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Differential-Difference Equations and Nonlinear Initial-Boundary Value Problems for Linear Hyperbolic Partial Differential Equations*

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1. INTRODUCTION

In several recent papers [2-4, 6] R. K. Brayton and W. L. Miranker have discussed certain nonlinear mixed initial-boundary problems arising from transmission line theory. Much of their work depends on the fact that the boundary problem can be replaced by an initial value problem for an associated differential-difference equation.[1] This replacement was achieved, however, by special physical considerations, and the published discussions deal only with rather special boundary problems.

The purpose of this paper is to generalize and systematize this approach to initial-boundary problems. We first of all formulate a more general class of such problems for transmission lines. This class contains linear hyperbolic systems of two linear partial differential equations with nonlinear integro-differential boundary conditions. Second, we present a systematic procedure for reducing such problems to initial value problems for differential-difference or integrodifferential-difference equations. This procedure is based on the method of characteristics, and is applicable to hyperbolic systems of the type indicated, irrespective of their physical origin.

It is clear that the replacement of a mixed initial-boundary problem for a partial differential system by a pure initial value problem for a differential-difference system has important theoretical and practical ramifications. For example, it should provide an efficient method for numerical integration. In later papers we shall investigate some of these ramifications, including uniqueness and global existence theorems, series and integral representations of solutions, stability questions, and the extension of the method to higher order systems corresponding to linked transmission lines.

In Section 2 of this paper, we shall describe the problem of Brayton and Miranker and briefly indicate their method of reducing it to a differential-

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[1] See Bellman and Cooke, [1], for the theory of such equations.

difference equation. Readers not interested in the physical problem can skip directly to Section 3, where we describe our method of reduction for systems of linear partial differential equations in normal hyperbolic form. In Section 4 we formulate a boundary problem of fairly general sort for a transmission line, and in Section 5 we convert this to a problem in normal form to which the technique of Section 3 is applicable. Finally, in Section 6 we briefly sketch an iterative technique applicable to nonlinear initial-boundary problems for semilinear systems of partial differential equations. This technique replaces the given problem by a sequence of initial value problems for differential-difference equations.

2. The Problem of Brayton and Miranker

The problem of Brayton and Miranker is to determine the current and voltage in a transmission line terminated at each end by linear or nonlinear circuit elements. Take an x-axis in the direction of the line, with the ends of the line at $x = 0$ and $x = 1$. Let $i(x, t)$ denote the current flowing in the line at time t and distance x down the line, and let $v(x, t)$ denote the voltage across the line at t and x. It is well-known that the functions v and i satisfy the partial differential equations

$$\begin{aligned} \frac{\partial v}{\partial x} &= -\left(Ri + L\frac{\partial i}{\partial t}\right) + e, \\ \frac{\partial i}{\partial x} &= -\left(Gv + C\frac{\partial v}{\partial t}\right), \end{aligned} \tag{1}$$

where R, L, G, C are the resistance, inductance, conductance, and capacitance per unit length of the line, and e is the voltage per unit length impressed along the line in series with it. In this paper, we shall assume that $R = G = 0$ (lossless line), so that the basic equations are

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} + e, \qquad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}. \tag{2}$$

This assumption results in a simpler case since if L and C are constants and $e = 0$, the variables v and i satisfy the wave equation. We shall later permit L and C to be functions of x and t, but in this section we assume that they are constants.

If v and i satisfy (2), with L and C constant, it is well-known that each can be represented as a superposition of traveling waves moving to the left and right with velocity $c = 1/(LC)^{1/2}$.

Thus we have

$$v(x, t) = \tfrac{1}{2} [\phi(x - ct) + \psi(x + ct)],$$
$$i(x, t) = \frac{1}{2Lc} [\phi(x - ct) - \psi(x + ct)], \tag{3}$$

where the functions ϕ and ψ must be determined from the boundary conditions, which will be certain integrodifferential relations. For example, for the terminations in the case treated by Brayton (see Fig. 1) they are

$$-v(0, t) = R_0 i(0, t)$$
$$i(1, t) = C_1 \frac{\partial v(1, t)}{\partial t} + f(v(1, t)). \tag{4}$$

Here $f(v(1, t))$ represents a nonlinear resistance.

