a) Any $R \in L^{-\infty}(\mathbf{g})$ is a compact operator $H^0(g_0) \to H^0(g_1)$.
(b) If $A \in L^0(\mathbf{g})$ is a Fredholm operator $H^0(g_0) \to H^0(g_1)$ then A is elliptic.

The algebra of classical pseudodifferential operators on a compact manifold, cf. Example 2.1 is asymptotically complete and has the Fredholm property in the sense of the previous definitions.

2.2. *Toeplitz subalgebras*

In the following let $\mathbf{g} = (g_0, g_1)$ and $\mathbf{g}_0 = (g_0, g_0)$, $\mathbf{g}_1 = (g_1, g_1)$.

Definition 2.5. Let $P_0 \in L^0(\mathbf{g}_0)$ and $P_1 \in L^0(\mathbf{g}_1)$ be two projections (i.e., $P_j^2 = P_j$). Then we define

$$T^\mu(\mathbf{g}, P_0, P_1) := \bigl\{ A \in L^\mu(\mathbf{g}) \bigm| (1 - P_1)A = 0,\ A(1 - P_0) = 0 \bigr\}$$

and we set

$$T^\star = \bigcup_{-\mu\in\mathbb{N}_0} \bigcup_{\substack{\mathbf{g}\in\mathbf{G},\\ P_j\in L^0(\mathbf{g}_j)}} T^\mu(\mathbf{g}, P_0, P_1).$$

We call $T^\mu(\mathbf{g}, P_0, P_1)$ a *Toeplitz subalgebra* of $L^\mu(\mathbf{g})$. We define

$$H^s(g_j, P_j) = P_j\big(H^s(g_j)\big), \quad s \in \mathbb{N}_0.$$

Note that $H^s(g_j, P_j)$ is a closed subspace of $H^s(g_j)$ and that any element $A \in T^\mu(\mathbf{g}, P_0, P_1)$ induces continuous operators

$$A : H^s(g_0, P_0) \to H^{s-\mu}(g_1, P_1).$$

Observe that the canonical map

$$\widetilde{A} \mapsto P_1\widetilde{A}P_0 : L^\mu(\mathbf{g}) \to T^\mu(\mathbf{g}, P_0, P_1)$$

is surjective; in other words, we can write

$$T^\mu(\mathbf{g}, P_0, P_1) = P_1 L^\mu(\mathbf{g}) P_0. \tag{8}$$

Definition 2.6. An operator $A \in T^0(\mathbf{g}, P_0, P_1)$ is called elliptic if there exists a $B \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$ such that

$$BA - P_0 \in T^{-\infty}(\mathbf{g}_0, P_0, P_0), \qquad AB - P_1 \in T^{-\infty}(\mathbf{g}_1, P_1, P_1).$$

Any such operator B is called a parametrix of A.

Note that P_j is the identity operator on $H^s(g_j, P_j)$. Therefore ellipticity in fact asks for the existence of $B \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$ and $R_j \in T^{-\infty}(\mathbf{g}_j, P_j, P_j)$ such that $R_0 = BA - 1$ on $H^s(g_0, P_0)$ and $R_1 = AB - 1$ on $H^s(g_1, P_1)$.

Remark 2.7. Referring to the previously used notation we could introduce new weight-data

$$\widehat{\mathbf{g}} = (\widehat{g}_0, \widehat{g}_1) := \big((g_0, P_0), (g_1, P_1)\big), \quad P_j \in L^0(\mathbf{g}_j) \text{ projection},$$

and then write $L^\mu(\widehat{\mathbf{g}}) := T^\mu(\mathbf{g}, P_0, P_1)$. Thus we could use for Toeplitz algebras the same formalism as above. However, we find it more intuitive to use the notation $T^\mu(\mathbf{g}, P_0, P_1)$ in the sequel.

3. Ellipticity in Toeplitz subalgebras

We shall investigate how the notion of ellipticity in the full algebra $L^0(\mathbf{g})$ descends to a Toeplitz subalgebra $T^0(\mathbf{g}, P_0, P_1)$.

3.1. Asymptotic summation and parametrices

A first simple observation is that asymptotic completeness passes over to Toeplitz subalgebras:

Lemma 3.1. *If $L^\star$ is asymptotically complete then so is $T^\star$.*

Proof. Let $A_j \in T^{-j}(\mathbf{g}, P_0, P_1)$ be a given sequence. Then there exists an $\widetilde{A} \in L^0(\mathbf{g})$ with $\widetilde{A} - \sum_{j=0}^{N-1} A_j \in L^{-N}(\mathbf{g})$ for any N. Choosing $A = P_1 \widetilde{A} P_0$ we have $A - \sum_{j=0}^{N-1} A_j \in T^{-N}(\mathbf{g}, P_0, P_1)$, since $A_j = P_1 A_j P_0$ for any j. $\square$

Proposition 3.2. *Let $L^\star$ be asymptotically complete. Then for $A \in T^0(\mathbf{g}, P_0, P_1)$ the following statements are equivalent*:

(1) *A is elliptic.*
(2) *There exist $B_0, B_1 \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$ such that $B_0 A - P_0 \in T^{-1}(\mathbf{g}_0, P_0, P_0)$ and $AB_1 - P_1 \in T^{-1}(\mathbf{g}_1, P_1, P_1)$.*
(3) *There exist $C_0, C_1 \in L^0(\mathbf{g}^{(-1)})$ such that $C_0 A - P_0 \in L^{-1}(\mathbf{g}_0)$ and $AC_1 - P_1 \in L^{-1}(\mathbf{g}_1)$.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

Let us now show that (3) implies (2). If $C_0 A - P_0 = R_0$ with $R_0 \in L^{-1}(\mathbf{g}_0)$ then multiplication from the left and the right with P_0 and the fact that $P_1 A = A = AP_0$ yield $P_0 C_0 P_1 A - P_0 = P_0 R_0 P_0$. Similarly we get $AP_0 C_1 P_1 - P_1 = P_1 R_1 P_1$ with $R_1 \in L^{-1}(\mathbf{g}_1)$. Thus (2) holds with $B_j = P_0 C_j P_1$ for $j = 0, 1$.

Finally, assume (2) is true. Hence there is an $R_0 \in T^{-1}(\mathbf{g}_0, P_0, P_0)$ with

$$B_0 A = P_0 - R_0 = (P_0 - R_0) P_0.$$

Since $T^\star$ is asymptotically complete by Lemma 3.1, we can choose an element $B_0' \in T^0(\mathbf{g}_0, P_0, P_0)$ with $B_0' \sim P_0 + \sum_{\ell=1}^\infty R_0^\ell$. Then $B_L := B_0' B_0$ is a left-parametrix of A, i.e. $B_L A - P_0 \in T^{-\infty}(\mathbf{g}_0, P_0, P_0)$. Analogously we can construct a right-parametrix B_R and then choose $B = B_L$ or $B = B_R$. $\square$

As an immediate consequence of part (3) of the previous proposition we obtain:

Corollary 3.3. *Let $L^\star$ be asymptotically complete, $\widetilde{A} \in L^0(\mathbf{g})$ and $Q_j, P_j \in L^0(\mathbf{g}_j)$ be projections with $P_j - Q_j \in L^{-1}(\mathbf{g}_j)$. Then $P_1 \widetilde{A} P_0 \in T^\mu(\mathbf{g}, P_0, P_1)$ is elliptic if, and only if, $Q_1 \widetilde{A} Q_0 \in T^\mu(\mathbf{g}, Q_0, Q_1)$ is elliptic.*

3.2. The Fredholm property

Assume that $L^0(\mathbf{g})$ has the Fredholm property. Since $H^0(g_j, P_j)$ is a closed subspace of $H^0(g_j)$ it is clear that smoothing operators from $T^{-\infty}(\mathbf{g}, P_0, P_1)$ induce compact operators $H^0(g_0, P_0) \to H^0(g_1, P_1)$. Therefore elliptic elements from $T^0(\mathbf{g}, P_0, P_1)$ induce Fredholm operators $H^0(g_0, P_0) \to H^0(g_1, P_1)$. For the reverse statement we shall need that the algebras are stable under taking adjoints.

Definition 3.4. We call $L^\star$ *$*$-closed* if for any admissible $\mathbf{g}$ and any $A \in L^0(\mathbf{g})$ there exists an $A^* \in L^0(\mathbf{g}^{(-1)})$ such that $A^* : H^0(g_1) \to H^0(g_0)$ coincides with the adjoint of $A : H^0(g_0) \to H^0(g_1)$.

