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# Revlex-initial 0/1-polytopes☆

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**Abstract**

We introduce revlex-initial 0/1-polytopes as the convex hulls of reverse-lexicographically initial subsets of 0/1-vectors. These polytopes are special knapsack-polytopes. It turns out that they have remarkable extremal properties. In particular, we use these polytopes in order to prove that the minimum numbers $g_{\mathrm{nfac}}(d,n)$ of facets and the minimum average degree $g_{\mathrm{avdeg}}(d,n)$ of the graph of a $d$-dimensional 0/1-polytope with $n$ vertices satisfy $g_{\mathrm{nfac}}(d,n) \leqslant 3d$ and $g_{\mathrm{avdeg}}(d,n) \leqslant d+4$. We furthermore show that, despite the sparsity of their graphs, revlex-initial 0/1-polytopes satisfy a conjecture due to Mihail and Vazirani, claiming that the graphs of 0/1-polytopes have edge-expansion at least one.



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## 1. Introduction

Let us call a subset $X$ of $\{0,1\}^d$ *revlex-initial* if, for every $x \in X$, all points in $\{0,1\}^d$ that are reverse-lexicographically smaller than $x$ are contained in $X$. The convex hulls of revlex-initial subsets of $\{0,1\}^d$ are the *revlex-initial* 0/1-*polytopes*. Phrased differently, the revlex-initial 0/1-polytopes are the convex hulls of those sets of 0/1-vectors of length $d$ that correspond to the binary representations of all numbers $0, 1, \ldots, n-1$ for some $n$.

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In particular, for every $1 \leqslant n \leqslant 2^d$ there is precisely one revlex-initial 0/1-polytope with $n$ vertices in $\mathbb{R}^d$.

Why should one be interested in such special polytopes? The general interest in 0/1-polytopes stems from their importance in combinatorial optimization. Investigations of 0/1-polytopes like traveling salesman polytopes, cut polytopes, stable set polytopes, and matching polytopes have not only led to beautiful insights into the interplay of combinatorics and geometry, but also to great algorithmic progress with respect to the corresponding optimization problems. From that work on such special 0/1-polytopes quite a few general questions on 0/1-polytopes have emerged, such as, e.g., the question for the maximal number of facets a $d$-dimensional 0/1-polytope may have (see Ziegler [15]).

With respect to this extremal question, Bárány and Pór [2] obtained a remarkable result. They showed that a random $d$-dimensional 0/1-polytope with roughly $2^{d/\log_2 d}$ vertices in expectation has at least (roughly) $2^{(1/4)d \log_2 d}$ facets. Recently, this bound was even improved to $2^{(1/2)d \log_2 d}$ by Gatzouras et al. [5]. The best known upper bound currently is $O((d-2)!)$ (due to Fleiner et al. [4]). It turns out that the revlex-initial 0/1-polytopes studied in this paper give some answers to two reverse extremal questions: How *few* facets or edges can a $d$-dimensional 0/1-polytope with a specified number of vertices have?

Note that, somewhat different to the class of general polytopes, the number of vertices of a 0/1-polytope may impose severe restrictions on the combinatorial type. For instance, a 0/1-polytope is simple if and only if it is the product of 0/1-simplices [8]. Thus, $d$-dimensional simple 0/1-polytopes with $n$ vertices do only exist if there is a factorization $n = \prod n_i$ of $n$ with $d = \sum(n_i - 1)$. Therefore, within the realm of 0/1-polytopes, it seems interesting to investigate extremal questions for *all* (reasonable) pairs $(d, n)$.

Our paper contains three main results.

(1) Revlex-initial 0/1-polytopes in $\mathbb{R}^d$ have no more than $3d$ facets (Theorem 2); from this we deduce that the smallest number of facets $g_{\text{nfac}}(d, n)$ of a $d$-dimensional 0/1-polytope with exactly $n$ vertices satisfies $g_{\text{nfac}}(d, n) \leqslant 3d$ for all $d$ and $n$ and $g_{\text{nfac}}(d, n(d)) \leqslant d + o(d)$ if $n(d)$ grows sub-exponentially with $d$ (Theorem 6).

(2) The average degree of every revlex-initial 0/1-polytope in $\mathbb{R}^d$ is at most $d+4$ (Theorem 4); from this we deduce that the smallest average degree $g_{\text{avdeg}}(d, n)$ of a $d$-dimensional 0/1-polytope with exactly $n$ vertices satisfies $g_{\text{avdeg}}(d, n) \leqslant d + 4$ (Theorem 7).

Since revlex-initial 0/1-polytopes have extremely sparse graphs, at first sight they look like candidates for counter-examples to an important conjecture due to Mihail and Vazirani (cited, e.g., in [3,9]) stating that the graph of every 0/1-polytope has edge-expansion at least one. However, supporting that conjecture, we prove:

(3) Revlex-initial 0/1-polytopes have edge-expansion at least one (Theorem 5); from this we deduce that, for every (reasonable) pair $(d, n)$, there are $d$-dimensional 0/1-polytopes with $n$ vertices, sparse graphs, and edge-expansion at least one (Theorem 8).

The context in which we came to study the special class of revlex-initial 0/1-polytopes is described in Section 3.4. They appeared from investigating an apparently strange behavior of certain convex hull algorithms on random 0/1-polytopes.

The notion of revlex-initial subsets of $\{0, 1\}^d$, or, equivalently, of a system of subsets of $\{1, \ldots, d\}$, is not new. It is related to the notion of *compression* of a set system, which plays an important role in the Kruskal–Katona theorem (see, e.g., [14, Theorem 8.32]) characterizing the $f$-vectors of simplicial complexes. Here, a system $\mathcal{S}$ of subsets of $\{1, \ldots, d\}$

(corresponding to a subset $X \subseteq \{0,1\}^d$) is called *compressed* if, for every $i$, the subsystem of $\mathcal{S}$ containing all sets from $\mathcal{S}$ of cardinality $i$ is reverse-lexicographically initial within the $i$-subsets of $\{1,\dots,d\}$. Clearly, every revlex-initial subset of $\{0,1\}^d$ corresponds to a compressed system of subsets of $\{1,\dots,d\}$, but the converse is not true.

In the context of the Kruskal–Katona theorem only compressed set systems that are closed under taking subsets are considered. Of course, all revlex-initial 0/1-polytopes correspond to compressed set systems with that property (i.e., revlex-initial 0/1-polytopes are *monotone*). But even more: Exploiting the interpretation in terms of binary representations of numbers, one finds that revlex-initial 0/1-polytopes are a special kind of knapsack polytopes (see Section 2).

Note that the terminus "compressed polytope" has already been coined with a different meaning (see, e.g., [13]).

## 2. Definitions

Throughout the paper, we assume that $d$ is a positive integer number. We start with fixing some notions and notation.

**Definition 1** (*Index ranges*). For a positive integer number $k$, let

$$[k] := \{1,2,\dots,k\} \quad \text{and} \quad [k]_0 := \{0,1,\dots,k-1\}.$$

We will identify $\mathbb{R}^d$ with $\mathbb{R}^{[d]_0}$, i.e. vectors $x \in \mathbb{R}^d$ have components $x_0, x_1, \dots, x_{d-1}$, similarly for $\mathbb{N}^d$.

**Definition 2** (*Reverse-lexicographical order*). A point $x \in \{0,1\}^d$ is *reverse-lexicographically smaller* than another point $y \in \{0,1\}^d \setminus \{x\}$ ($x \prec_{\text{rlex}} y$) if $x_{i_{\max}} < y_{i_{\max}}$ holds for $i_{\max} := \max\{i : x_i \neq y_i\}$. We denote $x \preceq_{\text{rlex}} y$ if $x = y$ or $x \prec_{\text{rlex}} y$ hold for $x, y \in \{0,1\}^d$.

For $x \in \{0,1\}^d$ denote $S(x) := \{i \in [d]_0 : x_i = 1\}$. Then we have

$$x \prec_{\text{rlex}} y \quad \Leftrightarrow \quad \max(S(x) \triangle S(y)) \in S(y)$$

for all $x, y \in \{0,1\}^d$ ($x \neq y$), where $\triangle$ denotes the symmetric difference of two sets.

**Definition 3** (*Revlex-initial 0/1-polytope*). A subset $X \subseteq \{0,1\}^d$ is *revlex-initial* if, for every $x \in X$, it contains all $y \in \{0,1\}^d$ with $y \prec_{\text{rlex}} x$. For $v \in \{0,1\}^d$ define

$$X^{\prec v} := \{x \in \{0,1\}^d : x \prec_{\text{rlex}} v\}.$$

A *revlex-initial* 0/1-*polytope* is the convex hull of any revlex-initial 0/1-set. We denote

$$P^{\prec v} := \operatorname{conv} X^{\prec v}.$$

Since $\prec_{\text{rlex}}$ defines a total ordering of $\{0,1\}^d$, every revlex-initial 0/1-set $X$ with $|X| < 2^d$ is of the form $X^{\prec v}$ for some $v \in \{0,1\}^d \setminus \{\mathbb{O}\}$. Note that $v \notin P^{\prec v}$.

