Affine Distance-transitive Groups with Alternating or Symmetric Point Stabiliser

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We determine all finite graphs which admit a distance-transitive, primitive, affine automorphism group $G$ such that a point stabilizer in $G$ is an alternating or symmetric group (modulo scalars). This work forms part of a programme to classify all finite distance-transitive graphs.

1. Introduction

This paper represents a contribution to the programme of classifying all finite primitive distance-transitive graphs, started in [15].

We refer to the introduction of [15] for basic definitions and examples concerning distance-transitive graphs and groups. Let $\Gamma$ be a finite distance-transitive graph with vertex set $\Omega$, and let $G$ be a subgroup of Aut $\Gamma$ acting distance-transitively on $\Gamma$. As explained in [15] (see also [5]), it is natural to assume that $G$ acts primitively on $\Omega$. It is also natural to assume that the valency of $\Gamma$ is at least 3. The main theorem of [15] states that with these assumptions, one of the following holds: (a) $\Gamma$ is a Hamming graph, or the complement of a Hamming graph of diameter 2; (b) $G$ is almost simple; (c) $G$ is an affine group. In this paper we contribute to the classification of the graphs in the affine case (c).

Suppose now that the distance-transitive group $G$ is an affine group. By definition, this means that the vertex set $\Omega$ can be identified with the vectors in a vector space $V$ of dimension $m$ over $\mathbb{F}_p$ ($p$ prime) in such a way that $G$ is a subgroup of the affine group $AGL(V) = AGL_m(p)$ containing the regular translation subgroup of order $p^m$. As $G$ is primitive, the stabiliser $G_0$ of the zero vector is an irreducible subgroup of $GL(V)$. Let $d$ be the diameter of $\Gamma$, so that $d$ is the number of orbits of $G_0$ on $V \setminus \{0\}$. The case in which $d \leq 2$ has been handled completely in [11]. The basis for the classification programme in the affine case is provided by the following result of van Bon, proved in [2] (see also [4]):

**Theorem 1.1 (van Bon).** With the above assumptions, suppose that the valency of $\Gamma$ is at least 3. Then one of the following holds:

(i) $d \leq 2$ (possibilities determined in [11]);
(ii) $\Gamma$ is a Hamming graph or a Bilinear Forms graph (Example 5(a) of [15, p. 3]);
(iii) $G_0 \leq GL_1(q)$, where $q = p^m$;
(iv) for some power $q = p^n$, $V$ carries an $\mathbb{F}_q$-structure such that the generalised Fitting subgroup $S = F^*(G_0/F(G_0))$ is a non-abelian simple group, the projective representation of which on $V$ is realised over $\mathbb{F}_q$ but over no proper subfield, and is absolutely irreducible (over $\mathbb{F}_q$).

Thus the major part of the classification programme involves dealing with the various types of simple group $S$ arising in case (iv) of Theorem 1.1. Our main result handles the case in which $S$ is an alternating group. Before stating the result, we describe the well known examples which will arise in the statement (for further details concerning these, see [6]).
EXAMPLES. Let \( n \geq 3 \) and let \( U = V_n(2) \), a vector space of dimension \( n \) over \( F_2 \). Define subspaces \( W = \{ (a_1, \ldots, a_n) \mid \sum a_i = 0 \} \) and \( X = \{ (a, a, \ldots, a) \mid a \in F_2 \} \).

Suppose that \( n \) is odd. Set \( V = W \) and define graphs \( \Gamma_1 \) and \( \Gamma_2 \) with vertex set \( V \) as follows: in \( \Gamma_1 \), two vectors are joined by an edge if they differ in precisely two entries; and in \( \Gamma_2 \) two vectors are joined iff they agree in precisely one entry. Then \( \Gamma_1 \) and \( \Gamma_2 \) are primitive and distance-transitive, and \( \text{Aut} \Gamma_1 = \text{Aut} \Gamma_2 = 2^{n-1} S_n \). The graph \( \Gamma_1 \) is a half-cube, and \( \Gamma_2 \) is a folded cube.

Now suppose that \( n \) is even, so that \( X \subseteq W \). Set \( V = W/X \) and define a graph \( \Gamma_3 \) with vertex set \( V \) by joining two vectors \( w_1 + X \) and \( w_2 + X \) iff \( w_1 \) and \( w_2 \) either differ in precisely two entries, or agree in precisely two entries. Then \( \Gamma_3 \) is primitive and distance-transitive, with \( \text{Aut} \Gamma_3 = 2^{n-2} S_n \); it is called a folded half-cube.

THEOREM. Suppose that \( G \) is an affine group acting primitively and distance-transitively on a graph \( \Gamma \), and that case (iv) of Theorem 1.1 holds, with \( S \) an alternating group. Then either the diameter \( d \leq 2 \), or \( \Gamma \) is a half-cube, a folded cube or a folded half-cube.

The paper has two further sections. In the first we collect various preliminary results needed in the proof of the theorem. These include a recent result of van Bon [3] which gives decisive information in the case in which the point stabiliser \( G_o \) fixes a quadratic form on \( V \). This result is of great use in our proof, which is given in the second section.