Since any $A \in T^0(\mathbf{g}, P_0, P_1)$ satisfies $A = P_1 A P_0$, taking the adjoint in the $*$-closed algebra $L^0(\mathbf{g})$ yields a map

$$T^0(\mathbf{g}, P_0, P_1) \to T^0\big(\mathbf{g}^{(-1)}, P_1^*, P_0^*\big).$$

This map preserves Fredholm operators:

Lemma 3.5. *Let $A \in T^0(\mathbf{g}, P_0, P_1)$ induce a Fredholm operator $A : H^0(g_0, P_0) \to H^0(g_1, P_1)$. Then $A^* : H^0(g_1, P_1^*) \to H^0(g_0, P_0^*)$ is also a Fredholm operator.*

Proof. First observe that there is a natural identification of the dual space of $H^0(g_j, P_j)$ with $H^0(g_j, P_j^*)$. In fact, any functional x in the dual space $H^0(g_j, P_j)'$ can be extended to one in $H^0(g_j)'$ by setting

$$\widetilde{x}(u) = x(P_j u), \quad u \in H^0(g_j).$$

If we denote by $I_j : H^0(g_j) \to H^0(g_j)'$ the standard Riesz map then

$$x \mapsto P_j^* I_j^{-1} \widetilde{x} : H^0(g_j, P_j)' \to H^0\big(g_j, P_j^*\big)$$

is a bijection. Under this identification the dual operator $A' : H^0(g_1, P_1)' \to H^0(g_0, P_0)'$ corresponds to $A^* : H^0(g_1, P_1^*) \to H^0(g_0, P_0^*)$. Now it remains to observe that duals of Fredholm operators are Fredholm operators. □

Lemma 3.6. *Let X and Y be two Hilbert spaces and $T : X \to Y$ be an upper semi-Fredholm operator, i.e., T has closed range and a finite-dimensional kernel. Then T^*T is a Fredholm operator.*

Proof. T^*T has finite-dimensional kernel, since $\ker T^*T = \ker T$. Since the range of T is closed, we have the orthogonal decomposition $Y = \operatorname{im} T \oplus \ker T^*$. Therefore $\operatorname{im} T^*T = \operatorname{im} T^*$. Since T^* is lower semi-Fredholm, its range has finite co-dimension. □

Theorem 3.7. *If $L^\star$ has the Fredholm property and is $*$-closed, then $T^\star$ has the Fredholm property.*

Proof. Let $A \in T^0(\mathbf{g}, P_0, P_1)$ induce a Fredholm operator $H^0(g_0, P_0) \to H^0(g_1, P_1)$; let us denote this operator by $\widehat{A}$. We have to show that A is elliptic. Due to the $*$-closedness of $L^\star$ we have

$$B := A^*A + (1 - P_0)^*(1 - P_0) \in L^0(\mathbf{g}_0). \tag{9}$$

We shall now show that $B : H^0(g_0) \to H^0(g_0)$ is a Fredholm operator. To this end define

$$T : H^0(g_0) \to H^0(g_1) \oplus H^0(g_0), \quad Tu = \big(Au, (1 - P_0)u\big).$$

Now $\ker T = \ker \widehat{A}$, and $\operatorname{im} T = \operatorname{im} \widehat{A} \oplus \operatorname{im}(1 - P_0)$ is closed in $H^0(g_1) \oplus H^0(g_0)$, i.e., T is upper semi-Fredholm. Due to Lemma 3.6, $B = T^*T$ is Fredholm.

Since $L^\star$ has the Fredholm property, B has a parametrix $C \in L^0(\mathbf{g}_0)$, i.e., $R := CB - 1 \in L^{-\infty}(\mathbf{g}_0)$. Therefore

$$P_0RP_0 = P_0C\big(A^*A + (1 - P_0)^*(1 - P_0)\big)P_0 - P_0 = P_0CA^*P_1A - P_0,$$

since $P_1A = AP_0 = A$. Therefore $B_L := P_0CA^*P_1 \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$ is a left-parametrix of A.

In view of Lemma 3.5 we can construct in the same way a left-parametrix to $A^* \in T^0(\mathbf{g}^{(-1)}, P_1^*, P_0^*)$. The adjoint of this left-parametrix yields a right-parametrix for A. Hence A is elliptic. □

Remark 3.8. For later purpose let us state here that the operator B defined in (9) is a self-adjoint Fredholm operator with

$$\ker\big(B : H^0(g_0) \to H^0(g_0)\big) = \ker\big(A : H^0(g_0, P_0) \to H^0(g_1, P_1)\big).$$

In fact, the kernel on the right-hand side is clearly contained in the kernel on the left-hand side. Moreover, $Bu = 0$ implies

$$0 = (Bu, u) = \big(A^*Au, u\big) + \big((1 - P_0)^*(1 - P_0)u, u\big) = \|Au\|^2 + \big\|(1 - P_0)u\big\|^2,$$

where inner-product and norm are those of $H^0(g_0)$. Thus $Au = 0$ and $(1 - P_0)u = 0$. In particular, $B : H^0(g_0) \to H^0(g_0)$ is an isomorphism if $A : H^0(g_0, P_0) \to H^0(g_1, P_1)$ is injective.

Another interesting property of many pseudodifferential calculi is their "spectral invariance", i.e., whenever an element of the algebra is invertible as a continuous operator between Sobolev spaces, then the inverse belongs to the calculus.

Theorem 3.9. *Let $L^\star$ have the Fredholm property and be $*$-closed. Furthermore assume that $R_1TR_0 \in L^{-\infty}(\mathbf{g})$ whenever $\mathbf{g}$ is admissible, $R_j \in L^{-\infty}(\mathbf{g}_j)$ are smoothing operators, and $T : H^0(g_0) \to H^0(g_1)$ continuously. Then $T^0(\mathbf{g}, P_0, P_1)$ is spectrally invariant for any admissible $\mathbf{g}$, i.e., if $A \in T^0(\mathbf{g}, P_0, P_1)$ induces an isomorphism $H^0(g_0, P_0) \to H^0(g_1, P_1)$ then there exists a $B \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$ such that $BA = P_0$ and $AB = P_1$.*

Proof. Let A be as stated. In particular, $A : H^0(g_0, P_0) \to H^0(g_1, P_1)$ is a Fredholm operator. By Theorem 3.7 there exists a parametrix $B \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$. In particular, $BA = 1 - R_0$ on $H^0(g_0, P_0)$ and $AB = 1 - R_1$ on $H^0(g_1, P_1)$ with smoothing operators $R_j \in T^{-\infty}(\mathbf{g}_j, P_j)$. These identities yield $A^{-1} = B + R_0A^{-1}$ and $A^{-1} = B + A^{-1}R_1$. Thus we get

$$A^{-1} = B + R_0B + R_0P_0A^{-1}P_1R_1.$$

The right-hand side belongs to $T^0(\mathbf{g}^{(-1)}, P_1, P_0)$. In fact, we can consider $T := P_0 A^{-1} P_1$ as a continuous map $H^0(g_1) \to H^0(g_0)$, hence $S := R_0 T R_1 \in L^{-\infty}(\mathbf{g}^{(-1)})$ by assumption. Moreover $P_1 S = S P_0 = 0$, showing that S belongs to $T^{-\infty}(\mathbf{g}^{(-1)}, P_1, P_0)$. $\square$

By abuse of notation we shall also write $A^{-1} := B$ for B from the previous theorem. This notation is reasonable, since $B : H^0(g_1, P_1) \to H^0(g_0, P_0)$ is the inverse of $A : H^0(g_0, P_0) \to H^0(g_1, P_1)$.

3.3. Reductions of orders

In applications typically the fitration in (5) uses a parameter $\mu \in \mathbb{Z}$ or $\mu \in \mathbb{R}$ and the scale of Sobolev spaces (6) admits regularities $s \in \mathbb{Z}$ or $s \in \mathbb{R}$. Of course, one is also interested in operators of order different from zero. A typical feature in pseudodifferential calculi is the existence of "reductions of orders", which allows to restrict ones attention to the zero order case. In the present general setting this means (to ask for) the existence of operators $S_\mu \in L^\mu(\mathbf{g})$ having an inverse $S_\mu^{-1} = S_{-\mu} \in L^{-\mu}(\mathbf{g})$, for any μ and any admissible $\mathbf{g} = (g, g)$.

In case of existence of such reductions of orders, $S_\mu^j \in L^\mu(\mathbf{g}_j)$, the study of $A \in T^\mu(\mathbf{g}, P_0, P_1)$ considered as an operator

$$A : H^s(g_0, P_0) \to H^{s-\mu}(g_1, P_1)$$

is equivalent to the study of

$$\widetilde{A} : H^0(g_0, \widetilde{P}_0) \to H^0(g_1, \widetilde{P}_1)$$

where

$$\widetilde{A} = S^1_{s-\mu} A S^0_{-s} \in L^0(\mathbf{g}, \widetilde{P}_0, \widetilde{P}_1)$$

with the two projections

$$\widetilde{P}_0 = S^0_s P_0 S^0_{-s} \in L^0(\mathbf{g}_0), \qquad \widetilde{P}_1 = S^1_{s-\mu} P_1 S^1_{\mu-s} \in L^0(\mathbf{g}_1).$$

In fact, the following diagram is commutative:

$$\begin{array}{ccc} H^s(g_0, P_0) & \xrightarrow{A} & H^{s-\mu}(g_1, P_1) \\ {\scriptstyle S^0_{-s}}\uparrow & & \downarrow{\scriptstyle S^1_{s-\mu}} \\ H^0(g_0, \widetilde{P}_0) & \xrightarrow{\widetilde{A}} & H^0(g_1, \widetilde{P}_1). \end{array}$$

3.4. Ellipticity and principal symbols

Above, ellipticity has been *defined* as the existence of a parametrix, i.e., an inverse modulo smoothing remainders. In applications it is of course desirable to characterize ellipticity in other terms that are easier to verify. Typically, with a given operator A one associates one or more "(homogeneous) principal symbols", which can be thought of bundle morphisms between finite- or infinite-dimensional vector bundles (we shall identify operator-valued functions $\sigma : M \to \mathscr{L}(X, Y)$ with morphism acting between the trivial vector-bundles $M \times X$ and $M \times Y$). Ellipticity is then aimed to be equivalent to the invertibility/bijectivity of the principal symbols.