Table 1
Example illustrating some of the definitions: we have $d = 10$, $w(v) = 5$, $S(v) = \{0, 2, 3, 6, 9\}$, and $\overline{S}(v) = \{1, 4, 5, 7, 8\}$

| $v$ | **1** | **0** | **1** | **1** | **0** | **0** | **1** | **0** | **0** | **1** |
|---|---|---|---|---|---|---|---|---|---|---|
| Indices | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Signature | $\sigma_5(v)$ | | $\sigma_4(v)$ | $\sigma_3(v)$ | | | $\sigma_2(v)$ | | | $\sigma_1(v)$ |
| $P_1^{\prec v}$ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | **0** |
| $P_2^{\prec v}$ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ | **0** | **0** | **0** | **1** |
| $P_3^{\prec v}$ | ⋆ | ⋆ | ⋆ | **0** | **0** | **0** | **1** | **0** | **0** | **1** |
| $P_4^{\prec v}$ | ⋆ | ⋆ | **0** | **1** | **0** | **0** | **1** | **0** | **0** | **1** |
| $P_5^{\prec v}$ | **0** | **0** | **1** | **1** | **0** | **0** | **1** | **0** | **0** | **1** |

**Definition 4** (*Signature of a 0/1-point*). Let $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$. Its *weight* $w(v) := \mathbb{1}^\top v$ is the number of ones of $v$. Its *signature* is the vector

$$(\sigma_1(v), \ldots, \sigma_{w(v)}(v))$$

with

$$S(v) = \{\sigma_1(v), \ldots, \sigma_{w(v)}(v)\} \quad \text{and} \quad \sigma_1(v) > \sigma_2(v) > \cdots > \sigma_{w(v)}(v).$$

Further we define the index set of all zero-components

$$\overline{S}(v) := [d]_0 \setminus S(v).$$

**Definition 5** (*Block decomposition*). For a 0/1-point $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$ with signature $(\sigma_1(v), \ldots, \sigma_{w(v)}(v))$, we call

$$X_q^{\prec v} := \{x \in \{0,1\}^d : x_{\sigma_q(v)} = 0, x_{\sigma_q(v)+1} = v_{\sigma_q(v)+1}, \ldots, x_{d-1} = v_{d-1}\}$$

(for $q \in [w(v)]$) the *blocks* of $P^{\prec v}$. Clearly, $X^{\prec v}$ is the disjoint union

$$X^{\prec v} = X_1^{\prec v} \uplus \cdots \uplus X_{w(v)}^{\prec v}$$

of its blocks. The faces $P_q^{\prec v} := \operatorname{conv} X_q^{\prec v}$ are the *block faces* of $P^{\prec v}$. The vector

$$(\dim P_1^{\prec v}, \ldots, \dim P_{w(v)}^{\prec v}) = (\sigma_1(v), \ldots, \sigma_{w(v)}(v))$$

is the *signature* of the revlex-initial 0/1-polytope $P^{\prec v}$ (Table 1).

As mentioned in the introduction, revlex-initial 0/1-polytopes are a special kind of knapsack polytopes. Indeed, for $d \in \mathbb{N}$ we define $a \in \mathbb{N}^d$ as $a_i := 2^i$. Then for two 0/1-vectors $v, w \in \{0,1\}^d$ we have $v \prec_{\text{rlex}} w$ if and only if $a^\top v < a^\top w$ holds. Thus we can identify each natural number $n \in \mathbb{N}$ with a unique 0/1-vector $v \in \{0,1\}^d$ for a unique $d$ such that $n = a^\top v$ and $v_{d-1} = 1$. Therefore we write $P^{<n}$ with $n \in \mathbb{N}$ instead of $P^{\prec v}$ with $v \in \{0,1\}^d$ with $v_{d-1} = 1$. With the above identification, $P^{<n}$ has exactly the $n$ vertices corresponding to the numbers $0, 1, \ldots, n-1$. In other words, $P^{\prec v}$ with $v \in \{0,1\}^d$ is the knapsack polytope $\operatorname{conv}\{x \in \{0,1\}^d : a^\top x \leqslant a^\top v - 1\}$.

## 3. The facets of revlex-initial 0/1-polytopes

### 3.1. Optimizing linear functions

For $c \in \mathbb{R}^d$ and $I \subseteq [d]_0$, define

$$c^+(I) := \sum_{i \in I} \max\{c_i, 0\}.$$

The following statement follows immediately from the block decomposition of revlex-initial 0/1-polytopes.

**Proposition 1.** *For every $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$ and $c \in \mathbb{R}^d$, we have*

$$\max\{c^\top x : x \in P^{\prec v}\} = \max\left\{\sum_{p=1}^{q-1} c_{\sigma_p(v)} + c^+([\sigma_q(v)]_0) : q \in [w(v)]\right\}.$$

In particular, the optimization problem $\max\{c^\top x : x \in P^{\prec v}\}$ (for given $v \in \{0,1\}^d$ and $c \in \mathbb{Q}^d$) can be solved in polynomial time.

### 3.2. A linear description

If $i \in \overline{S}(v)$ and $x \in X^{\prec v}$ with $x_i = 1$, then $x_j = 0$ must hold for some $j \in S(v)$ with $j > i$. Let us denote

$$S^{>i}(v) := \{j \in S(v) : j > i\} \quad \text{and} \quad \overline{S}^{>i}(v) := \{j \in \overline{S}(v) : j > i\}.$$

We will use similar notations with respect to $<$, $\leqslant$ and $\geqslant$. Thus, the inequalities

$$x_i + \sum_{j \in S^{>i}(v)} x_j \leqslant |S^{>i}(v)| \quad \text{for all } i \in \overline{S}(v) \tag{1}$$

and (since $v \notin P^{\prec v}$)

$$\sum_{j \in S(v)} x_j = v^\top x = \leqslant |S(v)| - 1 \tag{2}$$

are valid for $P^{\prec v}$. These inequalities are minimal cover inequalities. In fact, they are all minimal cover inequalities of the knapsack polytope $P^{\prec v}$.

**Theorem 1** (*Linear descriptions of revlex-initial 0/1-polytopes*). *For every $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$ the revlex-initial 0/1-polytope $P^{\prec v}$ has the following linear description*:

$$P^{\prec v} = \{x \in \mathbb{R}^d : \mathbb{0} \leqslant x \leqslant \mathbb{1},\ x \text{ satisfies (1) and (2)}\} \tag{3}$$

**Proof.** Denote the polytope defined by the right-hand side of (3) by $Q(v)$. Thus, $Q(v)$ is the set of all $x \in \mathbb{R}^d$ satisfying the following system of inequalities:

$$-x_i \leqslant 0 \quad \text{for all } i \in [d]_0 \tag{4}$$

$$x_i \leqslant 1 \quad \text{for all } i \in [d]_0 \tag{5}$$

$$x_i + \sum_{j \in S^{>i}(v)} x_j \leqslant |S^{>i}(v)| \quad \text{for all } i \in \overline{S}(v) \tag{6}$$

$$\sum_{j \in S(v)} x_j \leqslant w(v) - 1 \tag{7}$$

Denote by $A$ the matrix with the left-hand side coefficients of the inequalities in (6) and (7). The rows of $A$ can be put into an order, such that $A$ is an interval matrix. Thus $A$ is total unimodular (see, e.g., [11, Example 7, p. 279]), and appending the identity matrices $I_d$ and $-I_d$ does not change total unimodularity.

Since the right-hand sides of the inequalities in (4)–(7) are integers, all vertices of $Q(v)$ are integer vectors and by the inequalities of type (4) and (5) they are binary vectors. Therefore $Q(v) = P^{\prec v}$, since $Q(v) \cap \{0,1\}^d = X^{\prec v}$. □

### 3.3. The facet defining inequalities

Let us first describe the dimension of a revlex-initial 0/1-polytope.

**Proposition 2.** *For each $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$ the dimension of the revlex-initial 0/1-polytope $P^{\prec v}$ is*

$$\dim P^{\prec v} = 1 + \max\left(\{i \in [d]_0 : \mathbb{e}_i \prec_{\text{rlex}} v\} \cup \{-1\}\right),$$

*In our knapsack notation we have for $n \in \mathbb{N}$*

$$\dim P^{<n} = 1 + \max\{i \in \mathbb{N} \cup \{-1\} : 2^i < n\} = \min\{j \in \mathbb{N} : n \leqslant 2^j\}.$$

**Proof.** This follows from the block decomposition of $P^{\prec v}$. □

In particular, $P^{\prec v}$ is full-dimensional if and only if $\mathbb{e}_{d-1} \prec_{\text{rlex}} v$ (that is, $2^{d-1} < n \leqslant 2^d$). The following three propositions describe the facets of full-dimensional revlex-initial 0/1-polytopes.

**Proposition 3.** *For each $v \in \{0,1\}^d$ with $\mathbb{e}_{d-1} \prec_{\text{rlex}} v$ and for every $i \in [d]_0$, the inequality $x_i \geqslant 0$ defines a facet of $P^{\prec v}$.*

**Proof.** By Theorem 1, the inequalities (4)–(7) provide a linear description of $P^{\prec v}$. Since the trivial inequalities (4) are the only ones in this description which have negative coefficients, none of them can be conically combined from others. Hence, they all define facets of $P^{\prec v}$ (since $P^{\prec v}$ is full-dimensional). □

**Proposition 4.** *For each $v \in \{0,1\}^d$ with $\mathbb{e}_{d-1} \prec_{\text{rlex}} v$, the inequality $\sum_{j \in S(v)} x_j \leqslant w(v) - 1$ defines a facet of $P^{\prec v}$.*

**Proof.** The inequality $\sum_{j\in S(v)} x_j \leqslant w(v)-1$ is the only inequality in the linear description (4)–(7) of $P^{\prec v}$ provided by Theorem 1 that is violated by the point $v$, which is not contained in $P^{\prec v}$. Thus, that inequality must define a facet of $P^{\prec v}$. □

**Proposition 5.** *For each $v \in \{0,1\}^d$ with $\mathbb{e}_{d-1} \prec_{\text{rlex}} v$ and for every $i \in [d]_0$, the inequality $x_i \leqslant 1$ defines a facet of $P^{\prec v}$ unless*

$$w(v) = 2 \quad \textit{and} \quad i \in S(v)$$

*or*

$$\sigma_2(v) < d-2 \quad \textit{and} \quad \sigma_2(v) < i \leqslant d-1$$

(*in which cases they do not define facets*).