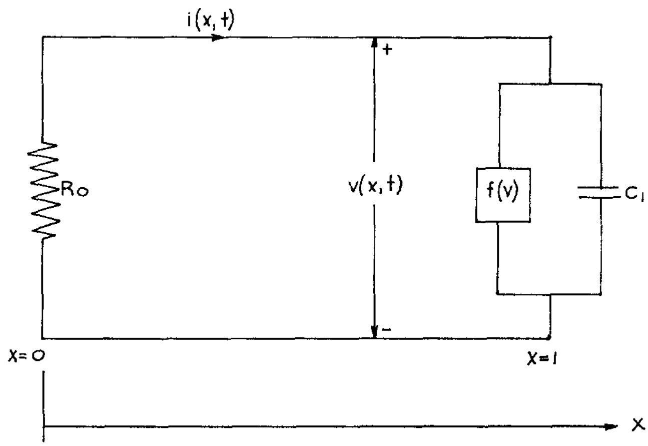

FIG. 1

The method of Brayton and Miranker for obtaining a differential-difference equation is based on the following physical ideas. The term $\phi(x - ct)$ represents a wave disturbance moving to the right with speed c, whereas $\psi(x + ct)$ represents a wave moving to the left. It is expected that these waves will be reflected at the endpoints, and in particular that a wave moving to the left at $x = 1$ will be reflected back as a wave moving to the right at $x = 1$ at time $T = 2/c$ later, but with an attenuation depending on the resistance R_0 at $x = 0$. That is, a solution should satisfy a relation

$$\psi(1 + ct) = a\phi(1 - c(t + T)) \tag{5}$$

for all t. Indeed, by using (3) in the first equation in (4) we obtain $\psi(ct) = a\phi(-ct)$ for $t > 0$, where $a = (R_0 + Lc)/(R_0 - Lc)$. Replacing t by $t + 1/c$ and using $T = 2/c$ we get (5). Now finally if we use (3) and (5) in the second equation of (4), we obtain an equation for the unknown ϕ and ϕ' evaluated at arguments $1 - ct$ and $1 - ct - cT$. That is, we have a differential-difference equation for ϕ.

In Section 3 we shall describe a way to make the reduction to a differential-difference equation which depends only on the well-known method of characteristics.

3. Reduction of Initial-Boundary Problems to Initial Value Problems for Differential-Difference Equations

In this section we shall present a method for reducing a class of initial-boundary-value problems for hyperbolic partial differential equations to initial value problems for integro-differential-difference equations. The method applies to a system of two linear partial differential equations which can be brought into the special normal hyperbolic form[2]

$$u_t + D(x, t)\, u_x = \phi(x, t), \qquad 0 \leqslant x \leqslant 1, \qquad 0 \leqslant t, \tag{1}$$

where

$$u(x, t) = \begin{pmatrix} u_1(x, t) \\ u_2(x, t) \end{pmatrix}, \qquad D = \begin{pmatrix} \tau_1(x, t) & 0 \\ 0 & \tau_2(x, t) \end{pmatrix}, \tag{2}$$

and where $\tau_1(x, t) > 0$ and $\tau_2(x, t) < 0$. The initial conditions are of the form

$$u(x, 0) = u_0(x), \qquad 0 \leqslant x \leqslant 1 \tag{3}$$

and the boundary conditions are of the form

$$f_i(u_1(0, t), u_1^{(1)}(0, t), u_1^{(-1)}(0, t), u_1(1, t), u_1^{(1)}(1, t), u_1^{(-1)}(1, t), u_2(0, t), u_2^{(1)}(0, t),$$

$$u_2^{(-1)}(0, t), u_2(1, t), u_2^{(1)}(1, t), u_2^{(-1)}(1, t), t) = 0 \qquad (i = 1, 2;\, t \geqslant 0). \tag{4}$$

Here the notation is defined by the equations

$$u_k^{(1)}(a, t) = \frac{\partial u}{\partial t}\, k(0, t) \qquad (k = 1, 2;\, a = 0 \text{ or } 1)$$

$$u_k^{(-1)}(a, t) = \int_0^t u_k(a, \tau)\, d\tau \qquad (k = 1, 2;\, a = 0 \text{ or } 1). \tag{5}$$

The functions f_1 and f_2 are given, as is the vector function u_0.

[2] Cf. R. Courant, [5], p. 424.

The form of the boundary conditions in (4) is suggested by the physical model of a transmission line which we formulate in Section 4. The requirement that the linear system (1) contain no term in u seems to be an essential limitation of the method as described in this section, but in Section 6 below we shall outline an extension of our method to general semilinear systems

$$u_t + D(x, t)\, u_x = \phi(x, t, u).$$

In any case, the boundary equations (4) can be very general nonlinear relations. The requirement that (1) be a system of only two equations is not fundamental, and in a later paper we plan to extend the method to systems of higher order with boundary conditions suggested by the consideration of linked transmission lines.