2. PRELIMINARY RESULTS

Let \( \Gamma \) be a finite primitive distance-transitive graph of diameter \( d \), and for vertices \( \alpha, \beta \) of \( \Gamma \), let \( d(\alpha, \beta) \) be the distance between \( \alpha \) and \( \beta \) (i.e. the length of the shortest path in \( \Gamma \) from \( \alpha \) to \( \beta \)). For \( 1 \leq i \leq d \), let \( I_i(\alpha) = \{ \beta \mid d(\alpha, \beta) = i \} \) and write \( k_i = |I_i(\alpha)| \). If \( d(\alpha, \beta) = i \), denote by \( c_i, a_i \) and \( b_i \) the numbers of vertices adjacent to \( \beta \) and at distances \( i - 1 \), \( i \) and \( i + 1 \) from \( \alpha \), respectively.

In the first proposition we collect some well known results from [6, 4.1.6, 5.1.1 and 5.4.1].

PROPOSITION 2.1. Suppose that \( k_1 \geq 3 \) and \( d \geq 3 \). Then:

(a) we have \( k_1 > b_1 \geq \cdots \geq b_{d-1} \) and \( 1 = c_1 \leq c_2 \leq \cdots \leq c_d \);
(b) there exist integers \( h, l \) with \( h \leq l \leq d \) such that
\[
1 < k_1 < \cdots < k_h = \cdots = k_l > \cdots > k_d,
\]
and, furthermore, \( k_1 < k_i \) for \( 1 < i < d \);
(c) if \( k_i = k_{i+1} \) then \( k_j \geq k_i \) for all \( j \), and if \( k_i = k_j \) for some \( i, j \) with \( 0 \leq i < j \) and \( i + j \leq d \), then \( k_{i+1} = k_{j-1} \);
(d) if \( d \geq 4 \) and \( c_2 > 1 \), then \( c_3 \geq 3c_2/2 \).

Now suppose that \( G \) is a primitive affine group acting distance-transitively on \( \Gamma \), so that \( G \leq AGL(V) = AGL_m(p) \), as described in the Introduction. The next proposition is elementary and well known, but we include a proof for completeness.

PROPOSITION 2.2. Suppose that \( k_1 \geq 3 \), \( d \geq 3 \). Then:

(a) \( \text{Aut} \Gamma \) contains the group \( F_p^* \) of non-zero scalar endomorphisms of \( V \);
(b) we have \( d \leq m \);
(c) \( |G_0| \geq (p^m - 1)/m \).

PROOF. (a) This is trivial if \( p = 2 \), so assume that \( p > 2 \). Let \( x \in \Gamma_i(0) \) and suppose that \( ix \notin \Gamma_i(0) \) for some \( i \in F_p^* \); choose \( i \) minimal in \( \{1, \ldots, p - 1\} \). Then \( (i - 1)x \in \Gamma_i(0) \), \( i \neq -1 \), and \( (i - 1)x \notin \Gamma_i(0) \). Then \( (i - 1)x \in \Gamma_i(0) \) and \( (i - 1)x \notin \Gamma_i(0) \).
$I_1(0)$, so $ix = x + (i - 1)x \in I_2(0)$. If $y \in I_2(0)$ then, by distance-transitivity, there exists $g \in G_0$ such that $(ix)g = y$; hence $i(xg) = y$. As $xg \in I_1(0)$, $y \in iI_1(0)$. Thus $I_2(0) \subseteq iI_1(0)$, and so $k_2 \leq k_1$. Since $d \geq 3$, this contradicts 2.1(b). Consequently, $iI_1(0) = I_1(0)$ for all $i \in \mathbb{F}_p^*$, and (a) follows.

(b) As $\Gamma$ is connected, the $\mathbb{F}_p$-span of $I_1(0)$ is $V$, and so $I_1(0)$ contains an $\mathbb{F}_p$-basis $v_1, \ldots, v_m$ of $V$. Any vector of weight $i$ relative to this basis (i.e. any $v = \sum \alpha_k v_k$ with $|\{k \mid \alpha_k \neq 0\}| = i$) has distance at most $i$ from $0$, and hence $V \subseteq \{0\} \cup I_1(0) \cup \cdots \cup I_m(0)$. Thus $d \leq m$.

(c) This follows immediately from (b).

The next result is a recent theorem of van Bon [3].

**Theorem 2.3 (van Bon).** Let $G \leq AGL(V) = AGL_m(p)$ be a primitive affine distance-transitive group, as above. Suppose that $V$ carries an $\mathbb{F}_p$-structure, $q = p^a$, preserved by $G_0$ (i.e. $G_0 \leq \Gamma L_n(q)$, where $q^* = p^m = p^m$). Assume further that the group of scalars $\mathbb{F}_p$ is contained in $G_0$, and that $G_0 \leq \Gamma O_n(q)$, the subgroup of $\Gamma L_n(q)$ preserving a non-degenerate quadratic form on $V$ up to scalar multiplication and field automorphisms. Then one of the following holds:

(i) $d \leq 2$;

(ii) $\Gamma$ is a Hamming graph;

(iii) $\Gamma$ is a half-cube, a folded cube or a folded half-cube.

We shall also need the following elementary result on Hamming graphs. We thank Dr van Bon for supplying the proof.

**Proposition 2.4.** Suppose that the affine group $G \leq AGL(V) = AGL_m(p)$ acts primitively and distance-transitively on a Hamming graph $\Gamma$ with vertex set $V$. Then $G_0$ acts imprimitively on $V$.