**Proof.** Unless one of the exceptions listed in the proposition holds, all inequalities from the linear description (4)–(7) of $P^{\prec v}$ provided by Theorem 1 that have a positive $i$th coefficient have right-hand side at least two. Since the only ones with negative $i$th coefficient have right-hand side zero, the inequality $x_i \leqslant 1$ cannot be conically combined from the others in that linear description. Hence it defines a facet of $P^{\prec v}$ (since $P^{\prec v}$ is full-dimensional).

In case of $w(v) = 2$ and $i \in S(v)$, let $j$ be such that $S(v) = \{i, j\}$. Thus, $x_i \leqslant 1$ is the sum of inequality (7) and $-x_j \leqslant 0$. Hence, it does not define a facet of $P^{\prec v}$.

Finally, consider the case $\sigma_2(v) < d-2$. If $\sigma_2(v) < i < d-1$, then the type-(6) inequality $x_i + x_{d-1} \leqslant 1$ implies $x_i \leqslant 1$ by adding $-x_{d-1} \leqslant 0$. If $i = d-1$ then the type-(6) inequality $x_j + x_{d-1} \leqslant 1$ for any $\sigma_2(v) < j < d-1$ implies $x_{d-1} \leqslant 1$ by adding $-x_j \leqslant 0$. Thus, in both cases, $x_i \leqslant 1$ does not define a facet of $P^{\prec v}$. □

**Proposition 6.** *For each $v \in \{0,1\}^d$ with $\mathbb{e}_{d-1} \prec_{\text{rlex}} v$ and for every $i \in \overline{S}(v)$, the inequality $x_i + \sum_{j\in S^{>i}(v)} x_j \leqslant |S^{>i}(v)|$ defines a facet of $P^{\prec v}$ unless $i < \sigma_{w(v)}(v)$ (in which case it does not define a facet*).

**Proof.** For each $i \in \overline{S}(v)$ with $i > \sigma_{w(v)}(v)$, the inequality $x_i + \sum_{j\in S^{>i}(v)} x_j \leqslant |S^{>i}(v)|$ is the only inequality in the linear description (4)–(7) of $P^{\prec v}$ that is violated by the point $v + \mathbb{e}_i - \mathbb{e}_{\sigma_{w(v)}(v)}$, which is not contained in $P^{\prec v}$. Thus, that inequality must define a facet of $P^{\prec v}$.

If $i < \sigma_{w(v)}(v)$, then $x_i + \sum_{j\in S^{>i}(v)} x_j \leqslant |S^{>i}(v)|$ does not define a facet since it equals the sum of the two inequalities $\sum_{j\in S(v)} x_j \leqslant w(v) - 1$ and $x_i \leqslant 1$. □

Combining Theorem 1 and the five preceding propositions, we obtain the following result.

**Theorem 2** (*Facets of revlex-initial 0/1-polytopes*). *Let $v \in \{0,1\}^d$ with $\mathbb{e}_{d-1} \prec_{\text{rlex}} v$, i.e., $P^{\prec v}$ is a full-dimensional revlex-initial 0/1-polytope. Let*

$$D(v) := D_1(v) \cup D_2(v)$$

*with*

$$D_1(v) \begin{cases} S(v) & \text{if } w(v) = 2, \\ \varnothing & \text{otherwise.} \end{cases}$$

*and*

$$D_2(v) \begin{cases} \{\sigma_2(v)+1, \dots, d-1\} & \text{if } \sigma_2(v) < d-2 \\ \varnothing & \text{otherwise} \end{cases}.$$

(1) *The following system is a minimal* (*with respect to. inclusion*) *linear description of* $P^{\prec v}$ *by facet defining inequalities*:

$$\begin{aligned} x_i &\geqslant 0 && \text{for all } i \in [d]_0, \\ x_i &\leqslant 1 && \text{for all } i \in [d]_0 \setminus D(v), \\ x_i + \sum_{j \in S^{>i}(v)} x_j &\leqslant |S^{>i}(v)| && \text{for all } i \in \overline{S}(v), i > \sigma_{w(v)}(v), \\ \sum_{j \in S(v)} x_j &\leqslant w(v) - 1. \end{aligned}$$

(2) *The number of facets of* $P^{\prec v}$ *is*

$$f_{d-1}(P^{\prec v}) = 2d + \left|\{\sigma_{w(v)}(v) < i < \sigma_2(v) : v_i = 0\}\right| + \varepsilon,$$

*where*

$$\varepsilon := \begin{cases} -1 & \text{if } w(v) = 2, \\ 0 & \text{if } w(v) > 2, v_{d-2} = 0, \\ 1 & \text{otherwise (i.e., } w(v) > 2, v_{d-2} = 1). \end{cases}$$

*We have*

$$2d - 1 \leqslant f_{d-1}(P^{\prec v}) \leqslant 3d - 2.$$

*The minimum number* $2d-1$ *of facets is attained if and only if* $w(v) = 2$, *and the maximum* $f_{d-1}(P^{\prec v}) = 3d-2$ *is achieved only by* $v = \mathbb{e}_0 + \mathbb{e}_{d-2} + \mathbb{e}_{d-1}$ (*for* $d \geqslant 3$).

See Fig. 2 for an illustration of the facet numbers of revlex-initial 0/1-polytopes.

### 3.4. Incremental convex-hull algorithms

The origin of our investigations on revlex-initial 0/1-polytopes lies in some experiments on computing the convex hulls of random 0/1-polytopes that we performed with the `polymake` system. Some of the results of the experiments are illustrated in Fig. 1, showing the running times for computing the convex hulls of (uniformly) random 0/1-polytopes in $\mathbb{R}^d$ depending on the number $n$ of vertices. The picture shows two curves, one for the *beneath-beyond* and one for the *double-description* method (where `polymake` uses Komei Fukuda's implementation `cdd` for the latter method).

These two methods are *incremental* in the sense that they iteratively compute the convex hull of the first $i+1$ vertices from the convex hull of the first $i$ vertices. Since $n-1$ vertices

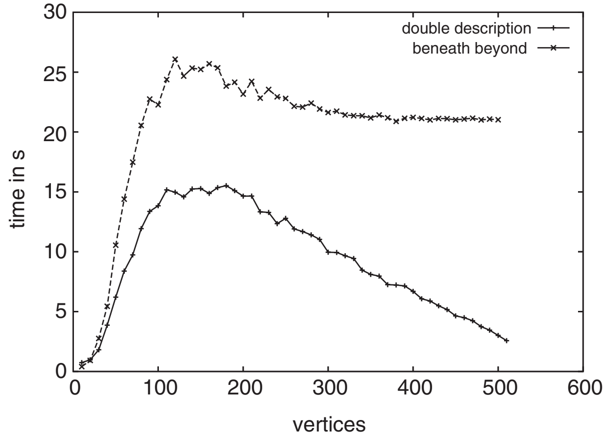


Fig. 1. Incremental convex hull algorithms: running times on 9-dimensional random 0/1-polytopes.

of a random 0/1-polytope with $n$ vertices should make a random 0/1-polytope with $n-1$ vertices, we had expected the curves to be monotonically increasing. However, the *first* $n-1$ vertices do only make a (uniform) random 0/1-polytope with $n-1$ vertices if the order of the $n$ vertices is (uniformly) random.

As it turned out, this is not the case for random 0/1-polytopes produced by the `polymake` system. Instead, the `rand01` client of `polymake` is implemented in such a way that the vertices of the random 0/1-polytope produced appear in lexicographic order. This led us to studying revlex-initial 0/1-polytopes.

And in fact, our results on the facet numbers of revlex-initial 0/1-polytopes make the curves in Fig. 1 plausible: For 0/1-polytopes with large numbers of vertices, which furthermore are lexicographically ordered, the intermediate polytopes appearing during the runs of incremental convex hull algorithms are quite close to revlex-initial 0/1-polytopes. Therefore, it is plausible that these intermediate polytopes have extremely few facets compared to random 0/1-polytopes with the same numbers of vertices (Fig. 2).

In particular, if the $2^d$ vertices of the entire cube are ordered lexicographically then the total number of facets of all intermediate polytopes produced by an incremental convex hull algorithm to compute the cube is bounded from above by $3d \cdot 2^d$, while for an arbitrary (even for a random) ordering there might be intermediate polytopes with super-exponentially many vertices (due to the results of Bárány and Pór [2] and Gatzouras et al. [5]).

These results indicate that it might be a good strategy to sort the vertices lexicographically before applying an incremental convex hull algorithm to a 0/1-polytope. However, we do not yet have any thorough computational study to support this.