The characteristics for (1) form two families of curves of slopes $dt/dx = 1/\tau_1$ and $dt/dx = 1/\tau_2$. We assume that through each point (x, t) in $0 \leqslant x \leqslant 1$, $0 \leqslant t$, there are two characteristics, C_1 with positive slope and C_2 with negative slope. The curve C_1 extends to the right until it intersects $x = 1$, and to the left until it intersects either $x = 0$ or $t = 0$, whereas C_2 intersects $x = 0$ on the left and either $x = 1$ or $t = 0$ on the right. Now if we introduce the directional differentiation

$$D_i = \frac{\partial}{\partial t} + \tau_i(x, t) \frac{\partial}{\partial x} \qquad (i = 1, 2) \tag{6}$$

along the characteristic C_i, the system (1) can be written in the form

$$D_i u_i = \phi_i(x, t) \qquad (i = 1, 2). \tag{7}$$

A characteristic C_1 through a point $(0, \hat{t})$ of the $x - t$ region intersects the boundary $x = 1$ at some point $(1, \hat{t} + T_1)$, where T_1 can be found by integration of the relation $dt/dx = 1/\tau_1$. For example, if the curves C_1 are described by the integral $g(x, t) = \text{const}$, then the curve through $(0, \hat{t})$ is $g(x, t) = g(0, \hat{t})$. Setting $x = 1$ and $t = \hat{t} + T_1$ we get $g(1, \hat{t} + T_1) = g(0, \hat{t})$. In any event, $T_1 = T_1(\hat{t})$ is a well-defined function of $\hat{t}$. Similarly, a characteristic C_2 through a point $(1, \hat{t})$ intersects $x = 0$ at a point $(0, \hat{t} + T_2)$ where $T_2 = T_2(\hat{t})$ depends only on $\tau_2(x, t)$. See Fig. 2.

Now let us integrate the equation $D_1 u_1 = \phi_1$ along a characteristic C_1 from a point $(0, t)$ to the point $(1, t + T_1(t))$. Clearly we obtain

$$u_1(1, t + T_1(t)) = u_1(0, t) + \psi_1(t), \qquad t \geqslant 0, \tag{8}$$

$$\psi_1(t) = \int_t^{t+T_1(t)} \phi_1(x_1, t_1)\, dt_1, \tag{9}$$

the integration being a line integral along C_1.

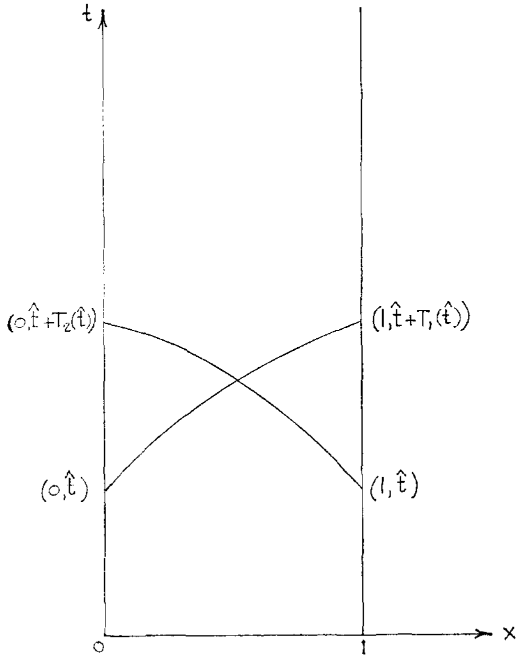

FIG. 2

In the same way by integrating $D_2 u_2 = \phi_2$, along a characteristic C_2 from $(0, t + T_2(t))$ to $(1, t)$ we obtain

$$u_2(1, t) = u_2(0, t + T_2(t)) + \psi_2(t), \qquad t \geqslant 0, \tag{10}$$

$$\psi_2(t) = \int_{t+T_2(t)}^{t} \phi_2(x_1, t_1)\, dt_1 . \tag{11}$$

Hence

$$u_1^{(1)}(0, t) = \frac{d}{dt} u_1(1, t + T_1(t)) - \psi_1'(t)$$

$$= [1 + T_1'(t)] \frac{\partial u_1}{\partial t}(1, t + T_1(t)) - \psi_1'(t).$$

Substitution of the relations (8) and (10) into the boundary conditions (4) will yield the equations we are seeking. In fact, let us define