Example 3.10. With a classical pseudodifferential operator $A \in L^{\mu}_{\mathrm{cl}}(M; E_0, E_1)$, cf. Example 2.1, we associate its homogeneous principal symbol, which is a vector bundle morphism

$$\sigma^{\mu}(A) : \pi^* E_0 \to \pi^* E_1,$$

where $\pi : S^*M \to M$ is the canonical projection of the unit co-sphere bundle S^*M of the (Riemannian) manifold M onto M itself, and $\pi^* E_j$ denotes the pull-back of the vector bundle E_j. A is elliptic if, and only if, $\sigma^{\mu}(A)$ is an isomorphism.

In this section we assume that in $L^{\star}$ we have such a characterisation of ellipticity in terms of principal symbols and investigate how this structure descends to Toeplitz subalgebras. To this end let us call $L^{\star}$ a *σ-algebra* if there exists a map

$$A \mapsto \sigma(A) = \big(\sigma_1(A), \ldots, \sigma_n(A)\big)$$

assigning to each $A \in L^0(\mathbf{g})$ an n-tuple of bundle morphisms

$$\sigma_{\ell}(A) : E_{\ell}(g_0) \to E_{\ell}(g_1)$$

between (finite- or infinite-dimensional) Hilbert space bundles $E_{\ell}(g_j)$, such that the following are true:

(1) The map respects the composition of operators, i.e.,

$$\sigma(AB) = \sigma(A)\sigma(B) := \big(\sigma_1(A)\sigma_1(B), \ldots, \sigma_n(A)\sigma_n(B)\big)$$

whenever $A \in L^0(\mathbf{g})$ and $B \in L^0(\mathbf{g}')$ as in (7).

(2) $\sigma(R) = 0$ for any smoothing operator R.

(3) A is elliptic if, and only if, $\sigma(A)$ is invertible, i.e., all $\sigma_{\ell}(A)$ are bundle isomorphisms.

If additionally $L^{\star}$ is $*$-closed we also ask that

(4) $\sigma(A^*) = \sigma(A)^*$, i.e., for any ℓ,

$$\sigma_{\ell}\big(A^*\big) = \sigma_{\ell}(A)^* : E_1^{\ell}(g_1) \to E_0^{\ell}(g_0),$$

where $\sigma_{\ell}(A)^*$ denotes the adjoint morphism (obtained by taking fibrewise the adjoint).

Definition 3.11. Let $L^\star$ be a σ-algebra and $A \in T^0(\mathbf{g}, P_0, P_1)$. Since the P_j are projections also the associated bundle morphisms $\sigma_\ell(P_j)$ are projections in $E_\ell(g_j)$. Therefore its range

$$E_\ell(g_j, P_j) := \sigma_\ell(P_j)\big(E_\ell(g_j)\big)$$

is a subbundle of $E_\ell(g_j)$. We now define

$$\sigma_\ell(A, P_0, P_1) : E_\ell(g_0, P_0) \to E_\ell(g_1, P_1)$$

by restriction of $\sigma_\ell(A)$ and then

$$\sigma(A, P_0, P_1) = \big(\sigma_1(A, P_0, P_1), \dots, \sigma_n(A, P_0, P_1)\big).$$

It is clear that if $A \in T^0(\mathbf{g}, P_0, P_1)$ is elliptic, then $\sigma(A, P_0, P_1)$ is invertible. In fact, if $B \in T^0(\mathbf{g}^{(-1)}, P_1, P_0)$ is a parametrix to A then $\sigma(B, P_1, P_0)$ is the inverse of $\sigma(A, P_0, P_1)$.

Theorem 3.12. *Let $L^\star$ be a $*$-closed σ-algebra. Then $T^\star$ is a σ-algebra. In particular, for $A \in T^0(\mathbf{g}, P_0, P_1)$ the following statements are equivalent*:

(a) *A is elliptic.*
(b) *$\sigma_\ell(A, P_0, P_1) : E_\ell(g_0, P_0) \to E_\ell(g_1, P_1)$ is an isomorphism for $\ell = 1, \dots, n$.*

Proof. Let us show that (b) implies (a) (the remaining statements are simple to see). Let us define the operator

$$B := A^*A + (1 - P_0)^*(1 - P_0) \in L^0(\mathbf{g}_0).$$

Applying the principal symbol map yields

$$\sigma_\ell(B) = \sigma_\ell(A)^*\sigma_\ell(A) + \sigma_\ell(1 - P_0)^*\sigma_\ell(1 - P_0).$$

Remark 3.8 (applied fibrewise) shows that any $\sigma_\ell(B)$ is an isomorphism, hence B is elliptic by assumption. Arguing as in the proof of Theorem 3.7 we find a left-parametrix to A and, by passing to adjoints, also a right-parametrix. Thus A is elliptic. □

The previous proof also shows that, under the assumptions of Theorem 3.12, a left-parametrix [right-parametrix] for A exists, provided $\sigma_\ell(A, P_0, P_1) : E_\ell(g_0, P_0) \to E_\ell(g_1, P_1)$ are (fibrewise) Fredholm monomorphisms [epimorphisms] for all $\ell = 1, \dots, n$.

Example 3.13. Let $P_j \in L^0_{\mathrm{cl}}(M; E_j, E_j)$ be two projections. Let us write

$$T^0_{\mathrm{cl}}\big(M; (E_0, P_0), (E_1, P_1)\big) = P_1 L^0_{\mathrm{cl}}(M; E_0, E_1) P_0$$

for the associated Toeplitz algebra and

$$H^s(M, E_j, P_j) = P_j\big(H^s(M, E_j)\big).$$

The principal symbol $\sigma(P_j) : \pi^* E_j \to \pi^* E_j$ is a projection, and its range is a subbundle of $\pi^* E_j$ which we denote by $E_j(P_j)$.

Then for $A \in T^0_{\mathrm{cl}}(M, (E_0, P_0), (E_1, P_1))$ the following statements are equivalent:

(a) A is elliptic, i.e., has a parametrix $B \in T^0_{\mathrm{cl}}(M; (E_1, P_1), (E_0, P_0))$.
(b) $A : H^0(M, E_0, P_0) \to H^0(M, E_1, P_1)$ is a Fredholm operator.
(c) $\sigma(A) : E_0(P_0) \to E_1(P_1)$ is an isomorphism.

Example 3.14. Let E_0, E_1 be smooth vector bundles over M. By Swan's theorem E_j is a subbundle of a trivial bundle; let us denote it by $\mathbb{C}^{N_j}$. Then the projections $p_j : \mathbb{C}^{N_j} \to E_j$ can be considered as zero-order pseudodifferential projections $P_j \in L^0_{\mathrm{cl}}(M; \mathbb{C}^{N_j}, \mathbb{C}^{N_j})$ and we can identify $L^\mu_{\mathrm{cl}}(M; E_0, E_1)$ with $L^\mu_{\mathrm{cl}}(M; (\mathbb{C}^{N_0}, P_0), (\mathbb{C}^{N_1}, P_1))$.

Similarly as discussed in Section 3.3, the existence of reductions of orders allows a straightforward extension of the above result from zero order operators to operators of general order. This fact we shall use below in our examples without further commenting on it.

3.5. Parameter-dependent operators

For the analysis of resolvents of differential operators, calculi of *parameter-dependent* pseudodifferential operators can be used very effectively. With suitable modifications, the above abstract approach can also capture features of parameter-dependent calculi. We shall give some details in this subsection.

In the following let Λ coincide with $\mathbb{R}^\ell$ or be a sectorial domain in the complex plane. We now assume that the elements of $L^\star$ are not single operators, but families/functions of operators $\lambda \mapsto A(\lambda)$. To make clear that we deal with families of operators we shall use notations like $L^\star(\Lambda)$, $L^\mu(\mathbf{g}; \Lambda)$ and denote elements by $A(\lambda)$, $B(\lambda)$, etc.