## 4. The graphs of revlex-initial 0/1-polytopes

### *4.1. Characterization of adjacency*

The one-dimensional faces of a polytope (forming its *1-skeleton* or *graph*) are particularly important, for instance, since the simplex algorithm for linear programming proceeds along

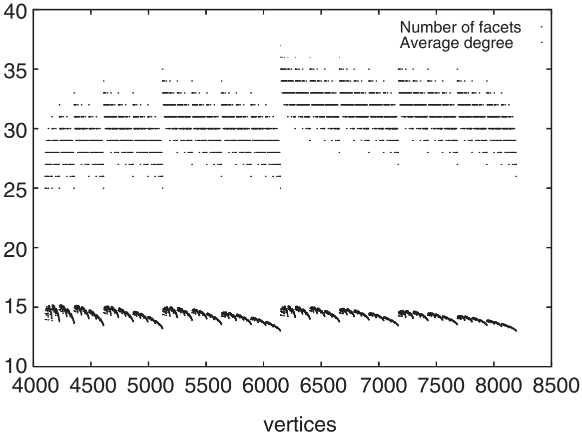


Fig. 2. The numbers of facets and the average degrees of all full-dimensional revlex-initial 0/1-polytopes for $d = 13$.

them. Moreover, in the special case of 0/1-polytopes, the graphs are important also for different reasons (see Section 4.3).

Here, we describe the graphs of revlex-initial 0/1-polytopes.

**Definition 6.** For $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$ and $1 \leqslant p < q \leqslant w(v)$ and $x \in \{0,1\}^d$ we define the sets (Table 2)

$$A^{\prec v}_{p,q}(x) := \big\{ z \in P^{\prec v}_p \;:\; z_i = x_i \text{ for all } 0 \leqslant i < \sigma_q(v),\\ z_{\sigma_q(v)} = 0,\; z_{\sigma_r(v)} = 1 \text{ for all } p < r < q \big\}$$

and

$$B^{\prec v}_{p,q}(x) := \big\{ z \in P^{\prec v}_p \;:\; zz_i = x_i \text{ for all } 0 \leqslant i < \sigma_q(v),\\ z_{\sigma_q(v)} = 1,\; z_{\sigma_r(v)} = 1 \text{ for all } p < r < q \big\}.$$

**Theorem 3** (*Graphs of revlex-initial 0/1-polytopes*). *For $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$, the graph of the corresponding revlex-initial* 0/1*-polytope $P^{\prec v}$ has the following structure.*

(1) *Let $x \in X^{\prec v}$ be a vertex of $P^{\prec v}$ contained in the block $P^{\prec v}_q$. Let p be some block number with $1 \leqslant p < q$.*
  (a) *The vertex x is adjacent to all vertices of $A^{\prec v}_{p,q}(x)$.*
  (b) *If $\max(\{i \in [\sigma_q(v)]_0 \;:\; x_i \neq v_i\} \cup \{-1\}) \notin S(v)$ then x is also adjacent to all vertices of $B^{\prec v}_{p,q}(x)$.*

(2) *The graph of $P^{\prec v}$ does not contain any other edges than the* (*cube-*)*edges of the blocks $P^{\prec v}_1, \ldots, P^{\prec v}_{w(v)}$ and the ones described in part* (i) *of this theorem.*

Table 2
Illustration of the definitions (with $p = 1$, $q = 4$, and $x = (1, 0, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \{0, 1\}^{10} \setminus \{\mathbb{0}\}$) on the example from Section 2

| $v$ | **1** | **0** | **1** | **1** | **0** | **0** | **1** | **0** | **0** | **1** |
|---|---|---|---|---|---|---|---|---|---|---|
| Indices | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Signature | $\sigma_5(v)$ | | $\sigma_4(v)$ | $\sigma_3(v)$ | | | $\sigma_2(v)$ | | | $\sigma_1(v)$ |
| $P_1^{\prec v}$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | **0** |
| $A_{1,4}^{\prec v}(x)$ | **1** | **0** | **0** | **1** | $\star$ | $\star$ | **1** | $\star$ | $\star$ | **0** |
| $B_{1,4}^{\prec v}(x)$ | **1** | **0** | **1** | **1** | $\star$ | $\star$ | **1** | $\star$ | $\star$ | **0** |
| $P_4^{\prec v}$ | $\star$ | $\star$ | **0** | **1** | **0** | **0** | **1** | **0** | **0** | **1** |

**Proof.** For the proof of part (1), let us denote by $F$ the face of $P^{\prec v}$ that is defined by the following equations:

$$z_i = x_i \quad (0 \leqslant i < \sigma_q(v)), \tag{8}$$

$$z_{\sigma_r(v)} = 1 \quad (p < r < q), \tag{9}$$

$$z_i = v_i \quad (\sigma_p(v) < i). \tag{10}$$

The claim in (a) follows from the fact that the only vertices of the face $\{z \in F : z_{\sigma_q(v)} = 0\}$ of $P^{\prec v}$ are the vertices of $A_{p,q}^{\prec v}(x)$ and $x$ itself. Since $A_{p,q}^{\prec v}(x)$ is contained in the hyperplane defined by $z_{\sigma_p(v)} = 0$, while $x$ is not, that face must be the pyramid with base $A_{p,q}^{\prec v}(x)$ and apex $x$.

In order to prove part (b), assume $\max(\{i \in [\sigma_q(v)]_0 : x_i \neq v_i\} \cup \{-1\}) \notin S(v)$. Thus, there is no block $P_r^{\prec v}$ with $r > q$ that has a common vertex with the face $F$. Hence, the only vertices of that face are the vertices of $A_{p,q}^{\prec v}(x)$, $B_{p,q}^{\prec v}(x)$, and $x$ itself. Again, since $A_{p,q}^{\prec v}(x)$ and $B_{p,q}^{\prec v}(x)$ are contained in the hyperplane defined by $z_{\sigma_p(v)} = 0$, while $x$ is not, that face must be the pyramid with base $\operatorname{conv}(A_{p,q}^{\prec v}(x) \cup B_{p,q}^{\prec v}(x))$ and apex $x$.

For the proof of part (2), suppose that $x$ and $y$ are adjacent vertices of $P^{\prec v}$ not contained in the same block. We may assume $x \in P_q^{\prec v}$ and $y \in P_p^{\prec v}$ with $1 \leqslant p < q \leqslant w(v)$.

We will first show that $y$ is contained in the face $F$ of $P^{\prec v}$ defined in the proof of part (1). Therefore, we have to prove that (8)–(10) is satisfied by $z = y$.

Let us assume (8) is not satisfied by $z = y$, i.e., there is some $0 \leqslant i < \sigma_q(v)$ with $x_i \neq y_i$. If we denote, for $a, b \in \{0, 1\}^d$, by $a \oplus b$ the component-wise addition modulo two, then we have $x \oplus \mathbb{e}_i \in P^{\prec v}$ (since $i < \sigma_q(v)$) and $y \oplus \mathbb{e}_i \in P^{\prec v}$ (since $i < \sigma_q(v) < \sigma_p(v)$) with

$$\{x \oplus \mathbb{e}_i, y \oplus \mathbb{e}_i\} \neq \{x, y\}$$

(since $x_{\sigma_p(v)} = 1 \neq 0 = y_{\sigma_p(v)}$). But then

$$\tfrac{1}{2}(x + y) = \tfrac{1}{2}(x \oplus \mathbb{e}_i + y \oplus \mathbb{e}_i)$$

contradicts the adjacency of $x$ and $y$. Thus, $z = y$ satisfies (8).

If (9) would not be satisfied by $z = y$, then there was some $p < r < q$ with $y_{\sigma_r(v)} = 0$. Due to $x \in P_q^{\prec v}$, $x_{\sigma_r(v)} = 1$ holds. Thus, we have $x - \mathbb{e}_{\sigma_r(v)} \in P^{\prec v}$ and $y + \mathbb{e}_{\sigma_r(v)} \in P^{\prec v}$

(since $y \in P_p^{\prec v}$ with $r < p$). Again,

$$\{x - \mathbb{e}_{\sigma_r(v)}, y + \mathbb{e}_{\sigma_r(v)}\} \neq \{x, y\}$$

holds, and therefore,

$$\tfrac{1}{2}(x + y) = \tfrac{1}{2}\big((x - \mathbb{e}_{\sigma_r(v)}) + (y + \mathbb{e}_{\sigma_r(v)})\big)$$

contradicts the adjacency of $x$ and $y$. Hence, (9) is satisfied by $z = y$.

Since $q > p$ and $x \in P_q^{\prec v}$, $y \in P_p^{\prec v}$, we clearly have $x_i = y_i = v_i$ for all $i > \sigma_p(v)$. Therefore, also (10) is satisfied by $z = y$, and thus, the claim $y \in F$ is proved.