**Proof.** In the Hamming graph $\Gamma$, each edge belongs to a unique maximal clique. We claim that the maximal cliques containing 0 are $\mathbb{F}_p$-subspaces of the vertex set $V$. For let $C$ be a maximal clique containing 0, and pick $0 \neq c \in C$. As $\mathbb{F}_p^* \leq \text{Aut } \Gamma$ by 2.2, $0 + c$ and $c + c$ both lie in $C$, so the clique $C + c$ intersects $C$. Hence $C + c = C$, and the claim follows.

Let $V_1, \ldots, V_n$ be the maximal cliques containing 0. If $x \in I_1(0)$ then $x$ is a sum of $i$ distinct vectors $v_1, \ldots, v_i$ in $I_1(0)$, and of no fewer than $i$ such vectors. Each $v_k$ lies in $I_1(0) \cap I_{k-1}(x)$. In the Hamming graph $\Gamma$, this set has size $i$, so the vectors $v_1, \ldots, v_i$ are uniquely determined by $x$. Moreover, no two of them are in the same clique $V_k$. Hence $V = V_1 \oplus \cdots \oplus V_n$, a direct decomposition of $V$ which is preserved by $G_0$. \qed

To conclude this section, we present some results on representations of alternating and symmetric groups. Let $F$ be a field of characteristic $p$, $n \geq 5$, and let the symmetric group $S_n$ act naturally on $F^n$ by permuting the co-ordinates. Define submodules $U$ and $W$ as follows:

$$U = \{(a_1, \ldots, a_n) \mid \sum a_i = 0\}, \quad W = \{(a, \ldots, a) \mid a \in F\}.$$

The $FA_n$-module $U/U \cap W$ is irreducible, and is called a **fully deleted permutation module**: its dimension is $n - 1$ if $p \nmid n$, $n - 2$ if $p | n$. Clearly, $S_n$ preserves the symmetric bilinear form induced by $((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum a_i b_i$. 
Note that for \( n \leq 13 \), almost all the modular character tables of \( A_n \) and its covering groups are known, and can be found in [14]. We use some of these tables for \( n = 13, 14 \), together with the results of [9, 16], to establish the following proposition.

**Proposition 2.5.** Suppose that \( n \geq 13 \) and that \( V \) is a faithful irreducible module for \( A_n \) or the covering group \( 2A_n \) in characteristic \( p \), for some prime \( p \). Suppose further that \( V \) is not a fully deleted permutation module. Then:

(i) if \( n \geq 16 \) then \( \dim V \geq n(n - 5)/4 \);
(ii) if \( n \geq 14 \) then \( \dim V \geq 32 \);
(iii) if \( 14 \leq n \leq 18 \) and \( p = 2 \), then \( \dim V \geq 64 \);
(iv) if \( n = 13 \) then either \( \dim V \geq 32 \) or \( p \geq 11 \), \( \dim V \geq 16 \).

**Proof.** Parts (i) and (ii) are immediate from [9, Theorem 7] when \( V \) is an \( A_n \)-module, and from [16] when \( V \) is a \( 2A_n \)-module. Part (iv) follows from the \( p \)-modular tables for \( A_{13} \) and \( 2A_{13} \) given in [14] when \( p < 11 \), and from [9, 16] again when \( p \geq 11 \).

It remains to prove (iii). The 2-modular table for \( A_{14} \) is given in [14] (or [1]), and it follows from this that \( \dim V \geq 64 \) when \( n = 14 \). Now consider \( n = 15 \). If the restriction \( V \downarrow A_{14} \) has a composition factor which is not trivial or a fully deleted permutation module, then \( \dim V \geq 64 \), so we may assume \( V \downarrow A_{14} \) has the trivial module or the fully deleted permutation module (\( D^{(14)} \) or \( D^{(13,1)} \) in the notation of [9]) as a submodule. Thus \( V \) is a composition factor of the induced module \( D^{(14)} \uparrow A_{15} \) or \( D^{(13,1)} \uparrow A_{15} \). By the Branching Theorem [8, 9.2], \( V \) is therefore a composition factor of one of the \( S_{15} \)-modules \( S^{(14)}(15) \) or \( S^{(13,2)}(15) \) (notation of [9] again). As \( V \neq D^{(14)} \), it follows that \( V \) is a composition factor of \( D^{(13,2)} \). This module is irreducible for \( A_{15} \), by [1, Theorem 8], so we conclude that \( V = D^{(13,2)} \). This has dimension at least \( 15.10/2 = 75 \) by [9, p. 420]. Repetition of this argument for \( n = 16, 17, 18 \) gives the result. □

3. Proof of the Theorem

Suppose that \( G \) is as in the statement of the Theorem in the Introduction. Thus \( G \leq AGL(V) = AGL_n(p) \) is an affine group acting primitively and distance-transitively on a graph \( \Gamma \) with vertex set \( V \). Moreover, for some power \( q = p^a \), \( V \) carries an \( \mathbb{F}_q \)-structure preserved by \( G_0 \), so that if \( n = m/a \), then \( G_0 \leq FL_n(q) \). Write \( G_0 = G_0/(G_0 \cap G_0) \). Then our assumption is that \( S = F^*(G_0) \) is an alternating group \( A_c \), \( c \geq 5 \), the projective representation of which on \( V = V_n(q) \) is absolutely irreducible and cannot be realised over a proper subfield of \( \mathbb{F}_q \). Let \( d \) be the diameter of \( \Gamma \). By 2.2(a), we may assume that the group \( \mathbb{F}_p^* \) of scalars lies in \( G_0 \).