With the notation from Section 2 we shall assume that $A(\lambda) \in L^\mu(\mathbf{g}; \Lambda)$ induces continuous operators

$$A(\lambda) : H^s(g_0) \to H^{s-\mu}(g_1)$$

for all λ and all s. Smoothing operators $R(\lambda) \in L^{-\infty}(\mathbf{g}; \Lambda)$ are required to satisfy

$$\|R(\lambda)\|_{H^s(g_0), H^t(g_1)} \xrightarrow{|\lambda|\to\infty} 0$$

for any s, t, where $\|\cdot\|_{X,Y}$ denotes the operator norm for operators $X \to Y$.

Definition 3.15. $A(\lambda) \in L^0(\mathbf{g}; \Lambda)$ is called *elliptic* (or *parameter-elliptic*) if there exists a $B(\lambda) \in L^0(\mathbf{g}^{(-1)}; \Lambda)$ such that

$$R_0(\lambda) := B(\lambda)A(\lambda) - 1 \in L^{-\infty}(\mathbf{g}_0; \Lambda),$$
$$R_1(\lambda) := A(\lambda)B(\lambda) - 1 \in L^{-\infty}(\mathbf{g}_1; \Lambda).$$

Any such operator $B(\lambda)$ is called a parametrix of $A(\lambda)$.

An elliptic $A(\lambda)$ induces isomorphisms $H^0(g_0) \to H^0(g_1)$ for sufficiently large λ, since $1 + R_j(\lambda)$ is invertible due to the decay property of smoothing remainders. If we assume that there exist smoothing $S_j(\lambda)$ such that $(1 + R_j(\lambda))^{-1} = 1 + S_j(\lambda)$ for sufficiently large λ, we can conclude that there exists a parametrix $B(\lambda)$ that equals $A(\lambda)^{-1}$ for large enough λ.

Definition 3.16. We call $L^\star(\Lambda)$ *inverse-closed*, if for any $R(\lambda) \in L^{-\infty}(\mathbf{g}; \Lambda)$ with admissible weight $\mathbf{g} = (g, g)$ there exists an $S(\lambda) \in L^{-\infty}(\mathbf{g}; \Lambda)$ such that

$$\bigl(1 + R(\lambda)\bigr)\bigl(1 + S(\lambda)\bigr) = \bigl(1 + S(\lambda)\bigr)\bigl(1 + R(\lambda)\bigr) = 1$$

for sufficiently large λ.

Using the notation of the previous definition, we have

$$\bigl(1 + R(\lambda)\bigr)^{-1} = 1 - R(\lambda) + R(\lambda)\bigl(1 + R(\lambda)\bigr)^{-1} R(\lambda)$$

whenever the inverse exists. Hence we see that $L^\star(\Lambda)$ is inverse closed if, and only if, for any such $R(\lambda)$ there exists an $R'(\lambda) \in L^{-\infty}(\mathbf{g}; \Lambda)$ such that

$$R'(\lambda) = R(\lambda)\bigl(1 + R(\lambda)\bigr)^{-1} R(\lambda)$$

for sufficiently large λ.

Example 3.17. With the notation introduced in Example 2.1, let

$$L^\mu(\mathbf{g}; \Lambda) = L^\mu_{\mathrm{cl}}(M; E_0, E_1; \Lambda)$$

be the space of classical parameter-dependent pseudodifferential operators of order μ, i.e., the local symbols of the operator-families satisfy uniform estimates of the form

$$\bigl|D^\alpha_\xi D^\beta_x D^\gamma_\lambda a(x, \xi, \lambda)\bigr| \leqslant C\bigl(1 + |\xi| + |\lambda|\bigr)^{\mu - |\alpha| - |\gamma|},$$

and have expansions into components homogeneous in (ξ, λ). The smoothing operators are rapidly decreasing in λ with values in the smoothing operators on M, i.e.,

$$L^{-\infty}(M; E_0, E_1; \Lambda) = \mathscr{S}(\Lambda, X),$$

where

$$X = L^{-\infty}(M; E_0, E_1) \cong \mathscr{C}^\infty(M, E_0 \boxtimes E_1).$$

Now it is straightforward to verify the inverse closedness, using the fact that for a rapidly decreasing function $r(\lambda)$ also

$$r'(\lambda) := \chi(\lambda) r(\lambda)\bigl(1 + r(\lambda)\bigr)^{-1} r(\lambda)$$

is rapidly decreasing, where χ is a zero excision function that vanishes where the inverse does not exist.

As before we can now consider Toeplitz algebras

$$T^{\mu}(\mathbf{g}, P_0, P_1; \Lambda) = P_1(\lambda)L^{\mu}(\mathbf{g}; \Lambda)P_0(\lambda)$$

with projections $P_j(\lambda) \in L^0(\mathbf{g}_j; \Lambda)$. An element $A(\lambda) \in T^0(\mathbf{g}, P_0, P_1; \Lambda)$ is called elliptic if there exists a $B(\lambda) \in T^0(\mathbf{g}, P_1, P_0; \Lambda)$ such that

$$B(\lambda)A(\lambda) - P_0(\lambda) \in T^{-\infty}(\mathbf{g}_0, P_0, P_0; \Lambda),$$
$$A(\lambda)B(\lambda) - P_1(\lambda) \in T^{-\infty}(\mathbf{g}_1, P_1, P_1; \Lambda).$$

Inverse-closedness of the Toeplitz algebras now means that to any projection $P(\lambda) \in L^0(\mathbf{g}; \Lambda)$, $\mathbf{g} = (g, g)$, and any $R(\lambda) \in T^{-\infty}(\mathbf{g}, P, P; \Lambda)$ there exists an $S(\lambda) \in T^{-\infty}(\mathbf{g}, P, P; \Lambda)$ such that

$$\big(P(\lambda) + R(\lambda)\big)\big(P(\lambda) + S(\lambda)\big) = \big(P(\lambda) + S(\lambda)\big)\big(P(\lambda) + R(\lambda)\big) = P(\lambda)$$

for sufficiently large λ. In this case, to any elliptic $A(\lambda)$ there always exists a parametrix $B(\lambda)$ such that

$$B(\lambda)A(\lambda) = P_0(\lambda), \qquad A(\lambda)B(\lambda) = P_1(\lambda)$$

for large λ. It is not difficult to see that inverse-closedness of $L^{\star}(\Lambda)$ implies that of $T^{\star}(\Lambda)$.

Assuming that ellipticity in $L^{\star}(\Lambda)$ is characterized by the invertibility of certain principal symbols, i.e., $L^{\star}(\Lambda)$ is a σ-algebra, we can now show as in Section 3.4:

Theorem 3.18. *If $L^{\star}(\Lambda)$ is an inverse-closed, $*$-closed σ-algebra, then $T^{\star}(\Lambda)$ is an inverse-closed σ-algebra.*

4. *SG*-pseudodifferential operators

If X is a Fréchet space let us denote by $S^{\mu}(\mathbb{R}^m, X)$ the Fréchet space of all smooth functions $a : \mathbb{R}^m \to X$ satisfying estimates

$$\big\| D_z^{\gamma} a(z) \big\| \leqslant C_{\gamma} \big(1 + |z|\big)^{\mu - |\gamma|}$$

uniformly in $z \in \mathbb{R}^m$ for any multi-index γ and any semi-norm $\| \cdot \|$ of X (the constant C_{γ} depends also on the semi-norm). With $S^{(\mu)}(\mathbb{R}^m, X)$ we denote the space of all smooth functions $a : \mathbb{R}^m \setminus \{0\} \to X$ of the form

$$a(z) = |z|^{\mu} \widetilde{a}\big(z/|z|\big), \quad \widetilde{a} : \mathbb{S}^{m-1} \to X,$$

where $\mathbb{S}^{m-1}$ denotes the unit-sphere in $\mathbb{R}^m$. Moreover, $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m, X)$ denotes the subspace of symbols $a \in S^{\mu}(\mathbb{R}^m, X)$ that have asymptotic expansions into homogeneous components: There exist $a^{(\mu-j)} \in S^{(\mu-j)}(\mathbb{R}^m, X)$ such that

$$a - \sum_{j=0}^{N-1} \chi a^{(\mu-j)} \in S^{\mu-N}\big(\mathbb{R}^m, X\big)$$

for any positive integer N and with $\chi(z)$ being a smooth zero-excision function. The function $a^{(\mu)}$ is called the principal component of a.

The class of pseudodifferential symbols we now consider are, roughly speaking, classical both in the x-variable and the corresponding co-variable ξ (for precise details we refer the reader to [3]).

Definition 4.1. For $\mu, m \in \mathbb{R}$ and $N_0, N_1 \in \mathbb{N}$ let us define

$$S_{\mathrm{cl}}^{\mu,m}\big(\mathbb{R}^n \times \mathbb{R}^n, N_0, N_1\big) := S_{\mathrm{cl}}^{m}\big(\mathbb{R}^n_x, S_{\mathrm{cl}}^{\mu}\big(\mathbb{R}^n_\xi, \mathscr{L}\big(\mathbb{C}^{N_0}, \mathbb{C}^{N_1}\big)\big)\big).$$

The space of associated pseudodifferential operators,

$$(Au)(x) = \big[\mathrm{op}(a)u\big](x) = \int e^{ix\xi} a(x,\xi)\widehat{u}(\xi)\, đ\xi,$$

we shall denote by $L_{\mathrm{cl}}^{\mu,m}(\mathbb{R}^n, N_0, N_1)$.