We obtain $y \in A_{p,q}^{\prec v}(x) \cup B_{p,q}^{\prec v}(x)$. It hence suffices to show that, in case of $y \in B_{p,q}^{\prec v}(x)$, we have

$$\max(\{i \in [\sigma_q(v)]_0 : x_i \neq v_i\} \cup \{-1\}) \notin S(v).$$

Therefore, suppose we have $y \in B_{p,q}^{\prec v}(x)$ and there is some $q < s \leqslant w(v)$ with $x_{\sigma_s(v)} = 0$ and $x_i = v_i$ for all $\sigma_s(v) < i < \sigma_q(v)$. Then we have $y - \mathbb{e}_{\sigma_q(v)} \in P^{\prec v}$ (due to $y \in P_p^{\prec v}$, $p < q$, and $y_{\sigma_q(v)} = 1$) and $x + \mathbb{e}_{\sigma_q(v)} \in P^{\prec v}$ (in fact: $x + \mathbb{e}_{\sigma_q(v)} \in P_s^{\prec v}$). Also here, we have

$$\{x + \mathbb{e}_{\sigma_q(v)}, y - \mathbb{e}_{\sigma_q(v)}\} \neq \{x, y\},$$

and thus,

$$\tfrac{1}{2}(x + y) = \tfrac{1}{2}\big((x + \mathbb{e}_{\sigma_q(v)}) + (y - \mathbb{e}_{\sigma_q(v)})\big)$$

contradicts the adjacency of $x$ and $y$. $\square$

### 4.2. *The number of edges*

Having the structural description given in Theorem 3 at hand, we can now derive a formula for the number of edges of a revlex-initial 0/1-polytope.

**Theorem 4** (*Edge numbers of revlex-initial 0/1-polytopes*). *For $v \in \{0,1\}^d \setminus \{\mathbb{0}\}$, the graph of the corresponding revlex-initial 0/1-polytope $P^{\prec v}$ has*

$$\sum_{p=1}^{w(v)} 2^{\sigma_p(v)} \left( \frac{\sigma_p(v)}{2} + \sum_{q=p+1}^{w(v)} 2^{p-q} \left( 2 - \left( \sum_{r=q+1}^{w(v)} 2^{\sigma_r(v)} \right) 2^{-\sigma_q(v)} \right) \right)$$

*edges. In particular, its average node degree is bounded from above by $d + 4$.*

**Proof.** The statement on the average degree follows from the exact expression for the number of edges: Inside the (outermost) brackets, the fraction $\frac{\sigma_p(v)}{2}$ is bounded from above by $\frac{d}{2}$ while the remaining sum clearly is at most 2. Thus the number of edges is at most $(\frac{d}{2}+2)$ times the number $\sum 2^{\sigma_p(v)}$ of vertices of $P^{\prec v}$.

In order to determine the total number of edges, let $1 \leqslant p < q \leqslant w(v)$. We have

$$\dim A_{p,q}^{\prec v}(x) \;=\; \dim B_{p,q}^{\prec v}(x) = (p + \sigma_p(v)) - (q + \sigma_q(v)) \;=:\; \delta_{p,q}$$

for each $x \in X_q^{\prec v}$.

Clearly, the number of edges between $P_q^{\prec v}$ and $P_p^{\prec v}$ described in part (1a) of Theorem 3, thus is

$$2^{\sigma_q(v)} \cdot 2^{\delta_{p,q}} \;=\; 2^{p+\sigma_p(v)-q}.$$

The number of $x \in X_q^{\prec v}$ that do not satisfy the condition of part (1b) of Theorem 3 is $\sum_{r=q+1}^{w(v)} 2^{\sigma_r(v)}$. Thus, the number of edges between $P_q^{\prec v}$ and $P_p^{\prec v}$ described in part (1b) is

$$\left(2^{\sigma_q(v)} - \sum_{r=q+1}^{w(v)} 2^{\sigma_r(v)}\right) \cdot 2^{\delta_{p,q}} = 2^{p+\sigma_p(v)-q} - \left(\sum_{r=q+1}^{w(v)} 2^{\sigma_r(v)}\right) 2^{\delta_{p,q}}.$$

Therefore, the total number of edges is

$$\sum_{p=1}^{w(v)} \sigma_p(v) 2^{\sigma_p(v)-1} + \sum_{1 \leqslant p<q \leqslant w(v)} \left(2 \cdot 2^{p+\sigma_p(v)-q} - \left(\sum_{r=q+1}^{w(v)} 2^{\sigma_r(v)}\right) 2^{(p+\sigma_p(v))-(q+\sigma_q(v))}\right),$$

where the first sum accounts for the edges inside the blocks and the second one (the double-sum) counts the edges running across different blocks. That expression can easily be simplified to the one stated in the theorem. □

### 4.3. *The edge-expansion*

The geometry of a 0/1-polytope $P$ (more precisely: its 1-skeleton, i.e., its graph) defines a natural neighborhood structure on the set system $\mathcal{S}$ corresponding to the vertices of $P$. Such a neighborhood structure can be used in order to design random walk algorithms for generating elements from $\mathcal{S}$ at random (according to a certain pre-specified probability distribution). Random walk algorithms are of great importance, for instance with respect to randomized approximative counting algorithms (see, e.g., [6]).

In many cases, the neighborhood structure defined geometrically via the associated 0/1-polytope has turned out to be quite appropriate for designing such random walk algorithms. A crucial parameter with respect to the time complexity of these methods is the *edge-expansion* of the neighborhood structure. The rule of thumb here is that the expansion should be bounded from below polynomially in $1/d$ (where $d$ is the dimension of the polytope) in order to achieve an efficient time algorithm.

**Definition 7.** The edge-expansion $\mathcal{X}(G)$ of a graph $G = (V, E)$ is defined as

$$\mathcal{X}(G) := \min\left\{\frac{|\delta(S)|}{|S|} \;:\; S \subset V,\ 0 < |S| \leqslant \frac{|V|}{2}\right\}$$

(with $\delta(S)$ denoting the set of all edges with one end node in $S$ and the other one in $V \setminus S$).

It has been conjectured by Mihail and Vazirani (cited, e.g., in [3,9]) that the graph of every 0/1-polytope has edge-expansion at least one. In fact, this conjecture is known to be true for several classes of 0/1-polytopes, including stable set polytopes, (perfect) matching polytopes, and polytopes associated with the bases of *balanced* (in particular: regular) matroids (see [9,3,7]). For more details and references, we refer to [7].

Here, further supporting the Mihail–Vazirani conjecture, we prove that despite the sparsity of their graphs, revlex-initial 0/1-polytopes have edge expansion at least one.

**Theorem 5** (*Edge-expansion of revlex-initial 0/1-polytopes*). *For $v \in \{0,1\}^d \setminus \{\mathbb{O}\}$, the graph of the corresponding revlex-initial* 0/1*-polytope $P^{\prec v}$ has edge-expansion at least one.*

In order to bound the edge expansion of a graph $G = (V, E)$ from below we will construct certain flows in the (uncapacitated) network $\mathcal{N}(G) = (V, A)$, where $A$ contains for each edge $\{u, v\} \in E$ both arcs $(u, v)$ and $(v, u)$. This strategy dates back to the method of "canonical paths" developed by Sinclair (see [12]). The extension to flows was explicitly exploited by Morris and Sinclair [10]. Feder and Mihail [3] use random canonical paths, which can equivalently be formulated in terms of flows.

The crucial idea is to construct for each ordered pair $(x, y) \in V \times V$ a flow $\phi_{(x,y)} : A \longrightarrow \mathbb{Q}^{\geqslant 0}$ in the network $\mathcal{N}(G)$ sending one unit of some commodity from $x$ to $y$. Define the multi-commodity flow (MCF) $\phi := \sum_{(x,y)\in V\times V} \phi_{(x,y)}$ as the sum of all the flows $\phi_{(x,y)}$. By

$$\phi_{\max} := \max\{\phi(a) \;:\; a \in A\}$$

we denote the maximal amount of $\phi$-flow on any arc. By construction of $\phi$, the total amount $\phi(S : V \setminus S)$ of $\phi$-flow leaving $S$ is at least $|S| \cdot (n - |S|)$, where $n = |V|$. On the other hand, we have $\phi(S : V \setminus S) \leqslant \phi_{\max} \cdot |\delta(S)|$. This implies $|S| \cdot (n - |S|) \leqslant \phi_{\max} \cdot |\delta(S)|$, and hence, if $|S| \leqslant \frac{n}{2}$ holds,

$$\frac{|\delta(S)|}{|S|} \geqslant \frac{n}{2 \cdot \phi_{\max}}.$$

Thus, we have proven

$$\mathcal{X}(G) \geqslant \frac{n}{2 \cdot \phi_{\max}}. \tag{11}$$

In the light of inequality (11) it is clear that the task is to construct a flow $\phi$ as above with $\phi_{\max} \leqslant \frac{n}{2}$ (where $n = |V|$).

**Proof of Theorem 5.** We will use the notations $P^{<n} := P^{\prec v} \subset \mathbb{R}^d$ and $X^{<n} := X^{\prec v}$, where $n \in \mathbb{N}$ is the number having binary representation $v$ (i.e., $n$ is the number of vertices of $P^{<n} = P^{\prec v}$). Clearly, we may assume $v_{d-1} = 1$, i.e., $n > 2^{d-1}$ and dim $P^{<n} = d$. Thus, in particular, the dimension $d$ and the 0/1-vector $v \in \{0,1\}^d$ are uniquely determined by the vertex number $n$.

We will prove the theorem by showing via induction on $n$ that, for every $n \in \mathbb{N}$, there is an MCF $\phi^n = \sum_{(x,y) \in X^{<n} \times X^{<n}} \phi^n_{(x,y)}$ on $\mathcal{N}(G(P^{<n}))$ such that $\phi^n_{\max} \leqslant \frac{n}{2}$.

The statement obviously holds for $n = 2$, since in that case, the polytope $P^{<n}$ consists of two vertices joint by an edge.