Suppose that the conclusion of the Theorem is false; that is, that \( d \geq 3 \) and \( \Gamma \) is not a half-cube, a folded cube or a folded half-cube.

**Lemma 3.1.** \( V \) is not a fully deleted permutation module for \( S = A_c \) (see Section 2 for definition).

**Proof.** Suppose false. Then \( n = c - \delta \), with \( \delta = 1 \) or \( 2 \), and as \( V \) is not realised over a proper subfield of \( \mathbb{F}_q \), we have \( q = p \).

Assume \( p \) is odd. We have \( \mathbb{F}_p^* \leq G_0 \), and \( G_0/\mathbb{F}_p^* \leq PGO_n(p) \), since \( S \) preserves a symmetric bilinear form on \( V \). By 2.4, \( \Gamma \) is not a Hamming graph, and so since \( d \geq 3 \), 2.3 gives a contradiction.

Thus \( p = 2 \). As \( d \geq 3 \), either \( c \geq 11 \) or \( c \in \{7, 9\} \). If \( c \) is odd then the two smallest orbits of \( G_0 \) on \( V \setminus \{0\} \) are those containing \( (1, 1, \ldots, 1, 0) \) and \( (1, 1, 0, \ldots, 0) \), of sizes
c and (\(\xi\)), and \(\Gamma_l(0)\) is one of these orbits, by 2.1(b). Then \(\Gamma\) is a folded cube or a half-cube, contrary to assumption. And if \(c\) is even then \(c \geq 12\) and the two smallest orbits are those containing the cosets \((1, 1, 0, \ldots, 0) + W\) and \((1, 1, 1, 0, \ldots, 0) + W\) (notation as in Section 2, just before 2.5), of sizes (\(\xi\)) and (\(\xi\)). If \(\Gamma_l(0)\) is the orbit of size (\(\xi\)), then \(\Gamma_2(0)\) contains the cosets of vectors of weights 2 and 6, contrary to the distance transitivity of \(\Gamma\). Hence \(\Gamma_l(0)\) is of size (\(\xi\)) and \(\Gamma\) is a folded half-cube, again a contradiction.

**Lemma 3.2.** We have \(|\tilde{G}_0| \geq (q^n - 1)/m(q - 1)\). Moreover, \(|\tilde{G}_0| \leq c!\), except possibly when \(c = 6\), in which case \(|\tilde{G}_0| \leq 1440\).

**Proof.** This is immediate from 2.2(c) and the fact that \(\tilde{G}_0 \leq \text{Aut}\, A_4\).

**Lemma 3.3.** We have \(c \leq 12\).

**Proof.** Suppose \(c > 12\). Assume first that \(p > 2\). If \(c \geq 14\) then, by 2.5 and 3.1, either \(n \geq c(c - 5)/4\) or \(c \leq 15\), \(n \geq 32\). Hence 3.2 gives either \(c! \geq 2(3^{c(c-5)/4} - 1)/c(c - 5)\) or \(15! \geq (3^{32} - 1)/64\), both of which are false. Hence \(c = 13\). By 2.5, either \(n \geq 32\) or \(p \geq 11\), \(n \geq 16\). In both cases 3.2 is again violated.

Now let \(p = 2\). If \(c \geq 19\) then \(n \geq c(c - 5)/4\) by 2.5(i); and if \(14 \leq c \leq 18\) then \(n \geq 64\) by 2.5(iii). In either case 3.2 gives a contradiction. Thus \(c = 13\). Here 3.2 forces \(n \leq 39\). The 2-modular table for \(A_{13}\) in [14] forces \(n = 32\). But the 32-dimensional 2-modular irreducible for \(A_{13}\) is realised over \(F_4\), not \(F_2\); so \(q = 4\) here, and 3.2 is violated.

**Lemma 3.4.** The embedding \(G_0 < \Gamma L_n(q)\) is one of those in the table below:

<table>
<thead>
<tr>
<th>(c)</th>
<th>(n)</th>
<th>(q)</th>
<th>Embedding of (G_0^{(\infty)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>16</td>
<td>3</td>
<td>(2A_1, &lt; SL_{16}(3))</td>
</tr>
<tr>
<td>11</td>
<td>16</td>
<td>3</td>
<td>(2A_{11}, &lt; SL_{11}(3))</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>2</td>
<td>(A_1 &lt; SL_{20}(2))</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>7</td>
<td>(3A_2, 6A_1 &lt; SL_{6}(7))</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7, 11</td>
<td>(2A_4 &lt; Sp_{7}(7), SL_{4}(11))</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>4, 9</td>
<td>(3A_6 &lt; SL_{3}(4), A_1 &lt; O_6(9))</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5, 7, 11, 13</td>
<td>(2A_6 &lt; Sp_7(q))</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(q &lt; 119)</td>
<td>(2A_4, &lt; SL_{4}(q))</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>9</td>
<td>(A_3 &lt; O_4(9))</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5, 7</td>
<td>(2A_4, &lt; Sp_4(q))</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td></td>
<td>(2A_3 &lt; Sp_3(3))</td>
</tr>
</tbody>
</table>