The class of regularizing operators,

$$L^{-\infty,-\infty}\big(\mathbb{R}^n, N_0, N_1\big) := \bigcap_{\mu,m\in\mathbb{R}} L_{\mathrm{cl}}^{\mu,m}\big(\mathbb{R}^n, N_0, N_1\big),$$

consists of all integral operators (with respect to the standard Lebesgue measure on $\mathbb{R}^n$) having an integral kernel

$$k(x,y) \in \mathscr{S}\big(\mathbb{R}^{2n}_{(x,y)}, \mathscr{L}\big(\mathbb{C}^{N_0}, \mathbb{C}^{N_1}\big)\big),$$

the space of rapidly decreasing functions with values in the linear operators $\mathbb{C}^{N_0} \to \mathbb{C}^{N_1}$.

The natural scale of Sobolev spaces such operators act in is given by

$$H^{s,\delta}\big(\mathbb{R}^n, N\big) := \big(1 + |x|^2\big)^{-\delta/2} H^s\big(\mathbb{R}^n, \mathbb{C}^N\big), \quad s, \delta \in \mathbb{R},$$

i.e., the standard $\mathbb{C}^N$-valued Sobolev spaces on $\mathbb{R}^n$ multiplied by a weight function. Then $A \in L_{\mathrm{cl}}^{\mu,m}(\mathbb{R}^n, N_0, N_1)$ induces continuous operators

$$A : H^{s,\delta}\big(\mathbb{R}^n, N_0\big) \to H^{s-\mu,\delta-m}\big(\mathbb{R}^n, N_1\big)$$

for any choice of s and δ.

By passing to the principal component with respect to x, or with respect to ξ, or simultaneously with respect to both x and ξ, we associate with $A = \mathrm{op}(a)$ three principal symbols, which are bundle morphisms

$$\sigma^\mu(A) : \big(\mathbb{R}^n_x \times \mathbb{S}^{n-1}_\xi\big) \times \mathbb{C}^{N_0} \to \big(\mathbb{R}^n_x \times \mathbb{S}^{n-1}_\xi\big) \times \mathbb{C}^{N_1},$$
$$\sigma_m(A) : \big(\mathbb{S}^{n-1}_x \times \mathbb{R}^n_\xi\big) \times \mathbb{C}^{N_0} \to \big(\mathbb{S}^{n-1}_x \times \mathbb{R}^n_\xi\big) \times \mathbb{C}^{N_1},$$
$$\sigma^\mu_m(A) : \big(\mathbb{S}^{n-1}_x \times \mathbb{S}^{n-1}_\xi\big) \times \mathbb{C}^{N_0} \to \big(\mathbb{S}^{n-1}_x \times \mathbb{S}^{n-1}_\xi\big) \times \mathbb{C}^{N_1}.$$

Let us then set

$$\sigma(A) = \big(\sigma^\mu(A), \sigma_m(A), \sigma_m^\mu(A)\big), \quad A \in L_{\mathrm{cl}}^{\mu,m}\big(\mathbb{R}^n, N_0, N_1\big).$$

Ellipticity of A is defined as the invertibility of (all three components of) $\sigma(A)$. It is well known that ellipticity is equivalent to the existence of a parametrix $B \in L_{\mathrm{cl}}^{-\mu,-m}(\mathbb{R}^n, N_1, N_0)$ modulo remainders in $L^{-\infty,-\infty}$, and it is equivalent to $A : H^{s,\delta}(\mathbb{R}^n, N_0) \to H^{s-\mu,\delta-m}(\mathbb{R}^n, N_1)$ being a Fredholm operator for some (and then for all) $s, \delta \in \mathbb{R}$; the latter result was probably first obtained by Hirschmann in the 1990s.

With two projections $P_j \in L_{\mathrm{cl}}^{0,0}(\mathbb{R}^n, N_j, N_j)$, $j = 0, 1$, let us write

$$T_{\mathrm{cl}}^{\mu,m}\big(\mathbb{R}^n, (N_0, P_0), (N_1, P_1)\big) = P_1 L_{\mathrm{cl}}^{\mu,m}\big(\mathbb{R}^n, N_0, N_1\big) P_0$$

for the associated Toeplitz subalgebra and

$$H^{s,\delta}\big(\mathbb{R}^n, N_j, P_j\big) = P_j\big(H^{s,\delta}\big(\mathbb{R}^n, N_j\big)\big)$$

for the associated scale of projected Sobolev spaces. The principal symbols $\sigma^0(P_j)$, $\sigma_0(P_j)$ and $\sigma_0^0(P_j)$ of P_j are itself projections, thus their ranges define subbundles

$$\begin{aligned} E^0(N_j, P_j) &\subset \big(\mathbb{R}_x^n \times \mathbb{S}_\xi^{n-1}\big) \times \mathbb{C}^{N_j}, \\ E_0(N_j, P_j) &\subset \big(\mathbb{S}_x^{n-1} \times \mathbb{R}_\xi^n\big) \times \mathbb{C}^{N_j}, \\ E_0^0(N_j, P_j) &\subset \big(\mathbb{S}_x^{n-1} \times \mathbb{S}_\xi^{n-1}\big) \times \mathbb{C}^{N_j}. \end{aligned}$$

The principal symbol $\sigma(A, P_0, P_1)$ consists of the three components

$$\begin{aligned} \sigma^\mu(A, P_0, P_1) &: E^0(N_0, P_0) \to E^0(N_1, P_1), \\ \sigma_m(A, P_0, P_1) &: E_0(N_0, P_0) \to E_0(N_1, P_1), \\ \sigma_m^\mu(A, P_0, P_1) &: E_0^0(N_0, P_0) \to E_0^0(N_1, P_1), \end{aligned}$$

obtained by the restriction of the corresponding symbols of A.

Theorem 4.2. *For $A \in T_{\mathrm{cl}}^{\mu,m}(\mathbb{R}^n, (N_0, P_0), (N_1, P_1))$ the following statements are equivalent*:

(a) *A has a parametrix $B \in T_{\mathrm{cl}}^{-\mu,-m}(M, (N_1, P_1), (N_0, P_0))$.*
(b) *$A : H^{s,\delta}(\mathbb{R}^n, N_0, P_0) \to H^{s-\mu,\delta-m}(\mathbb{R}^n, N_1, P_1)$ is a Fredholm operator for some $s, \delta \in \mathbb{R}$.*
(c) *The morphisms $\sigma^\mu(A, P_0, P_1)$, $\sigma_m(A, P_0, P_1)$, and $\sigma_m^\mu(A, P_0, P_1)$ are isomorphisms.*

In this case, (b) *is true for arbitrary $s, \delta \in \mathbb{R}$.*

In case $\mu = m = 0$ and $s = \delta = 0$ this theorem is just a particular case of Theorems 3.7 and 3.12, with

$$L^{\mu}(\mathbf{g}) = L^{\mu,\mu}_{\mathrm{cl}}\bigl(\mathbb{R}^n, N_0, N_1\bigr), \quad \mathbf{g} = \bigl(\bigl(\mathbb{R}^n, N_0\bigr), \bigl(\mathbb{R}^n, N_1\bigr)\bigr),$$
$$H^s(g) = H^{s,s}\bigl(\mathbb{R}^n, N\bigr), \quad g = \bigl(\mathbb{R}^n, N\bigr).$$

The general case is obtained by the use of order reductions, analogously as described in Section 3.3 (with the minor modification that we have here two parameters μ and m). In fact, the operators having symbol $[x]^m[\xi]^\mu I_N$, where I_N is the $N \times N$-unit matrix and $[\cdot] : \mathbb{R}^n \to (0,\infty)$ is a smooth function that coincides with $|\cdot|$ outside the unit-ball, induce isomorphisms $H^{s,\delta}(\mathbb{R}^n, N) \to H^{s-\mu,\delta-m}(\mathbb{R}^n, N)$ for arbitrary s and δ.

Theorem 4.3. *If $A \in T^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n, (N_0, P_0), (N_1, P_1))$ induces an isomorphism*

$$H^{s,\delta}\bigl(\mathbb{R}^n, N_0, P_0\bigr) \to H^{s-\mu,\delta-m}\bigl(\mathbb{R}^n, N_1, P_1\bigr)$$

for some $s, \delta \in \mathbb{R}$, then this is true for any $s, \delta \in \mathbb{R}$ and the inverse A^{-1} belongs to $T^{-\mu,-m}_{\mathrm{cl}}(M, (N_1, P_1), (N_0, P_0))$.