Thus let us suppose that for all $2 \leqslant n' < n$ there is such an MCF $\phi^{n'}$ on $\mathcal{N}\Big(G(P^{<n'})\Big)$ with $\phi^{n'}_{\max} \leqslant \frac{n'}{2}$. The induction step, i.e., the construction of an appropriate MCF $\phi^n$, will be subdivided into two cases.

Let $G := G(P^{<n})$. For a subset $A$ of the nodes of $G$, we denote by $G[A]$ the subgraph of $G$ induced by $A$ (similarly, we use $\mathcal{N}(G)\,[A]$). Two 0/1-polytopes $P$ and $Q$ are called 0/1-*equivalent* if they can be transformed into each other by (potentially) lifting one of them into the space of the other and applying a symmetry of the cube (i.e., by flipping and permuting coordinates). Of course, such a transformation induces an isomorphism between the graphs of $P$ and $Q$. Note that for $w \in \{0,1\}^d$ the vertex $w \oplus \mathbb{e}_0$ is the one obtained by flipping the first coordinate of $w$.

*Case* 1: ($v_0 = 0$). Define the following faces of $P^{<n}$ and the corresponding vertex sets:

$$F_A := \{w \in P^{<n} : w_0 = 0\}, \quad F_B := \{w \in P^{<n} : w_0 = 1\},$$
$$X_A := \{w \in X^{<n} : w_0 = 0\}, \quad X_B := \{w \in X^{<n} : w_0 = 1\}.$$

Then, for every $x \in X_A$, we have $x \oplus \mathbb{e}_0 \in X_B$ (and vice versa). Thus $P^{<n}$ is a prism over $F_A$. In particular, $F_A$ and $F_B$ are 0/1-equivalent. Furthermore, they both are 0/1-equivalent to $P^{<n'}$ with $n' = \frac{n}{2}$. Thus, $G[X_A]$ and $G[X_B]$ both are isomorphic to $G(P^{<n'})$.

Let $\phi^A$ and $\phi^B$ be the MCFs induced by $\phi^{n'}$ on $\mathcal{N}(G)\,[X_A]$ and $\mathcal{N}(G)\,[X_B]$, respectively. Thus $\phi^A_{\max} = \phi^B_{\max} = \phi^{n'}_{\max} \leqslant \frac{n}{4}$ by the induction hypothesis. Now we construct the MCF $\phi^n$ on $\mathcal{N}(G)$ by defining each $\phi^n_{(x,y)}$ in the following way (note that $G[X_A]$ and $G[X_B]$ are edge-disjoint):

$$\begin{aligned}
x, y \in X_A: &\quad \phi^n_{(x,y)} := \phi^A_{(x,y)},\\
x, y \in X_B: &\quad \phi^n_{(x,y)} := \phi^B_{(x,y)},\\
x \in X_A, y \in X_B: &\quad \phi^n_{(x,y)} := \Psi_{(x,x\oplus\mathbb{e}_0)} + \phi^B_{(x\oplus\mathbb{e}_0,y)},\\
x \in X_B, y \in X_A: &\quad \phi^n_{(x,y)} := \Psi_{(x,x\oplus\mathbb{e}_0)} + \phi^A_{(x\oplus\mathbb{e}_0,y)}.
\end{aligned}$$

Here, $\Psi_{(x,x\oplus\mathbb{e}_0)}$ denotes the flow just sending one unit along the arc $(x, x \oplus \mathbb{e}_0) \in A$ (and nothing along any other arc). In the resulting MCF $\phi^n$, every arc $(x, x \oplus \mathbb{e}_0)$ with $x \in X_A$ carries one unit of flow for each of the $|X_B| = \frac{n}{2}$ pairs $(x, y)$, $y \in X_B$. The same holds for the reverse arcs $(x, x \oplus \mathbb{e}_0)$, $x \in X_B$. Thus we have $\phi(x, x \oplus \mathbb{e}_0) = \frac{n}{2}$ for every such arc, and we conclude

$$\phi^n_{\max} \leqslant \max\left\{\tfrac{n}{2}, 2 \cdot \phi^{n'}_{\max}\right\} = \frac{n}{2}.$$

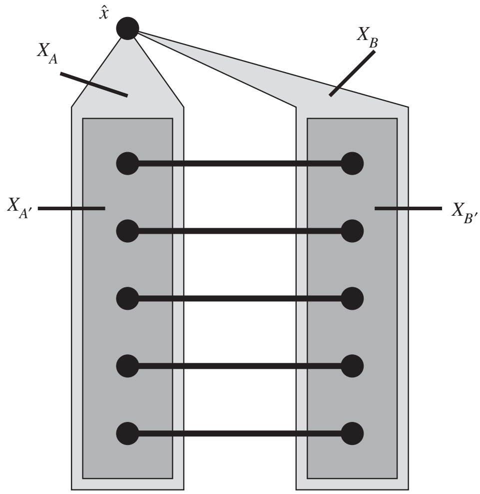


Fig. 3. Illustration of the sets used in case 2 of the proof of Theorem 5.

*Case* 2: ($v_0 = 1$) Let $\hat{x} \in \{0,1\}^d$ be the revlex-predecessor of $v$, i.e., the 0/1-vector corresponding to the number $(n-1)$. Then $\hat{x}$ is the "last" vertex of $P^{<n}$ and $\{\hat{x}\} = P^{<n}_{w(n)}$ is the "last" block of $P^{<n}$. Define the following faces of $P^{<n}$ and the corresponding vertex sets (see Fig. 3):

$$\begin{aligned} F_A &:= \{w \in P^{<n} : w_0 = 0\}, & F_B &:= \{w \in P^{<n} : w_0 = 1\}, \\ X_A &:= \{w \in X^{<n} : w_0 = 0\}, & X_B &:= \{w \in X^{<n} : w_0 = 1\} \cup \{\hat{x}\}, \\ X'_A &:= X_A \setminus \{\hat{x}\}, & X'_B &:= X_B \setminus \{\hat{x}\}. \end{aligned}$$

Thus $P^{<n}$ is a *partial prism* over $F_A$, i.e., $P^{<n}$ arises from a true prism over $F_A$ via removing the vertex $(\hat{x} + \mathbb{e}_0)$ corresponding to $\hat{x}$ (and taking the convex hull).

We will first prove that there is a spanning subgraph of $G[X_B]$ that is isomorphic to $G[X_A]$. Indeed, this is a simple consequence of the fact that $P^{<n}$ is a partial prism over $F_A$ (with $\hat{x}$ being the "not duplicated vertex"): Every edge $\{a, a'\}$ of $F_A$ with $a, a' \neq \hat{x}$ gives rise to a quadrangular 2-face $\{a, a', a \oplus \mathbb{e}_0, a' \oplus \mathbb{e}_0\}$ of the partial prism (showing that $\{a \oplus \mathbb{e}_0, a' \oplus \mathbb{e}_0\}$ is an edge of $G$), and every edge $\{\hat{x}, a\}$ of $F_A$ yields a triangular 2-face $\{\hat{x}, a, a \oplus \mathbb{e}_0\}$ of the partial prism (showing that $\{\hat{x}, a \oplus \mathbb{e}_0\}$ is an edge of $G$).

Hence, there is a spanning subgraph of $G[X_B]$ that is isomorphic to $G[X_A]$. Furthermore, the face $F_A$ of $P^{<n}$ is 0/1-equivalent to $P^{<n'}$ with $n' = (n+1)/2$. Therefore, the MCF $\phi^{n'}$ induces MCFs $\phi^A$ and $\phi^B$ on $\mathcal{N}(G)\,[X_A]$ and $\mathcal{N}(G)\,[X_B]$, respectively, with

$$\phi^A_{\max} = \phi^B_{\max} = \phi^{n'}_{\max} \leqslant \tfrac{n+1}{4}$$

by the induction hypothesis.

With $\alpha := \frac{n-1}{n+1} < 1$ we have $(1+\alpha)\phi^{n'}_{\max} \leqslant \frac{n}{2}$. Thus we can increase each of the flows $\phi^A$ and $\phi^B$ by an $\alpha$-fraction without making the flow exceed the desired limit of $n/2$ at any arc. We construct the MCF $\phi^n$ on $\mathcal{N}(G)$ by defining each $\phi^n_{(x,y)}$ in the following way (note that $G[X_A]$ and $G[X_B]$ are edge-disjoint):

$$\begin{aligned}
x, y \in X_A: \quad & \phi^n_{(x,y)} := \phi^A_{(x,y)}, \\
x, y \in X_B: \quad & \phi^n_{(x,y)} := \phi^B_{(x,y)}, \\
x \in X'_A, y \in X'_B: \quad & \phi^n_{(x,y)} := \alpha\left(\Psi_{(x,x\oplus e_0)} + \phi^B_{(x\oplus e_0,y)}\right) \\
& \qquad + (1-\alpha)\left(\phi^A_{(x,\hat{x})} + \phi^B_{(\hat{x},y)}\right), \\
x \in X'_B, y \in X'_A \quad & \phi^n_{(x,y)} := \alpha\left(\Psi_{(x,x\oplus e_0)} + \phi^A_{(x\oplus e_0,y)}\right) \\
& \qquad + (1-\alpha)\left(\phi^B_{(x,\hat{x})} + \phi^A_{(\hat{x},y)}\right).
\end{aligned}$$

Here, as in the first case, $\Psi_{(x,x\oplus e_0)}$ is the flow sending one unit along the arc $(x, x\oplus e_0) \in A$ and nothing along any other arc.