$$y_1(t) = u_1(1, t), \qquad y_2(t) = u_2(0, t). \tag{12}$$

Then by (8) we have

$$
\begin{aligned}
u_1(0,t) &= y_1(t + T_1(t)) - \psi_1(t),\\
u_1^{(1)}(0,t) &= [1 + T_1'(t)]\, y_1'(t + T_1(t)) - \psi_1'(t)\\
u_1^{(-1)}(0,t) &= \int_0^t y_1(\tau + T_1(\tau))\, d\tau - \int_0^t \psi_1(\tau)\, d\tau.
\end{aligned} \tag{13}
$$

Likewise from (10) we get

$$
\begin{aligned}
u_2(1,t) &= y_2(t + T_2(t)) + \psi_2(t)\\
u_2^{(1)}(1,t) &= [1 + T_2'(t)]\, y_2'(t + T_2(t)) + \psi_2'(t)\\
u_2^{(-1)}(1,t) &= \int_0^t y_2(\tau + T_2(\tau))\, d\tau + \int_0^t \psi_2(\tau)\, d\tau.
\end{aligned} \tag{14}
$$

If we substitute these relations into (4), we obtain two equations

$$
\begin{aligned}
F_i \Big(& y_1(t), y_1(t + T_1(t)), y_1'(t), y_1'(t + T_1(t)), \int_0^t y_1(\tau)\, d\tau, \int_0^t y_1(\tau + T_1(\tau))\, d\tau,\\
& y_2(t), y_2(t + T_2(t)), y_2'(t), y_2'(t + T_2(t)),\\
& \int_0^t y_2(\tau)\, d\tau, \int_0^t y_2(\tau + T_2(\tau))\, d\tau, t \Big) = 0 \qquad (t \geqslant 0), \qquad (i = 1, 2).
\end{aligned} \tag{15}
$$

In other words, the result is a system of two integrodifferential-difference equations in two unknown functions.

Moreover, the system (15) is equivalent to the original problem. We have already shown that a solution of (1), (3), (4) yields a solution of (15). Conversely, suppose that $y_1(t)$ and $y_2(t)$ satisfy (15) for $t \geqslant 0$. Define $u_1(1, t)$ and $u_2(0, t)$ for $t \geqslant 0$ by (12). Define $u_1(x, t)$ and $u_2(x, t)$ throughout the strip $0 \leqslant x \leqslant 1, 0 < t$ by integrating (7) along characteristics and using the known values of $u_1(1, t)$ and of $u_2(0, t)$, respectively. Clearly Eqs. (8) and (10) are then valid, and also (13) and (14). Therefore Eq. (15) leads at once to (4). Equation (1) is satisfied, since $u_1(x, t)$, $u_2(x, t)$ satisfy (7). Thus, every solution of (15) yields in a unique way a solution of (1) and (4). (We assume here that $T_i'(t) \neq -1$ and that the characteristics are of the kind previously assumed).

We also observe that an initial condition (3) leads, by integration along characteristics C_1 emanating from the segment $0 \leqslant x \leqslant 1$, $t = 0$, to values of $u_1(1, t)$ (or $y_1(t)$) on $x = 1, 0 \leqslant t \leqslant T_1(0)$, and by integration along characteristics C_2 emanating from this segment to values of $u_2(0, t)$ (or $y_2(t)$) on $x = 0$, $0 \leqslant t \leqslant T_2(0)$. (See Fig. 3.) These values of $y_1(t)$ and $y_2(t)$ will in general

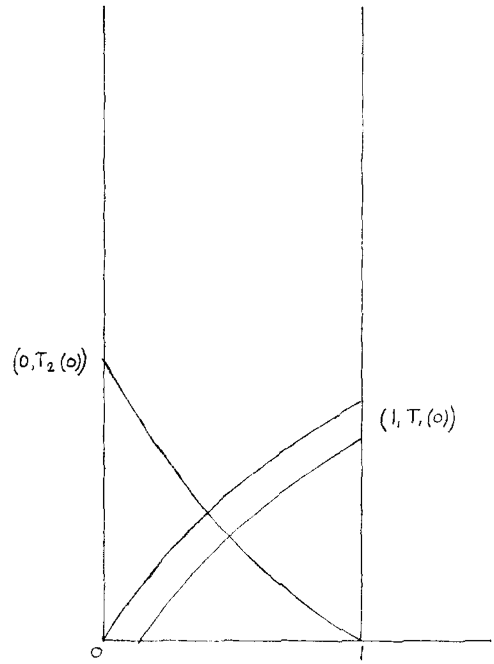

FIG. 3

be appropriate initial values for the system (15). However, we shall defer a detailed analysis of the existence-uniqueness question for the resulting initial-value problem to a later occasion.