**Proof.** We know by 3.3 that \(c \leq 12\). The \(p\)-modular tables of \(A_4\) and its covering groups are all given in [14] (in [7] when \(p > c\)). We use these tables to obtain a list of the representations satisfying 3.1 and 3.2. Apart from those in the table in the conclusion, the possibilities for \(c, n\) and \(q\) are as follows:

<table>
<thead>
<tr>
<th>(c)</th>
<th>((n, q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>11, 12</td>
<td>(16, 4)</td>
</tr>
<tr>
<td>10</td>
<td>(16, 2), (26, 2), (16, 3), (8, 5)</td>
</tr>
<tr>
<td>9</td>
<td>(8, (p), (p \leq 7))</td>
</tr>
<tr>
<td>8</td>
<td>(4, 2), (2, 2), (14, 2), (8, 3), (8, 5)</td>
</tr>
<tr>
<td>7</td>
<td>(4, 2), (14, 2), (6, 4), (4, 9), (3, 25), (4, 25)</td>
</tr>
<tr>
<td>6</td>
<td>(4, 2), (2, 9), (3, 25)</td>
</tr>
<tr>
<td>5</td>
<td>(3, 11)</td>
</tr>
</tbody>
</table>
When \((n, q)\) is \((4, 2)\) or \((2, 9)\), \(G_0\) is transitive on \(V\backslash\{0\}\), contrary to the fact that \(d \geq 3\). By [14], in the remaining cases, when \(q = p^2\) we have \(G_0 \leq \Gamma U_n(p) \leq GO_n^\times(p)\), and when \(q = p\) we have \(G_0 \leq GO_n^\times(p)\) (where \(GO_n(p)\) denotes the group preserving a non-degenerate quadratic form up to scalar multiplication). As \(F_n^* \leq G_0\), we can apply 2.3 here. Conclusions (i) and (iii) of 2.3 are excluded by assumption. Hence 2.3(ii) holds: that is, \(\Gamma\) is a Hamming graph. Then, by 2.4, \(G_0\) acts imprimitively on \(V\backslash\{0\}\), and hence \(A_\xi\) or \(S_\xi\) has a non-trivial transitive action on a set of \(b\) points, where \(b \mid n\); and \(b > 2\) as \(A_\xi\) is irreducible. The only possibility is \(c = 8\), \((n, q) = (8, 3)\) or \((8, 5)\). Here \(G_0^{(c)} = 2A_8 < GO_8^\times(p)\), \(p = 3\) or \(5\), and there is an orthonormal basis of \(V\) such that \(G_0^{(c)}\) stabilises \(\{ \pm v \mid v \in B\}\). This embeds \(2A_8 < 2^8\). \(A_8 < S_{16}\), which is impossible as \(2A_8\) has no subgroup of index 16.

The rest of the proof consists of ruling out the possibilities in 3.4.

**Lemma 3.5.** The group \(G_0 = 2.A_12 \leq SL_{16}(3)\), acting irreducibly on \(V = V_{16}(3)\), has precisely eight orbits on \(V\backslash\{0\}\), of sizes 60480, 123200, 475200, 475200, 665280, 1330560, 13305600 and 26611200.

**Proof.** We have

\[
\tilde{G}_0 = A_{12} < P\Omega_{16}^\times(3) < L_{16}(3) = PSL(V),
\]

where the embedding of \(A_{12}\) in \(P\Omega_{16}^\times(3)\) is given by the fully deleted permutation module over \(F_3\) (see Section 2), and the embedding \(P\Omega_{16}^\times(3) < L_{16}(3)\) by a spin representation of \(P\Omega_{16}^*\) (see, for example, [10, 5.4.9]). Write \(A = \tilde{G}_0\), \(H = P\Omega_{16}^\times(3)\) in the above embedding, and let \(W\) be the fully deleted permutation module for \(A\) over \(F_3\), so that \(H = P\Omega(W)\). As in Section 2, we write elements of \(W\) as 12-tuples \((a_1, \ldots, a_{12})\) (mod \(\langle (1, 1, \ldots, 1) \rangle\)), where \(\sum a_i = 0\). The orbits of \(H\) on the 1-spaces of \(V\) are given by [11, 2.9]: there are exactly two orbits, \(A_1\) and \(A_2\), where

- \(|A_1| = 91840\), \(H_{\delta_1}\) is an \(A_4\)-parabolic subgroup \((\delta_1 \in A_1)\);
- \(|A_2| = 21431520\), \(H_{\delta_2} = 3^8\). Spin\(_7\) \((3) \ (\delta_2 \in A_2)\).