It is easy to see that this is a valid MCF (i.e. for each pair $(x, y)$ the flow $\phi^n_{(x,y)}$ really sends one unit of flow). Thus let us check $\phi^n_{\max}$. Firstly, in order to estimate the flow on the arcs inside $G[X_A]$ and $G[X_B]$, we determine the multiplier by which each flow $\phi^A_{(s,t)}$ respectively $\phi^B_{(s,t)}$ appears in the definition of $\phi^n$. By symmetry, it suffices to do this for all pairs $s, t \in X_A$.

Each pair $s, t \in X'_A$ is used once with multiplier one (for $(x, y) = (s, t)$) and once with multiplier $\alpha$ (for $(x, y) = (s \oplus e_0, t)$). Thus, each $\phi^A_{(s,t)}$ appears with multiplier $(1+\alpha)$ for $s, t \neq \hat{x}$.

Each pair $s = \hat{x}$ and $t \in X'_A$ is used once with multiplier one (for $(x, y) = (s, t)$) and, for each of the $(n-1)/2$ pairs $x \in X_B$ and $y = t$, with multiplier $(1-\alpha)$. Each pair $s \in X'_A$ and $t = \hat{x}$ is used once with multiplier one (for $(x, y) = (s, t)$) and, for each of the $(n-1)/2$ pairs $x = s$ and $y \in X_B$, with multiplier $(1-\alpha)$.

Thus, due to

$$(1-\alpha)\frac{n-1}{2} = \frac{n+1-(n-1)}{n+1}\,\frac{n-1}{2} = \frac{n-1}{n+1} = \alpha,$$

each $\phi^A_{(s,t)}$ with $s = \hat{x}$ or $t = \hat{x}$ appears with multiplier $(1+\alpha)$.

Secondly, we estimate the flow along the arcs $(x, x\oplus e_0)$ with $x \neq \hat{x}$. By symmetry we restrict our attention to the case $x \in X'_A$ and $y \in X'_B$, and we find that each arc $(x, x\oplus e_0)$ is used $(n-1)/2$ times with flow-value $\alpha$.

Altogether, this yields

$$\phi^n_{\max} \leqslant \max\left\{(1+\alpha)\cdot\phi^{n'}_{\max}, \frac{n-1}{2}\cdot\alpha\right\} \leqslant \frac{n}{2},$$

which concludes the inductive step, and thus, the proof. □

## 5. Towards a lower-bound-theorem for 0/1-polytopes

In the following, we will exploit the following construction (using revlex-initial 0/1-polytopes) several times.

**Proposition 7.** *For $d, n \in \mathbb{N}$ with $d+1 \leqslant n \leqslant 2^d$ there exists $\tilde{d} \in \mathbb{N}$ such that for $\tilde{n} := n - (d - \tilde{d})$ the following inequalities hold.*

$$0 \leqslant \tilde{d} \leqslant d, \tag{12}$$

$$2^{\tilde{d}-1} < \tilde{n} \leqslant 2^{\tilde{d}}, \tag{13}$$

$$\tilde{d} \leqslant 1 + \log_2 n. \tag{14}$$

*Furthermore $P^{<\tilde{n}}$ is a $\tilde{d}$-dimensional revlex-initial 0/1-polytope with $\tilde{n}$ vertices.*

**Proof.** To see that such a $\tilde{d}$ and $\tilde{n}$ exist, observe that with $\tilde{n}(k) := n - (d - k)$ we have $\tilde{n}(k) > 2^{k-1}$ for $k = 0$ and $\tilde{n}(k) \leqslant 2^k$ for $k = d$; note that for these estimates we need $d + 1 \leqslant n \leqslant 2^d$. Then, we have that

$$\tilde{d} := \min\{k \in \mathbb{N} : \tilde{n}(k) \leqslant 2^k\}$$

satisfies (12).

By definition, we have $\tilde{n}(\tilde{d}) \leqslant 2^{\tilde{d}}$. If $\tilde{d} = 0$, then (as stated above) also $\tilde{n}(\tilde{d}) > 2^{\tilde{d}-1}$ is true, and otherwise, from the minimality of $\tilde{d}$ we conclude $\tilde{n}(\tilde{d} - 1) > 2^{\tilde{d}-1}$, which, of course, implies $\tilde{n}(\tilde{d}) > 2^{\tilde{d}-1}$. Hence, $\tilde{d}$ also satisfies (13).

Finally, (14) trivially follows from (13).

Thus, with $\tilde{n} := \tilde{n}(\tilde{d})$, by (13) and Proposition 2 the revlex-initial 0/1-polytope $P^{<\tilde{n}}$ has dimension $\tilde{d}$. □

**Definition 8.** For arbitrary $d, n \in \mathbb{N}$ with $d + 1 \leqslant n \leqslant 2^d$ and $\tilde{d}, \tilde{n} \in \mathbb{N}$ as in Proposition 7 we define $P(d, n)$ to be the $d$-dimensional 0/1-polytope with $n$ vertices obtained by building the $(d - \tilde{d})$-fold pyramid over $P^{<\tilde{n}}$.

We denote the parameters $\tilde{d}$ and $\tilde{n}$ by $\tilde{d}(d, n)$ and $\tilde{n}(d, n)$.

Note that Proposition 7 guarantees that this construction always works as claimed in the definition of $P(d, n)$.

### 5.1. An upper bound on the minimal number of facets

**Definition 9.** For $d, n \in \mathbb{N}$ with $d + 1 \leqslant n \leqslant 2^d$, denote by $g_{\text{nfac}}(d, n)$ the minimal number of facets of a $d$-dimensional 0/1-polytope with $n$ vertices.

Note that a $k$-dimensional 0/1-polytope in $\mathbb{R}^d$ (with $k < d$) can isometrically be projected to a $k$-dimensional 0/1-polytope in $\mathbb{R}^k$. Thus, the definition is independent of the ambient spaces of the polytopes.

**Proposition 8.** *For every $d + 1 \leqslant n \leqslant 2^d$ we have $g_{\text{nfac}}(d, n) \leqslant d + 2\log_2 n$.*

**Proof.** By Theorem 2(2), the revlex-initial 0/1-polytope $P^{<\tilde{n}(d,n)}$ has at most $3\tilde{d} - 2$ facets. Thus $P(d, n)$ has at most $3\tilde{d} - 2 + n - \tilde{n} = 2\tilde{d} + d - 2$ facets. The claim of the proposition follows by (14). □

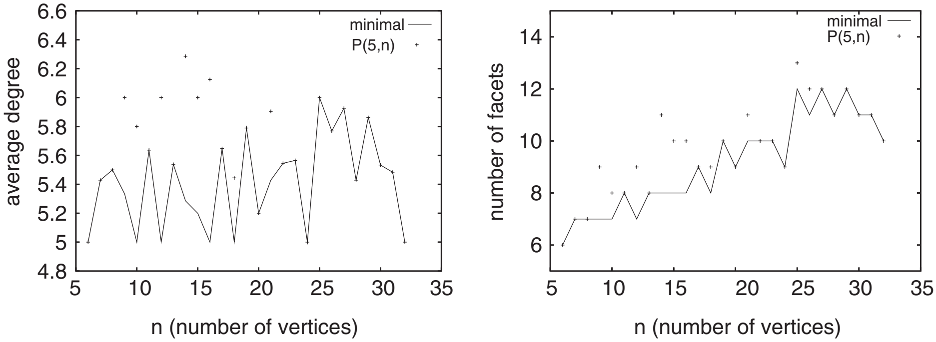


Fig. 4. Comparison of the lower bounds on $g_{\mathrm{nfac}}(5, n)$ and $g_{\mathrm{avdeg}}(5, n)$ obtained from the polytopes $P(5, n)$ in the proofs of Propositions 7 and 8 with the true values of $g_{\mathrm{nfac}}(5, n)$ and $g_{\mathrm{avdeg}}(5, n)$ obtained from Aichholzer's enumeration [1].

The proposition immediately implies the following results:

**Theorem 6.** (1) *For every* $d + 1 \leqslant n \leqslant 2^d$ *we have* $g_{\mathrm{nfac}}(d, n) \leqslant 3d$.
(2) *For* $d + 1 \leqslant n(d) \leqslant 2^{o(d)}$ *we have* $g_{\mathrm{nfac}}(d, n(d)) = d + o(d)$.
(3) *For* $1 < \alpha < 2$ *and* $n(d) := \lfloor \alpha^d \rfloor$ *we have* $g_{\mathrm{nfac}}(d, n(d)) \leqslant (1 + 2\log_2 \alpha)d + o(d)$.

The upper bounds on $g_{\mathrm{nfac}}(d, n)$ provided by the polytopes $P(d, n)$ in Proposition 8 are not sharp, at least not for all parameters $d$ and $n$. This follows, for instance, from the examples of Cartesian products of $r$ 0/1-simplices of dimension $d_1, \ldots, d_r$ (which are precisely the simple 0/1-polytopes, see Kaibel and Wolff [8]). Such a product is a 0/1-polytope of dimension $d = \sum d_i$ with $\prod(d_i + 1)$ vertices and $d + r$ facets. In particular for $n = (\lfloor \frac{d}{2} \rfloor + 1)(\lceil \frac{d}{2} \rceil + 1)$, this yields

$$g(d, n) = d + 2,$$

while the polytopes $P(d, n)$ have $d + \Omega(\log_2 d)$ facets.