We may summarize by stating that in this section we have described a general method for establishing a correspondence between mixed problems of the form (1), (3), (4) and initial-value problems for systems of ordinary differential-difference equations. In this way, known theory and technique for either kind of problem can be applied to the other.

4. TRANSMISSION LINE WITH GENERAL TERMINATIONS

We shall now illustrate our method by considering the physical problem of a lossless transmission line terminated by more general circuit elements in series or parallel at the ends of the line. In this section we formulate the problem, and in the next section we transform it to normal hyperbolic form so that the method of Section 3 can be applied.

Let us consider a linear lossless line described by equations

$$\frac{\partial v}{\partial x} = -L(x,t)\frac{\partial i}{\partial t} + e(x,t), \qquad \frac{\partial i}{\partial x} = -C(x,t)\frac{\partial v}{\partial t}, \tag{1}$$

with initial conditions

$$v(x, 0) = v_0(x), \qquad i(x, 0) = i_0(x), \qquad 0 \leqslant x \leqslant 1, \tag{2}$$

where $v_0(x)$ and $i_0(x)$ are given functions. We shall suppose that the line is terminated by parallel or series circuits as in Fig. 4 or 5. For the sake of

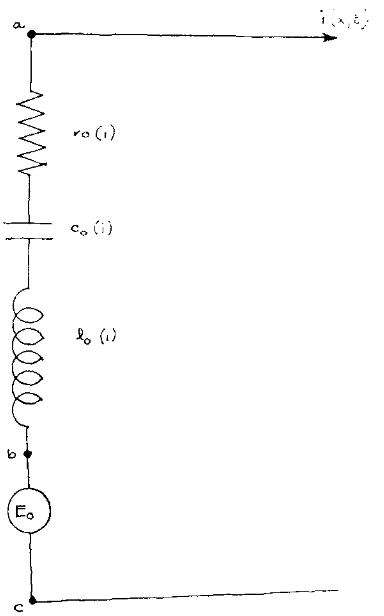

FIG. 4a

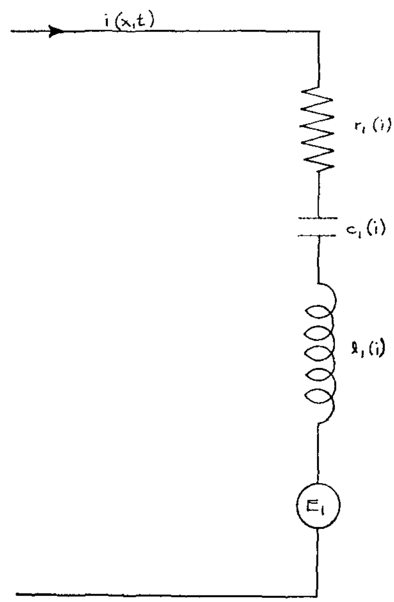

FIG. 4b

generality, we allow all the lumped elements to be arbitrary functions of the voltage or current, rather than requiring linear elements. For a series termination as in Figure 4, we have for the voltages between nodes[3]

$$- v(0, t) = - v_{ac} = v_{ba} - v_{bc} = v_{ba} - E_0(t),$$

$$- v(0, t) = r_0(i(0, t)) + c_0 \left(\int_0^t i(0, \tau) \, d\tau \right) + \ell_0 \left(\frac{di}{dt} (0, t) \right) - E_0(t), \tag{3}$$

and similarly

$$v(1, t) = r_1(i(1, t)) + c_1 \left(\int_0^t i(1, \tau) \, d\tau \right) + \ell_1 \left(\frac{di}{dt} (1, t) \right) + E_1(t). \tag{4}$$

[3] Here we assume that $i(0,0) = i(1,0) = 0$.