Consider first the orbits of \(A\) on \(A_1\). Now \(A_1\) corresponds to an orbit of \(H\) on totally singular 5-spaces in \(W\), for which a representative can be taken to be

\[
U_1 = \langle (1^3, 0^0), (0^3, 1^3, 0^0), (0^6, 1^3, 0^3), (1, -1, 0, 1, -1, 0, 1, -1, 0, 0^2), (0^3, 1, 0, -1, 1, -1, 0, 1, -1, 0) \rangle
\]

The stabiliser of this 5-space in \(A\) has order \(2^4 \cdot 3^5\), so

\(|(U_1)^A| = 61600.\)

Now let \(M\) be a transitive subgroup \(M_{11}\) of \(A\). From the 3-modular table of \(M\) in [14], we see that \(W \downarrow M\) (the restriction of \(W\) to \(M\)) has two composition factors of dimension 5, neither of which is self-dual; hence \(M\) fixes a totally singular 5-space \(U_2\) in \(W\). Replacing \(M\) by the \(S_{12}\)-conjugate \(M^{(1,2)}\) if necessary, we may take it that \(U_2 \in A_1\). Then

\(|(U_2)^A| = |A_{12}: M_{11}| = 30240.\)

Since \(|A_1| = 61600 + 30240\), this shows that \(A\) has just two orbits on \(A_1\), with orbit representatives \(U_1\) and \(U_2\).

Now consider the orbits of \(A\) on \(A_2\). Here

\(H_{\delta_2} = 3^8 \cdot \text{Spin}_7(3) < 3^8 \cdot \Omega_7^+\). \(\Omega_8^6(3) < 3^8 \cdot \Omega_8^6(3)\) \(< 2 \cdot P\Omega_{16}(3) = H\).
Write this chain as $H_5 < H_1 < H_2 < H$. Then $H$ has a block system $\Sigma$ on $\Delta_2$ $H$-isomorphic to the coset space $(H:H_2)$, which is $H$-isomorphic to the set of totally singular 1-spaces in $W$. Thus $\Sigma$ consists of 9922 blocks, each of size 2160. Now $A$ has 3 orbits on totally singular 1-spaces in $W$, with representatives, stabilisers and sizes as follows:

- representative $\langle (1^3, 0^6) \rangle$, stabiliser $(A_9 \times A_3)$, 2, size 220,
- representative $\langle (1^6, 0^6) \rangle$, stabiliser $(S_6 \wr S_2) \cap A_{12}$, size 462,
- representative $\langle (1^3, -1^3, 0^6) \rangle$, stabiliser $((S_3 \wr S_2) \times S_6) \cap A_{12}$, size 9240.

Let $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ be the corresponding orbits of $A$ on $\Sigma$, of sizes 220, 462 and 9240 respectively. We now analyse the orbits of $A$ on the points of $\Delta_2$ lying in the blocks in $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$.

First we consider $\Sigma_1$. Pick $B \in \Sigma_1$ corresponding to the 1-space $\langle x \rangle = \langle (1^3, 0^6) \rangle$, and write $X = x^{-1}/\langle x \rangle$, an 8-space of type $O_8^+(3)$. Then $A_B$ has a block system $(B_1, B_2)$ on $B$ corresponding to $\{x, -x\}$. As $A_B = (A_9 \times A_3)$, 2 fixes both $x$ and $-x$, $A_B$ fixes $B_1$ and $B_2$. The 8-space $X$ admits the natural action of $(H_1)^X = \Omega_8^+(3)$, and $H_1$ acts on $B_1$ as on the coset space $(Q_8^+(3):\text{Spin}(3))$, where the subgroup $\text{Spin}(3)$ is irreducible on $X$. We have $(A_B)^X = S_9 \lhd (H_1)^X$. As $A_B$ fixes the non-isotropic 1-space $\langle (1, -1, 0^{10}) \rangle$ modulo $\langle x \rangle$, $(A_B)^X$ lies in a reducible subgroup $N_1 = \Omega_7(3)$ of $(H_1)^X$. By [13, p. 105, Lemma A], $N_1$ is transitive on $B_1$, and $N_1 \cap \text{Spin}(3) = G_2(3)$. And by [13, p. 101, Lemma B], $N_1 = S_6G_2(3)$. Hence $(A_B)^X = S_9$ is transitive on $B_1$. We conclude that $A$ has just two orbits on the points of $\Delta_2$ lying in blocks of $\Sigma_1$, both of size $220 \times 1080$.