Proof. Again we can restrict ourselves to the case $\mu = m = 0$. Then the theorem is a direct consequence of Theorem 3.9, provided we can show that $R_1 T R_0 \in L^{-\infty,\infty}(\mathbb{R}^n, N_0, N_1)$ for any choice of $R_j \in L^{-\infty,\infty}(\mathbb{R}^n, N_j, N_j)$ and $T : L^2(\mathbb{R}^n, N_0) \to L^2(\mathbb{R}^n, N_1)$. However, this is true since the R_j have integral kernel $k_j(x, y) \in \mathscr{S}(\mathbb{R}^{2n}_{(x,y)}, \mathscr{L}(\mathbb{C}^{N_j}, \mathbb{C}^{N_j}))$, and thus $R_1 T R_0$ has the integral kernel

$$k(x, y) = \int_{\mathbb{R}^n} k_1(x, z)\bigl[T_z k_0(z, y)\bigr]\, dz$$

(where T_z indicates application of T to functions of the z-variable) which is rapidly decreasing, since $k_1(x, z)$ and $T_z k_0(z, y)$ are square integrable in z and rapidly decreasing in x and y, respectively. □

5. Operators on manifolds with conic singularities

We are now going to discuss the cone algebra of Schulze. We shall not enter too much in details here but refer the reader to [3] and [24] for detailed presentations of the cone algebra. We shall focus on a version of the cone algebra which is sufficient for the characterization of the Fredholm property in certain weighted Sobolev spaces.

Let $\mathbb{B}$ be a smooth compact $(n+1)$-dimensional manifold with boundary $X := \partial\mathbb{B}$. We identify a collar neighborhood of X with $[0, 1) \times X$ and fix corresponding variables (t, x) near the boundary. The typical differential operators we consider on (the interior) of $\mathbb{B}$ are away from the boundary usual differential operators with smooth coefficients, while near the boundary they can be written in the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j, \quad a_j \in \mathscr{C}^\infty\bigl([0, 1), \mathrm{Diff}^{\mu-j}(X)\bigr),$$

with coefficients taking values in the space of differential operators on X. The Laplacian with respect to a Riemannian metric which near the boundary has the form $dt^2 + t^2\,dx^2$ is of that form, with $\mu = 2$. Such "cone differential operators" act in a scale of weighted Sobolev spaces

$$\mathscr{H}^{s,\gamma}(\mathbb{B}) = k^\gamma \mathscr{H}^{s,0}(\mathbb{B}), \quad s, \gamma \in \mathbb{R},$$

where k is a smooth positive function on the interior of $\mathbb{B}$ that coincides with $k(t,x) = t$ near the boundary, while $u \in \mathscr{H}^{s,0}(\mathbb{B})$ for $s \in \mathbb{N}_0$ if, and only if, $u \in H^s_{\mathrm{loc}}(\operatorname{int}\mathbb{B})$ and

$$(t\partial_t)^j D_x^\alpha u(t,x) \in L^2\big((0,1) \times X, t^n\,dt\,dx\big)$$

for all $j + |\alpha| \leqslant s$; this definition can be extended to real s by interpolation and duality. A differential operator as above induces then maps

$$A : \mathscr{H}^{s,\gamma}(\mathbb{B}) \to \mathscr{H}^{s-\mu,\gamma-\mu}(\mathbb{B}).$$

Roughly speaking, the cone algebra consists of pseudodifferential operators where the polynomial $h(t,z) = \sum_{j=0}^{\mu} a_j(t) z^j$ for the differential case described above is replaced by more general $L^\mu_{\mathrm{cl}}(X)$-valued functions $h(t,z)$ that are smooth up to $t = 0$ and holomorphic in a vertical strip of arbitrarily small width centered around the line $\operatorname{Re} z = (n+1)/2 - \gamma$. The residual operators in the calculus are integral operators with respect to the measure $t^n\,dt\,dx$ having kernel in $\mathscr{H}^{\infty,\gamma-\mu+\varepsilon}(\mathbb{B}) \widehat{\otimes}_\pi \mathscr{H}^{\infty,-\gamma+\varepsilon}(\mathbb{B})$. One also can extend this concept to operators acting on sections in vector bundles over $\mathbb{B}$, finally yielding the operator spaces

$$C^{\mu-j}\big(\mathbb{B}, (\gamma, E_0), (\gamma-\mu, E_1)\big), \quad j \in \mathbb{N}_0,\ \gamma \in \mathbb{R}. \tag{10}$$

With any element A of (10) with $j = 0$ we associate two principal symbols. The first is

$$\sigma_c^\mu(A) : \pi_c^* E_0 \to \pi_c^* E_1,$$

where $\pi_c : T_c^*\mathbb{B} \setminus 0 \to \mathbb{B}$ denotes the canonical projection of the "compressed" co-tangent bundle of $\mathbb{B}$ onto $\mathbb{B}$; for a precise definition see [5]. Roughly speaking, over the interior of $\mathbb{B}$ this symbol recovers the usual principal symbol, while for $t \to 0$ the product $t\tau$ (arising from the totally characteristic derivative $t\partial_t$) is replaced by a single variable $\widetilde{\tau}$. The second is the so-called *conormal symbol*

$$\sigma_M^\mu(A)(\tau) = h\left(0, \tfrac{n+1}{2} - \gamma + i\tau\right) : H^s\big(X, E_0'\big) \to H^{s-\mu}\big(X, E_1'\big), \quad \tau \in \mathbb{R},$$

where E_j' denotes the restriction of E_j to $X = \partial\mathbb{B}$ and the choice of s does not play a role. Note that this definition differs slightly from the definition usually used in the literature; it is convenient for us because in this way the conormal symbol becomes multiplicative, i.e.,

$$\sigma_M^{\mu_1+\mu_0}(A_1 A_0)(\tau) = \sigma_M^{\mu_1}(A_1)(\tau)\sigma_M^{\mu_0}(A_0)(\tau), \quad \tau \in \mathbb{R},$$

whenever $A_0 \in C^{\mu_0}(\mathbb{B},(\gamma,E_0),(\gamma-\mu_0,E_1))$, $A_1 \in C^{\mu_1}(\mathbb{B},(\gamma-\mu_0,E_1),(\gamma-\mu_0-\mu_1,E_2))$ and, similarly, it is compatible with taking adjoints. Identifying operator-valued functions with morphisms in trivial bundles, for an A from (10) with $j=0$ we get

$$\sigma_M^\mu(A) : \mathbb{R} \times H^s\big(X, E_0'\big) \to \mathbb{R} \times H^{s-\mu}\big(X, E_1'\big).$$

For convenience of notation let us now set $\gamma_0 = \gamma$ and $\gamma_1 = \gamma - \mu$. Given two projections $P_j \in C^0(\mathbb{B},(\gamma_j,E_j),(\gamma_j,E_j))$, the associated Toeplitz subalgebras are

$$T^{\mu-j}\big(\mathbb{B},(\gamma_0,E_0,P_0),(\gamma_1,E_1,P_1)\big) = P_1 C^{\mu-j}\big(\mathbb{B},(\gamma_0,E_0),(\gamma_1,E_1)\big)P_0.$$

The scales of projected Sobolev spaces are

$$\mathscr{H}^{s,\gamma_j}(\mathbb{B},E_j,P_j) = P_j\big(\mathscr{H}^{s,\gamma_j}(\mathbb{B},E_j)\big).$$

Since both $\sigma_c^0(P_j)$ and $\sigma_M^0(P_j)$ are projections we obtain subbundles

$$E_j^c(P_j) := \sigma_c^0(P_j)\big(\pi_c^* E_j\big) \subset \pi_c^* E_j,$$
$$E_j^{M,s}(P_j) := \sigma_M^0(P_j)\big(\mathbb{R} \times H^s\big(X, E_j'\big)\big) \subset \mathbb{R} \times H^s\big(X, E_j'\big).$$

The principal symbol $\sigma(A, P_0, P_1)$ consists of the two components

$$\sigma_c^\mu(A, P_0, P_1) : E_0^c(P_0) \to E_1^c(P_1),$$
$$\sigma_M^\mu(A, P_0, P_1) : E_0^{M,s}(P_0) \to E_1^{M,s-\mu}(P_1),$$

induced by the restriction of $\sigma_c^\mu(A)$ and $\sigma_M^\mu(A)$, respectively.

Theorem 5.1. *For $A \in T^\mu(\mathbb{B},(\gamma,E_0,P_0),(\gamma-\mu,E_1,P_1))$ the following statements are equivalent*:

(a) *A has a parametrix $B \in T^{-\mu}(\mathbb{B},(\gamma-\mu,E_1,P_1),(\gamma,E_0,P_0))$.*
(b) *$A : \mathscr{H}^{s,\gamma}(\mathbb{B},E_0,P_0) \to \mathscr{H}^{s-\mu,\gamma-\mu}(\mathbb{B},E_1,P_1)$ is a Fredholm operator for some $s \in \mathbb{R}$.*
(c) *Both morphisms $\sigma_c^\mu(A,P_0,P_1)$ and $\sigma_M^\mu(A,P_0,P_1)$ are isomorphisms.*

In this case, (b) *is true for arbitrary $s \in \mathbb{R}$.*

In fact, by the existence of suitable reductions of orders, we can reduce the proof of the previous theorem to the case $\mu = s = \gamma = 0$. This is then a particular case of Theorems 3.7 and 3.12, with

$$L^\mu(\mathbf{g}) = C^\mu\big(\mathbb{B},(0,E_0),(0,E_1)\big), \quad \mathbf{g} = \big((\mathbb{B},E_0),(\mathbb{B},E_1)\big),$$
$$H^s(g) = \mathscr{H}^{s,0}(\mathbb{B},E), \quad g = (\mathbb{B},E).$$

The Toeplitz algebras are spectrally invariant, as a consequence of Theorem 3.9 (with a proof very similar to that of Theorem 4.3).