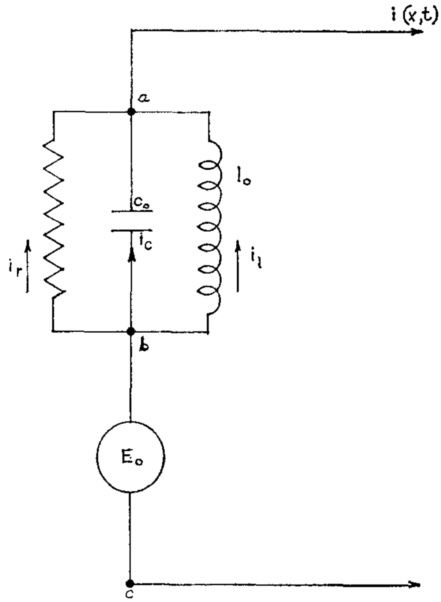

FIG. 5a

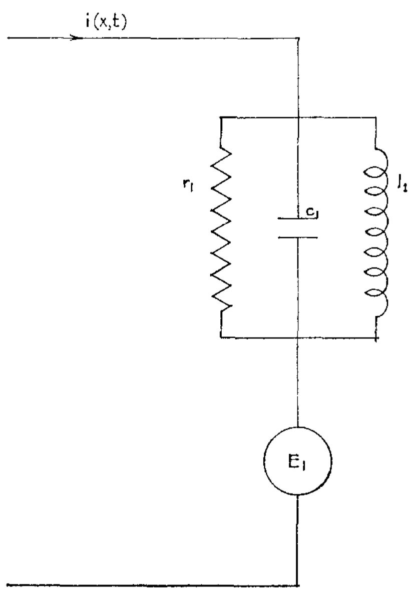

FIG. 5b

In these equations, $r_0(i)$, $c_0(i)$, $\ell_0(i)$, $r_1(i)$, $c_1(i)$, and $\ell_1(i)$ are considered to be arbitrary functions of the real variable i. If all circuit elements are linear, we have

$$r_k(i(k, t)) = R_k i(k, t) \qquad (k = 1, 2)$$

$$c_k\left(\int_0^t i(k, \tau)\, d\tau\right) = \frac{1}{C_k} \int_0^t i(k, \tau)\, d\tau \qquad (k = 1, 2)$$

$$\ell_k\left(\frac{di}{dt}(k, t)\right) = L_k \frac{di}{dt}(k, t) \qquad (k = 1, 2), \tag{5}$$

where R_0, R_1, C_0, C_1, L_0, L_1 are constants, the ordinary resistance, capacitance, and inductance in the terminations.

For a parallel termination as in Fig. 5a, we have $v(0, t) = v_{ac}$, $E_0(t) = v_{bc}$, and $i(0, t) = i_\ell + i_c + i_r$, where i_ℓ, i_c, i_r are the currents through the inductive, capacitative, and resistive branches. We assume that these currents are related to the impressed voltage by relations of the form

$$\begin{aligned} i_c &= c_0\left(\frac{d}{dt} v_{ba}\right), \\ i_l &= \ell_0\left(\int_0^t v_{ba}(\tau)\, d\tau\right), \\ i_r &= r_0(v_{ba}), \end{aligned} \tag{6}$$

wher c_0, ℓ_0, and r_0 are known functions of one independent variable. Using the relation $v_{ab} = v(0, t) - E_0(t)$, we therefore obtain

$$\begin{aligned} -i(0, t) = &- c_0\left(E_0'(t) - \frac{d}{dt} v(0, t)\right) - \ell_0\left(\int_0^t [E_0(\tau) - v(0, \tau)]\, d\tau\right) \\ &- r_0(E_0(t) - v(0, t)). \end{aligned} \tag{7}$$

Similarly

$$\begin{aligned} i(1, t) = &- c_1\left(E_1'(t) - \frac{d}{dt} v(1, t)\right) - \ell_1\left(\int_0^t [E_1(\tau) - v(1, \tau)]\, d\tau\right) \\ &- r_1(E_1(t) - v(1, t)). \end{aligned} \tag{8}$$

In case the circuit elements are all linear, the relations (6) reduce to

$$\begin{aligned} i_c &= C_0\left(E_0'(t) - \frac{d}{dt} v(0, t)\right), \\ i_\ell &= \frac{1}{L_0}\left(\int_0^t (E_0(\tau) - v(0, \tau))\, d\tau\right), \\ i_r &= \frac{1}{R_0} [E_0(t) - v(0, t)], \end{aligned} \tag{9}$$

where L_0, C_0, R_0 are the ordinary inductance, capacitance, and resistance, and (7) reduces to

$$-i(0,t) = C_0 \frac{dv(0,t)}{dt} + \frac{1}{R_0} v(0,t) + \frac{1}{L_0}\int_0^t v(0,\tau)\,d\tau$$
$$- C_0 E_0'(t) - \frac{1}{R_0} E_0(t) - \frac{1}{L_0}\int_0^t E_0(\tau)\,d\tau. \tag{10}$$