Now consider $\Sigma_2$. Choose a block $C \in \Sigma_2$ corresponding to the 1-space $\langle y \rangle = \langle (1^6, 0^6) \rangle$. As above, $A_C$ has a block system $(C_1, C_2)$ on $C$. The group $A_C = (S_6 \wr S_2) \cap A$ contains an element interchanging $y$ with $-y$, and this element interchanges $C_1$ with $C_2$. Now consider $A_{C_1} = (S_6 \wr S_2) \cap A = (A_6 \times A_6)$. 2. Write $Y$ for the $O_8^+(3)$-space $y^{-1}/\langle y \rangle$. Then $(A_{C_1})^Y = (\Omega_7^+(3) \times \Omega_7^+(3))$. 2 acting naturally on a decomposition $Y = Y_1 \perp Y_2$, where the $Y_i$ are non-degenerate 4-spaces. Again, $(H_1)^Y = \Omega_7^+(3)$ acts on $C_1$ as on $(\Omega_8^+(3):\text{Spin}(3))$. Let $\tau$ be a triality automorphism of $(H_1)^Y/(\langle -1 \rangle) = P\Omega_8^+(3)$ such that $(\text{Spin}(3)/(\langle -1 \rangle))^\tau$ is the stabiliser of a non-isotropic 1-space in $Y$ (see [7, p. 140]). Then $(A_{C_1})^\tau$ is an irreducible subgroup $P\Omega_7^+(9). \langle \sigma \rangle$, where $\sigma$ is a field automorphism. Thus the orbit sizes of $A_{C_1}$, on $C_1$ are the same as those of $P\Omega_7^+(9). \langle \sigma \rangle$ on an $(H_1)^\tau$-orbit of non-isotropic 1-spaces in $Y$. Letting $Q$ be the quadratic form on $Y$ fixed by $(H_1)^Y$, we may take it that $Q = TP$, where $T$ is the trace function from $F_6$ to $F_3$ and $P$ is a quadratic form on $Y$, regarded as $O_7^+(9)$-space, fixed by $(A_{C_1})^\tau$ (see [10, §4.3]). The orbits of $P\Omega_7^+(9)$ on the non-zero vectors in $O_7^+(9)$-space are of the form $\Phi_\lambda = \{v \mid P(v) = \lambda \}$ ($\lambda \in F_6$). For $v \in \Phi_\lambda$, $Q(v) = 0$ iff $\lambda + \lambda^2 = 0$, which holds for $\lambda = 0$ and two other values $\lambda_1, \lambda_2$, where $\lambda_1$ has order 4. Thus $\Phi_\lambda$ consists of $Q$-non-isotropic vectors when $\lambda = 1, -1, \lambda_2, \lambda_2^2, \lambda_3$ or $\lambda_3^2$, where $\lambda_2, \lambda_3$ have order 8. The field automorphism $\sigma$ fuses the orbit pairs $\Phi_{\lambda_1}, \Phi_{\lambda_2}, \Phi_{\lambda_3}$. Thus $(A_{C_1})^\tau = P\Omega_7^+(9). \langle \sigma \rangle$ has precisely four orbits on the non-isotropic 1-spaces over $F_3$ in $Y$. As $|\Phi_1| = 720$ for $\lambda \neq 0$ (recall that $\Phi_0$ consists of vectors rather than 1-spaces), two of these four orbits have size 360 (those containing 1-spaces inside $\Phi_1$ or $\Phi_{-1}$), and the other two have size 720. On a single $(H_1)^\tau$-orbit of non-isotropic 1-spaces, therefore, $A_{C_1}$ has two orbits, of sizes 360 and 720. Consequently, $A$ has just two orbits on the points of $\Delta_2$ lying in blocks of $\Sigma_2$, of sizes $462 \times 2 \times 360$ and $462 \times 2 \times 720$.

Finally, we find the orbits of $A$ on points of $\Delta_2$ lying in blocks in $\Sigma_3$. Pick a block $D \in \Sigma_3$ corresponding to $\langle z \rangle = \langle (1^3, -1^3, 0^6) \rangle$, and write $Z = z^{-1}/\langle z \rangle$. Again, $A_D = ((S_6 \wr S_2) \times S_6) \cap A$ contains an element interchanging $z$ with $-z$, hence interchanging
the two blocks $D_1, D_2$ in a block system for $A_D$ on $D$. Then $A_{D_1} = (S_1 \times S_1 \times S_1) \cap A < (S_1 \times S_1) \cap A$ which acts, as before, as $(\Omega_4(3) \times \Omega_4(3)) \cdot 2$ in $(H_1)^2 = \Omega_8^+(3)$. Applying triality as above, the orbit sizes of $A_{D_1}$ on $D_1$ are the orbit sizes of the subgroup $(3^2 \cdot 2 \times L_2(9)). \langle \alpha \rangle < P\Omega_8^+(3). \langle \alpha \rangle$ on an $(H_1)^2$-orbit of non-isotropic 1-spaces. We know from the previous paragraph that $M = P\Omega_8^+(3). \langle \alpha \rangle$ has two orbits $\Pi_1, \Pi_2$ on such 1-spaces, with point-stabilisers $M_{\pi_i} = \Omega_3(9). \langle \alpha \rangle$ and $M_{\pi_i} = \Omega_3(9)$ (where $\pi_i \in \Pi_i$). The intersection of $(3^2 \cdot 2 \times L_2(9)). \langle \alpha \rangle$ with $M_{\pi_i}$ is a diagonal copy of $A_6$ in $A_6 \times A_6 = P\Omega_8^+(9)$. Hence $(3^2 \cdot 2 \times L_2(9)). \langle \alpha \rangle$ is transitive on both $\Pi_1$ and $\Pi_2$. We conclude that $A$ has two orbits on the points of $\Delta_2$ lying in blocks of $\Sigma_3$, of sizes $9240 \times 2 \times 720$ and $9240 \times 2 \times 360$.

We have now shown that the full list of orbit sizes of $A$ on the set of 1-spaces in $V$ is

$30240, 61600, 237600, 332640, 665280, 6652800, 13305600$.

As $G_0 = 2A_{12}$, the orbit sizes of $G_0$ on $V \setminus \{0\}$ are twice these numbers, giving the result.