The right part of Fig. 4 shows that for $d = 5$ the polytopes $P(5, n)$ achieve the respective minimum number of facets in all but 10 cases (i.e., in 17 out of 27 cases). Fig. 5 depicts the numbers of facets (and the average degrees) of the polytopes $P(13, n)$.

For sub-exponential numbers of vertices, Part (2) of Theorem 6 shows that the minimum number of facets is asymptotically as small as the number of facets of any $d$-dimensional polytope can be (up to an additive o(1)-term). The range of sub-exponential vertex numbers is particularly interesting for two reasons: Firstly, many 0/1-polytopes that are relevant in combinatorial optimization have sub-exponentially many vertices (e.g., cut polytopes of complete graphs and traveling salesman polytopes). Secondly, the papers by Bárány and Pór [2] and Gatzouras et al. [5] show that within sub-exponential ranges of vertex numbers a random 0/1-polytope has very many facets. In fact, it may well be that the maximum numbers of facets of 0/1-polytopes is (roughly) attained by these polytopes.

The examples of products of simplices (i.e., simple 0/1-polytopes) seem to indicate that it might be hopeless to derive an explicit formula for $g_{\mathrm{nfac}}(d, n)$, i.e., a sharp lower

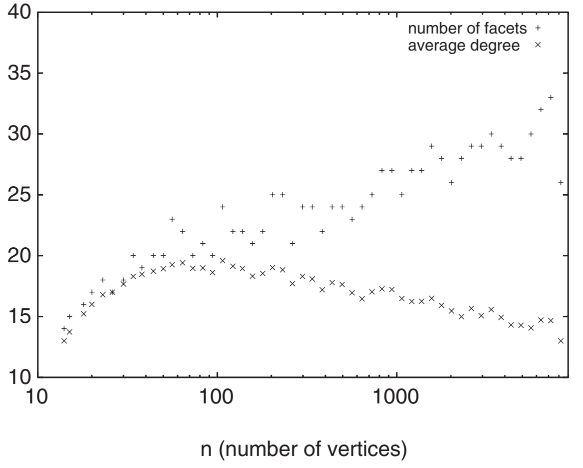


Fig. 5. Numbers of facets and average degrees of the polytopes $P(13,n)$ providing the upper bounds on $g_{\text{nfac}}(13,n)$ and $g_{\text{avdeg}}(13,n)$.

bound theorem for the facet numbers of 0/1-polytopes. Nevertheless, the question for the (asymptotic) best upper bound on $g_{\text{nfac}}(d,n)$ that does only depend on $d$ (and not on $n$) might be within reach. In particular, we do not know whether there is some constant $\alpha < 3$ such that $g_{\text{nfac}}(d,n) \leqslant \alpha d + \mathrm{o}(d)$ holds for all $d$ and $n$. This might even be true for $\alpha = 2$.

*5.2. An upper bound on the minimal number of edges*

**Definition 10.** For $d, n \in \mathbb{N}$ with $d+1 \leqslant n \leqslant 2^d$, denote by $g_{\text{avdeg}}(d,n)$ the minimal average degree among all graphs of $d$-dimensional 0/1-polytopes with $n$ vertices.

Revlex-initial 0/1-polytopes and the pyramidal construction yield the following bound of the minimum average degrees.

**Theorem 7.** *For $d+1 \leqslant n \leqslant 2^d$, we have $g_{\text{avdeg}}(d,n) \leqslant d+4$.*

**Proof.** Set $\tilde{d} := \tilde{d}(d,n)$ and $\tilde{n} := \tilde{n}(d,n)$. By Theorem 4, the revlex-initial 0/1-polytope $P^{<\tilde{n}}$ has at most $(\tilde{d}+4)\tilde{n}$ edges. Thus, $P(d,n)$ (the $(d-\tilde{d})$-fold pyramid over $P^{<\tilde{n}}$) has at most

$$(\tilde{d}+4)\tilde{n} + (d-\tilde{d})n \leqslant (d+4)n$$

edges. □

The left part of Fig. 4 shows that for $d = 5$ the polytopes $P(5,n)$ achieve the respective minimum average degree in all but 8 cases (i.e., in 19 out of 27 cases). Comparisons of the average degrees and the graph densities of the polytopes $P(d,n)$ and of random 0/1-polytopes are depicted in Figs. 6 and 7, respectively.

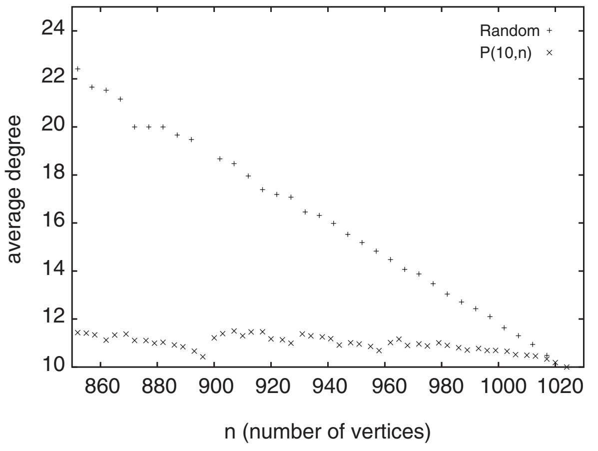


Fig. 6. Average degrees of the polytopes $P(10, n)$ and uniformly random 10-dimensional 0/1-polytopes (by sampling).

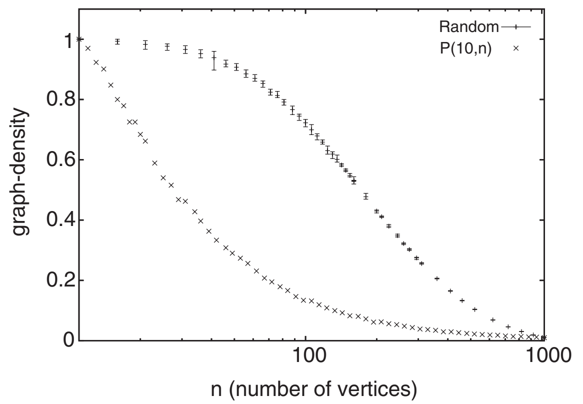


Fig. 7. The graph densities of the polytopes $P(10, n)$ used in the proof of Theorem 7 versus the graph densities of respective random 0/1-polytopes (by sampling).

Finally, the polytopes $P(d, n)$ yield examples of 0/1-polytopes with remarkably sparse graphs, satisfying, nevertheless, the Mihail–Vazirani conjecture.

**Theorem 8.** *For every $d+1 \leqslant n \leqslant 2^d$, there is a d-dimensional* 0/1-p*olytope with n vertices, at most $(d+4)n$ edges, and edge expansion at least one.*

**Proof.** By Theorem 7 the polytope $P(d, n)$ has at most $(d+4)n$ edges. Since $P(d, n)$ is a $k$-fold pyramid over the revlex-initial 0/1-polytope $P^{<\tilde{n}(d,n)}$ the multi-commodity flow constructed in the proof of Theorem 5 can be easily extended to a multi-commodity flow of $P(d, n)$ sending one unit of flow from every vertex to every other vertex. $\square$

## 6. Concluding remarks

The contributions of this paper concern three topics: (1) Investigations of a "natural"–class of 0/1-polytopes, (2) lower bound theorem(s) for 0/1-polytopes, and (3) support of the Mihail–Vazirani conjecture on the edge expansion of the graphs of 0/1-polytopes.

With respect to the first topic, one may be interested also in studying the convex hulls of sets of 0/1-vectors that are only *gradually* revlex-initial (the 0/1-polytopes corresponding to compressed set systems), i.e., convex hulls of sets $X$ of 0/1-vectors which, with every $x \in X$, contain all 0/1-vectors $y$ which have the same number of ones as $x$ and are revlex-smaller than $x$. Due to the important role played by the *monotone* ones among them (more precisely: by the corresponding set systems) in the theory of simplicial complexes, it might be that these objects bear some connections between 0/1-polytopes and combinatorial topology. This would be quite interesting.

It seems that precise lower bound theorems on the number of facets (edges, or even other-dimensional faces) are hard to obtain. Nevertheless, with respect to topic (2) some questions remain open that may be tractable, e.g., the question whether there is some $\alpha < 3$ (maybe $\alpha = 2$?) with $g_{\text{nfac}}(d, n) \leqslant \alpha d + \text{o}(d)$.

Perhaps the most interesting and promising line to follow up this research concerns topic (3) Extending our techniques for construction of the multi-commodity flows showing that revlex-initial 0/1-polytopes (as special knapsack-polytopes) have edge expansion at least one to all knapsack polytopes (or even to all monotone polytopes) would be a big support for the Mihail–Vazirani conjecture (which itself is of great importance in the theory of random generation and approximate counting, as mentioned in Section 4.3). It follows from work of Morris and Sinclair [10] that the edge-expansion of the graphs of $d$-dimensional 0/1-knapsack polytopes is bounded from below by a polynomial in $1/d$. Their proof in fact shows that this is true even for the subgraph that is formed by those edges which are also edges of the cube. Since our flows extensively use non-cube edges, the techniques used in the proof of Theorem 5 seem to have good potential to improve the current lower bound, maybe even to 'one' as conjectured by Mihail and Vazirani.

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