Similarly, (8) becomes

$$i(1,t) = C_1 \frac{dv(1,t)}{dt} + \frac{1}{R_1} v(1,t) + \frac{1}{L_1}\int_0^t v(0,\tau)\,d\tau$$
$$- C_1 E_1'(t) - \frac{1}{R_1} E_1(t) - \frac{1}{L_1}\int_0^t E_1(\tau)\,d\tau. \tag{11}$$

The various boundary conditions above can all be subsumed under the general form

$$\sum_{j=-1}^{1} [\alpha_j(v^{(j)}(0,t),t) + \beta_j(i^{(j)}(0,t),t)] = f_1(t)$$

$$\sum_{j=-1}^{1} [\gamma_j(v^{(j)}(1,t),t) + \delta\,(i^{(j)}(1,t),t)] = f_2(t), \qquad t > 0 \tag{12}$$

where

$$v^{(-1)}(k,t) = \int_0^t v(k,\tau)\,d\tau,$$
$$i^{(-1)}(k,t) = \int_0^t i(k,\tau)\,d\tau, \qquad (k = 0, 1) \tag{13}$$

and where $\alpha_j(v,t)$, $\beta_j(i,t)$, $\gamma_j(v,t)$, and $\delta_j(i,t)$ are given functions of their arguments. In the special case in which the line is terminated with linear time-invariant circuit elements, Eqs. (12) take the simpler form

$$\sum_{j=-1}^{1} [\alpha\, v^{(j)}(0,t) + \beta_j i^{(j)}(0,t)] = f_1(t),$$

$$\sum_{j=-1}^{1} [\gamma_j v^{(j)}(1,t) + \delta_j i^{(j)}(1,t)] = f_2(t), \qquad t > 0 \tag{14}$$

where α_j, β_j, γ_j, and δ_j are constants. Moreover, for the special terminations in Figures 4 and 5 we have the following special values:

Parallel on left:

$$\beta_{-1} = \beta_1 = 0, \qquad \beta_0 = 1, \qquad \text{all} \qquad \alpha_j \geqslant 0;$$

Parallel on right:

$$\delta_{-1} = \delta_1 = 0, \qquad \delta_0 = -1, \qquad \text{all} \qquad \gamma_j \geqslant 0;$$

Series on left:

$$\alpha_{-1} = \alpha_1 = 0, \qquad \alpha_0 = 1, \qquad \text{all} \qquad \beta_j \geqslant 0;$$

Series on right:

$$\gamma_{-1} = \gamma_1 = 0, \qquad \gamma_0 = -1, \qquad \text{all} \qquad \delta_j \geqslant 0. \tag{15}$$

5. Reduction to Normal Hyperbolic Form

In this section we shall use standard methods to replace the initial-boundary-value problem (4-1), (4-2), (4-12) by a problem in the normal hyperbolic form of Section 3.

If we let

$$w = \begin{pmatrix} v \\ i \end{pmatrix} \tag{1}$$

Eq. (4-1) can be written

$$w_t + Aw_x = \epsilon, \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 1/C \\ 1/L & 0 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} 0 \\ e(x, t)/L(x, t) \end{pmatrix}. \tag{3}$$

The eigenvalues of A are $\tau_1(x, t) = 1/\sqrt{LC}$, $\tau_2(x, t) = -1/\sqrt{LC}$, and corresponding row eigenvectors are $(\sqrt{C}, \sqrt{L})$ and $(-\sqrt{C}, \sqrt{L})$. Consequently we let

$$H = \begin{pmatrix} \sqrt{C} & \sqrt{L} \\ -\sqrt{C} & \sqrt{L} \end{pmatrix}, \qquad D = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \tag{4}$$

and make the substitution

$$z = Hw. \tag{5}$$

Since $HA = DH$, we obtain the equation (in normal hyperbolic form)

$$z_t + Dz_x = Bz + \phi, \tag{6}$$

where

$$\phi(x, t) = H\epsilon = \begin{pmatrix} e(x, t)/(L(x, t))^{1/2} \\ e(x, t)/(L(x, t))^{1/2} \end{pmatrix} \tag{7}$$

$$B = \left(\frac{\partial H}{\partial t} + D \frac{\partial H}{\partial x}\right) H^{-1}. \tag{8}$$