Theorem 5.2. *If $A \in T^{\mu}(\mathbb{B},(\gamma,E_0,P_0),(\gamma-\mu,E_1,P_1))$ induces an isomorphism from $\mathscr{H}^{s,\gamma}(\mathbb{B},E_0,P_0)$ to $\mathscr{H}^{s-\mu,\gamma-\mu}(\mathbb{B},E_1,P_1)$ for some s, then for all s and*

$$A^{-1} \in T^{-\mu}\big(\mathbb{B},(\gamma-\mu,E_1,P_1),(\gamma,E_0,P_0)\big).$$

6. Boundary value problems

Let us denote by $\mathcal{B}^{\mu,d}(M;(E_0,J_0),(E_1,J_1))$ the space of all operators $\mathcal{A}=\begin{pmatrix} A_+ + G & K \\ T & Q\end{pmatrix}$ of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}_0$ as shortly described in the introduction. Note that the type d only enters in the structure of the singular Green operators G and the trace operators T and counts, roughly speaking, the number of derivatives in direction normal to the boundary contained in the action of G and T, respectively. Recall that any such $\mathcal{A}$ induces continuous mappings (1) whenever $s > d - 1/2$. The homogeneous principal symbol

$$\sigma_{\psi}^{\mu}(\mathcal{A}) : \pi^* E_0 \to \pi^* E_1,$$

coincides with the usual homogeneous principal symbol of the pseudodifferential operator A in the upper left corner of $\mathcal{A}$, acting between the pull-backs of E_j to the unit co-sphere bundle of M. One associates with $\mathcal{A}$ a so-called principal *boundary symbol*

$$\sigma_{\partial}^{\mu}(\mathcal{A}) : \pi_{\partial}^* \begin{pmatrix} E_0' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_0 \end{pmatrix} \to \pi_{\partial}^* \begin{pmatrix} E_1' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_1 \end{pmatrix},$$

where π_∂ denotes the canonical projection of the unit co-sphere bundle of ∂M to ∂M and E_j' denotes the restriction of E_j to the boundary; the choice of $s > d - 1/2$ does not play a role. The composition of operators induces a map

$$\begin{aligned} &\mathcal{B}^{\mu_1,d_1}\big(M;(E_1,J_1),(E_2,J_2)\big) \times \mathcal{B}^{\mu_0,d_0}\big(M;(E_0,J_0),(E_1,J_1)\big) \\ &\quad \to \mathcal{B}^{\mu_0+\mu_1,d}\big(M;(E_0,J_0),(E_2,J_2)\big), \quad d = \max(d_0, d_1+\mu_0). \end{aligned} \tag{11}$$

Both principal symbols are multiplicative under composition. The equivalence of ellipticity and Fredholm property in Boutet de Monvel's algebra has been shown in [13], the spectral invariance in [18].

6.1. Toeplitz subalgebras

Let $\mathcal{P}_j \in \mathcal{B}^{0,0}(M;(E_j,J_j),(E_j,J_j))$, $j=0,1$, be two projections and set

$$\mathcal{T}^{\mu,d}\big(M;(E_0,J_0,P_0),(E_1,J_1,P_1)\big) = \mathcal{P}_1 \mathcal{B}^{\mu,d}\big(M;(E_j,J_j),(E_j,J_j)\big)\mathcal{P}_0.$$

Note that, according to (11),

$$\mathcal{T}^{\mu,d}\big(M;(E_0,J_0,\mathcal{P}_0),(E_1,J_1,\mathcal{P}_1)\big) \subset \mathcal{B}^{\mu,\max(\mu,d)}\big(M;(E_0,J_0),(E_1,J_1)\big).$$

We shall thus focus on the case $\mu \leqslant d$. Since both $\sigma_{\psi}^{0}(\mathcal{P}_j)$ and $\sigma_{\partial}^{0}(\mathcal{P}_j)$ are projections we obtain subbundles

$$E_j^{\psi}(\mathcal{P}_j) := \sigma_\psi^0(\mathcal{P}_j)\big(\pi^* E_j\big) \subset \pi^* E_j,$$
$$E_j^{\partial,s}(\mathcal{P}_j) := \sigma_\partial^0(\mathcal{P}_j)\left(\pi_\partial^*\begin{pmatrix} E_j' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_j \end{pmatrix}\right) \subset \pi_\partial^*\begin{pmatrix} E_j' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_j \end{pmatrix}.$$

The principal symbol $\sigma(\mathcal{A}, \mathcal{P}_0, \mathcal{P}_1)$ consists of the two components

$$\sigma_\psi^\mu(\mathcal{A}, \mathcal{P}_0, \mathcal{P}_1) : E_0^\psi(\mathcal{P}_0) \to E_1^\psi(\mathcal{P}_1),$$
$$\sigma_\partial^\mu(\mathcal{A}, \mathcal{P}_0, \mathcal{P}_1) : E_0^{\partial,s}(\mathcal{P}_0) \to E_1^{\partial,s-\mu}(\mathcal{P}_1),$$

induced by the restriction of $\sigma_\psi^\mu(\mathcal{A})$ and $\sigma_\partial^\mu(\mathcal{A})$, respectively. In the following let us write $t_+ = \max(0, t)$.

Theorem 6.1. *For $\mathcal{A} \in \mathcal{T}^{\mu,\mu_+}(M; (E_0, J_0, \mathcal{P}_0), (E_1, J_1, \mathcal{P}_1))$ the following statements are equivalent*:

(a) *$\mathcal{A}$ has a parametrix $\mathcal{B} \in \mathcal{T}^{-\mu,(-\mu)_+}(M; (E_1, J_1, \mathcal{P}_1), (E_0, J_0, \mathcal{P}_0))$, i.e.,*

$$\mathcal{B}\mathcal{A} - \mathcal{P}_0 \in \mathcal{T}^{-\infty,\mu_+}\big(M; (E_0, J_0, \mathcal{P}_0), (E_0, J_0, \mathcal{P}_0)\big),$$
$$\mathcal{A}\mathcal{B} - \mathcal{P}_1 \in \mathcal{T}^{-\infty,(-\mu)_+}\big(M; (E_1, J_1, \mathcal{P}_1), (E_1, J_1, \mathcal{P}_1)\big).$$

(b) *For some $s > \mu_+ - 1/2$*

$$\mathcal{A} : \mathcal{P}_0\begin{pmatrix} H^s(M, E_0) \\ \oplus \\ H^s(\partial M, J_0) \end{pmatrix} \to \mathcal{P}_1\begin{pmatrix} H^{s-\mu}(M, E_1) \\ \oplus \\ H^{s-\mu}(\partial M, J_1) \end{pmatrix}$$

is a Fredholm operator.

(c) *Both morphisms $\sigma_\psi^\mu(\mathcal{A}, \mathcal{P}_0, \mathcal{P}_1)$ and $\sigma_\partial^\mu(\mathcal{A}, \mathcal{P}_0, \mathcal{P}_1)$ are isomorphisms.*

In this case, (b) *is true for arbitrary $s > \mu_+ - 1/2$.*

Note that this result is both a generalization and strengthening of Theorem 2.2 of [20] in the case $d = \mu_+$. Also here the crucial point is to reduce to the case of $\mu = s = 0$. This is then a particular case of Theorems 3.7 and 3.12, with

$$L^\mu(\mathbf{g}) = \mathcal{B}^{\mu,0}\big(M; (E_0, J_0), (E_1, J_1)\big), \qquad \mathbf{g} = \big((M, E_0, J_0), (M, E_1, J_1)\big),$$
$$H^s(g) = \begin{matrix} H^s(M, E) \\ \oplus \\ H^s(\partial M, J) \end{matrix}, \qquad g = (M, E, J).$$

In fact, this reduction is possible, since one can show that for any bundles E and J there exist elements $\mathcal{R}^\mu \in \mathcal{B}^{\mu,0}(M; (E, J), (E, J))$, $\mu \in \mathbb{Z}$, that induce isomorphisms $H^s(M, E) \oplus H^s(\partial M, J) \to H^{s-\mu}(M, E) \oplus H^{s-\mu}(\partial M, J)$ and such that $(\mathcal{R}^\mu)^{-1} = \mathcal{R}_{-\mu}$; see [1,13,6]. This also allows to obtain the spectral invariance, using Theorem 3.9 (with a proof very similar to that of Theorem 4.3, using that smoothing operators of type 0 are integral operators with smooth kernel).