If B is zero, Eq. (6) is of the form discussed in Section 3. Moreover, since

$$2\sqrt{C}\, v = z_1 - z_2, \qquad 2\sqrt{L}\, i = z_1 + z_2, \tag{9}$$

the initial conditions (4-2) yield

$$z_1(x, 0) = \sqrt{C(x, 0)}\, v_0(x) + \sqrt{L(x, 0)}\, i_0(x),$$

$$z_2(x, 0) = -\sqrt{C(x, 0)}\, v_0(x) + \sqrt{L(x, 0)}\, i_0(x), \tag{10}$$

and the boundary conditions (4-12) take the form

$$\sum_{j=-1}^{1} [\hat{\alpha}_j(z_1^{(j)}(0, t) - z_2^{(j)}(0, t), t) + \hat{\beta}_j(z_1^{(j)}(0, t) + z_2^{(j)}(0, t), t)] = f_1(t)$$

$$\sum_{j=-1}^{1} [\hat{\gamma}_j(z_1^{(j)}(1, t) - z_2^{(j)}(1, t), t) + \hat{\delta}_j(z_1^{(j)}(1, t) + z_2^{(j)}(1, t), t)] = f_2(t), \tag{11}$$

where $\hat{\alpha}_j(z, t)$, $\hat{\beta}_j(z, t)$, $\hat{\gamma}_j(z, t)$, $\hat{\delta}_j(z, t)$ are given functions. For the linear case (4-14), the boundary conditions may be written as

$$\sum_{j=-1}^{1} [a_{1j} z_1^{(j)}(0, t) + b_{1j} z_2^{(j)}(0, t)] = f_1(t),$$

$$\sum_{j=-1}^{1} [a_{2j} z_1^{(j)}(1, t) + b_{2j} z_2^{(j)}(1, t)] = f_2(t) \qquad t > 0 \tag{12}$$

where

$$a_{1j} = \frac{\alpha_j}{2\sqrt{C}} + \frac{\beta_j}{2\sqrt{L}}, \qquad b_{1j} = -\frac{\alpha_j}{2\sqrt{C}} + \frac{\beta_j}{2\sqrt{L}}$$

$$a_{2j} = \frac{\gamma_j}{2\sqrt{C}} + \frac{\delta_j}{2\sqrt{L}}, \qquad b_{2j} = -\frac{\gamma_j}{2\sqrt{C}} + \frac{\delta_j}{2\sqrt{L}}. \tag{13}$$

Thus, if $B \equiv 0$, the problem is reduced to one of the form treated in Section 3. A sufficient condition for $B \equiv 0$ is that C and L be constants.

6. Semilinear Systems and Differential-Difference Equations

The method of Section 3 cannot be applied in the indicated form to semilinear systems

$$u_t + D(x, t)\, u_x = \phi(x, t, u), \qquad 0 \leqslant x \leqslant 1, \qquad 0 \leqslant t. \tag{1}$$

Indeed, integration along a characteristic C_1 yields

$$u_1(1, t + T_1(t)) = u_1(0, t) + \int_t^{t+T_1(t)} \phi_1(x_1, t_1, u(x_1, t_1))\, dt_1,$$

where the integral is a line integral. It is no longer true that the values of u_1 on $x = 1$ depend only on the values of u_1 on $x = 0$ at earlier times.

However, it appears to be possible to replace a mixed initial boundary problem for (1) by a sequence of initial value problems for a system of differential-difference equations, by using a method of successive approximations. Suppose that the initial and boundary conditions are those in (3-3) and (3-4). Let $u = v(x, t)$ be an initial approximation, defined in $0 \leqslant x \leqslant 1$, $0 \leqslant t$. The problem

$$u_t + D(x, t)\, u_x = \phi(x, t, v(x, t)), \tag{2}$$

where u satisfies (3-3) and (3-4), is then of the type considered in Section 3, and can therefore be replaced by a differential-difference system of the type in (3-15). The solution of this problem yields $u_1(1, t)$ and $u_2(0, t)$ and then, by integration along characteristics, $u_1(x, t)$ and $u_2(x, t)$. These components now define a new $v(x, t)$. Inserting in (2) we obtain a new problem, again of the type treated in Section 3.

Continuing in this way, we can generate a sequence of initial-value problems for differential-difference systems and a sequence of solutions thereto. We leave for a later paper the discussion of whether this sequence converges to a solution of the mixed initial boundary problem for (1) with boundary conditions as in (3-4), and of numerical methods based on these ideas. It is also possible to apply similar ideas to a nonlinear system in which D depends on x, t, u.

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