Theorem 6.2. *If $\mathcal{A} \in \mathcal{T}^{\mu,\mu_+}(M; (E_0, J_0, \mathcal{P}_0), (E_1, J_1, \mathcal{P}_1))$ induces an isomorphism*

$$\mathcal{A} : \mathcal{P}_0 \begin{pmatrix} H^s(M, E_0) \\ \oplus \\ H^s(\partial M, J_0) \end{pmatrix} \to \mathcal{P}_1 \begin{pmatrix} H^{s-\mu}(M, E_1) \\ \oplus \\ H^{s-\mu}(\partial M, J_1) \end{pmatrix}$$

for some $s > \mu_+ - 1/2$, then for all $s > \mu_+ - 1/2$ and

$$\mathcal{A}^{-1} \in \mathcal{T}^{-\mu,(-\mu)_+}\big(M; (E_1, J_1, \mathcal{P}_1), (E_0, J_0, \mathcal{P}_0)\big).$$

6.2. The Stokes operator

We shall need the following result on the existence of reductions of orders in Toeplitz subalgebras.

Lemma 6.3. *Let Y be a closed Riemannian manifold and E a Hermitian vector bundle over Y. Moreover, let $P \in L^0_{\mathrm{cl}}(Y; E, E)$ be an orthogonal projection and $\mu \in \mathbb{R}$. Then there exist $R^t \in T^t(Y; (E, P), (E, P))$ for $t = \mu$ and $t = -\mu$ such that $R^\mu R^{-\mu} = R^{-\mu} R^\mu = P$.*

Proof. We can assume $\mu > 0$. Choose an $S \in L^\mu_{\mathrm{cl}}(Y; E, E)$ which is invertible, symmetric, and satisfies

$$(Su, u) > 0, \quad \text{for all } 0 \neq u \in \mathscr{C}^\infty(Y, E),$$

where $(\cdot,\cdot)$ is a scalar-product of $L^2(Y, E)$. Then $R := PSP + (1-P)S(1-P)$ is also symmetric and

$$(Ru, u) = (SPu, Pu) + \big(S(1-P)u, (1-P)u\big) > 0, \quad 0 \neq u \in \mathscr{C}^\infty(Y, E).$$

Since the spectrum of elliptic and positive operators consists of isolated positive eigenvalues, we conclude that R is invertible with inverse in $L^{-\mu}_{\mathrm{cl}}(Y; E, E)$. Then $R^\mu := PSP$ induces isomorphisms $H^s(Y, E, P) \to H^{s-\mu}(Y, E, P)$. Due to spectral invariance, cf. Theorem 3.9, the claim follows. □

Now let $n \geqslant 2$ be the dimension of M and let

$$L^2_\sigma\big(M, \mathbb{C}^n\big) = \big\{u \in L^2\big(M, \mathbb{C}^n\big) \bigm| \operatorname{div} u = 0,\ \gamma_\nu u = 0\big\}$$

denote the space of square integrable solenoidal vector fields; here we use the notation

$$\gamma u = u|_{\partial M}, \qquad \gamma_\nu u = \nu \cdot \gamma u,$$

where ν denotes the outer normal of M. Also let us set

$$H^s_\sigma\big(M, \mathbb{C}^n\big) = H^s\big(M, \mathbb{C}^n\big) \cap L^2_\sigma\big(M, \mathbb{C}^n\big).$$

It has been shown in [8] that there is a projection $P \in \mathcal{B}^{0,0}(M; (\mathbb{C}^n, 0), (\mathbb{C}^n, 0))$ (the Helmholtz projection, of course) such that

$$H^s_\sigma\big(M, \mathbb{C}^n\big) = P\big(H^s\big(M, \mathbb{C}^n\big)\big).$$

Let us define the projection $Q \in L^0_{\mathrm{cl}}(\partial M; \mathbb{C}^n, \mathbb{C}^n)$ by

$$Qv = v - (v \cdot \nu)\nu. \tag{12}$$

This induces a projection of $H^s(\partial M, \mathbb{C}^n)$ onto

$$H^s_\nu\big(\partial M, \mathbb{C}^n\big) := \big\{v \in H^s\big(\partial M, \mathbb{C}^n\big) \bigm| v \cdot \nu = 0\big\}.$$

The latter space arises by restricting solenoidal vector fields to the boundary, i.e.,

$$\gamma : H^s_\sigma\big(M, \mathbb{C}^n\big) \to H^s_\nu\big(\partial M, \mathbb{C}^n\big) \tag{13}$$

surjectively, cf. Proposition 2.1 in [4]. The Stokes operator (with Dirichlet boundary conditions) is now

$$\begin{pmatrix} P\Delta \\ \gamma \end{pmatrix} : H^s_\sigma\big(M, \mathbb{C}^n\big) \to \begin{matrix} H^{s-2}_\sigma(M, \mathbb{C}^n) \\ \oplus \\ H^{s-1/2}_\nu(\partial M, \mathbb{C}^n) \end{matrix}. \tag{14}$$

Due to Lemma 6.3 we can choose an $R \in T^{3/2}(\partial M; \mathbb{C}^n, \mathbb{C}^n)$ inducing isomorphisms $H^s_\nu(\partial M, \mathbb{C}^n) \to H^{s-3/2}_\nu(\partial M, \mathbb{C}^n)$ for all s, and then consider

$$\begin{pmatrix} P\Delta \\ T \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} P\Delta \\ \gamma \end{pmatrix} : H^s_\sigma\big(M, \mathbb{C}^n\big) \to \begin{matrix} H^{s-2}_\sigma(M, \mathbb{C}^n) \\ \oplus \\ H^{s-2}_\nu(\partial M, \mathbb{C}^n) \end{matrix}. \tag{15}$$

Now we can rewrite (15) in the form

$$\mathcal{A} := \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \Delta \\ T \end{pmatrix} P.$$

Setting

$$\mathcal{P}_0 = P \in B^{0,0}\big(M; \big(\mathbb{C}^n, 0\big), \mathbb{C}^n, 0\big),$$
$$\mathcal{P}_1 = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in \mathcal{B}^{0,0}\big(M; \big(\mathbb{C}^n, \mathbb{C}^n\big), \big(\mathbb{C}^n, \mathbb{C}^n\big)\big),$$

we obtain that $\mathcal{A}$ belongs to a Toeplitz subalgebra,

$$\mathcal{A} \in \mathcal{T}^{2,2}\big(M; \big(\mathbb{C}^n, 0, \mathcal{P}_0\big), \big(\mathbb{C}^n, \mathbb{C}^n, \mathcal{P}_1\big)\big).$$

Now (14) is an isomorphism for $s = 2$, hence so is (15). In fact, (13) is surjective and the Dirichlet realization (4) is known to be self-adjoint and positive, hence is an isomorphism. Then we can use the elementary fact that a linear map

$$\begin{pmatrix} A \\ T \end{pmatrix} : H \to \begin{matrix} E \\ \oplus \\ F \end{matrix}$$

is an isomorphism if, and only if, $T : H \to F$ is surjective and $A : \ker T \to E$ is an isomorphism.

From Theorem 6.2 we can conclude that the invertibility of $\mathcal{A}$ holds for any $s > 3/2$ and that the inverse is realized by an element belonging to $\mathcal{T}^{-2,0}(M; (\mathbb{C}^n, \mathbb{C}^n, \mathcal{P}_1), (\mathbb{C}^n, 0, \mathcal{P}_0))$. Reformulating this result for the original Stokes operator (14) yields the following:

Theorem 6.4. *The Stokes operator with Dirichlet boundary conditions,*

$$\begin{pmatrix} P\Delta \\ \gamma \end{pmatrix} : H^s_\sigma\big(M, \mathbb{C}^n\big) \to \begin{matrix} H^{s-2}_\sigma(M, \mathbb{C}^n) \\ \oplus \\ H^{s-1/2}_\nu(\partial M, \mathbb{C}^n) \end{matrix},$$

is invertible for any $s > 3/2$ and

$$\begin{pmatrix} P\Delta \\ \gamma \end{pmatrix}^{-1} = P\begin{pmatrix} A_+ + G & KS \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P(A_+ + G)P & PKSQ \end{pmatrix},$$

where P is the Helmholtz projection, Q the projection introduced in (12), $S \in L^{-3/2}_{\mathrm{cl}}(\partial M; \mathbb{C}^n, \mathbb{C}^n)$, and

$$\begin{pmatrix} A_+ + G & K \end{pmatrix} \in \mathcal{B}^{-2,0}\big(M; \big(\mathbb{C}^n, \mathbb{C}^n\big), \big(\mathbb{C}^n, 0\big)\big).$$

Since operators from Boutet de Monvel's algebra are known to act continuously in L_p-Sobolev and Besov spaces, cf. [7], the previous theorem remains valid in a corresponding L_p-version with $1 < p < \infty$.

Acknowledgments

The author thanks Robert Denk (Konstanz) and Jürgen Saal (Darmstadt) for helpful discussions concerning the Stokes operator.

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