\text{Lemma 3.6.} \ c \text{ is not 12.}

\textbf{Proof.} Suppose $c = 12$, so that $(n, q) = (16, 3)$ by 3.4. By [14], an irreducible 16-dimensional 3-modular representation of $2.A_{12}$ does not extend to $2.A_{12}$, so $G_0 = 2.A_{12}$ and the orbit sizes of $G_0$ on $V \setminus \{0\}$ are as in 3.5. By 2.1(b), $k_1 = |\Gamma(0)|$ is 60480 or 123200. Write $\tilde{\Gamma}$ for the set of 1-spaces in $\Gamma_0(0) (i = 1, \ldots, 8)$ and, as in 3.5, write $A = \tilde{G}_0 (= A_{12})$.

Assume first that $k_1 = 60480$. Pick $<u> \in \tilde{\Gamma}$. From the proof of 3.5 we have $A_{\langle u \rangle} = M_{11}$. As $M_{11} < M_{12} < A$, $A$ has a block system on $\tilde{\Gamma}$ consisting of 2520 blocks of size 12. Let $B$ be a block. As $A_B = M_{12}$ acts primitively on the set of unordered pairs of elements of $B$, the set $\{(a, b) \mid (a), (b) \in B, (a) \neq (b)\}$ consists of 66 distinct 2-spaces. Consequently, if $(a), (b) \in B$ then $(a \pm b) \notin B$. If either $(a + b)$ or $(a - b)$ lies in $\tilde{\Gamma}$, then $A_B = M_{12}$ fixes a set of at most $66 \times 2$ 1-spaces in $\tilde{\Gamma} \setminus B$; but the orbit sizes of $M_{12}$ on the 2520 blocks $(A_{12}: M_{12})$ are 1, 440, 495, 1584 (see [12, p. 17]). Thus both $(a + b)$ and $(a - b)$ lie in $\tilde{\Gamma}$. This gives $2520 \times 66 \times 2$ ordered pairs $\{(a), (b)\} \times \{(c)\}$ such that $\langle c \rangle \in \tilde{\Gamma}_2, (c) \subseteq (a, b), (a) \neq (b)$ and $(a), (b)$ lie in the same block in $\tilde{\Gamma}$. Each 1-space of $\tilde{\Gamma}$ occurs as the second entry of one of these pairs a constant number of times, and so $|\tilde{\Gamma}_2|$ divides $2520 \times 66 \times 2 = 332640$. From the orbit sizes given in 3.5, we conclude that $|\tilde{\Gamma}_2| = 332640$. Then, by 2.1(b), there exists $j$ such that $k_j = k_{j+1} = 475200$. However, by 2.1(c) this implies that $k_j$ is the largest orbit size, which is not the case.

Thus $k_1 = 123200$. From 2.1(b) we deduce that $k_8 = 60480$ and $k_2 = k_7 = 475200$. Thus $7b_1 = 27c_2$ (where $b_i$, $c_i$ are as defined in Section 2). By 2.1(a, d), $7b_2 \leq 7b_1 = 27c_2 \leq 18c_3 = 475200 \times 18b_2/k_3$. Hence $k_3 < 1221943$, and it follows that $k_3 = 665280$. Then, by 2.1(a) again, $5b_3 \leq 5b_2 = 7c_3 \leq 7c_4 = 7 \times 665280b_3/k_4$, so that $k_4 < 931392$, which is impossible.

\text{Lemma 3.7.} \ (a) The group $G_0 = 2.A_{11} < SL_{16}(3)$ has more that 16 orbits on nonzero vectors.

(b) $c$ is not 11.

\textbf{Proof.} (a) Using [14], we have

$G_0 = A_{11} < A_{12} < L_{16}(3)$,
where \( A_{12} \) has the orbits given in 3.5. We examine the restrictions of these orbits to \( G_0 \). Let \( A = A_{12} \), and let \( \Phi_1, \ldots, \Phi_8 \) be the \( A \)-orbits on 1-spaces of \( V \) of sizes 30240, 61600, 237600, 237600, 322640, 665280, 6652800 and 13305600 respectively. Pick \( \phi_i \in \Phi_i \) (\( 1 \leq i \leq 8 \)). The stabilisers \( A_{\phi_i} \) are given in the proof of 3.5.

As \( A_{\phi_i} \) is a transitive subgroup \( M_{11} \), \( A_{11} \cap M_{11} = L_2(11) \) here, and so \( G_0 \) is transitive on \( \Phi_1 \).

We have \( A_{\phi_2} = ((S_5 \wr S_5) \times S_3) \cap A \). This can intersect \( A_{11} \) in subgroups of index 3 or 9; hence \( G_0 \) has two orbits on \( \Phi_2 \), of sizes 15400 and 46200.

Now consider the orbits \( \Phi_3 \) and \( \Phi_4 \). In the notation of the proof of 3.5, \( \Phi_3 \cup \Phi_4 \) consists of the 1-spaces in blocks in the block system \( \Sigma_1 \), which consists of 220 blocks of size 2160. For \( B \in \Sigma_1 \), \( A_B = (A_9 \times A_3) \cdot 2 \), and \( A_B \) fixes both blocks \( B_1, B_2 \) in the block system \( \{B_1, B_2\} \) on \( B \). The group \( (A_9 \times A_3) \cdot 2 \) can intersect \( A_{11} \) in subgroups of index 3 or 9 in \( A_B \), so \( G_0 \) has two orbits on \( \Sigma_1 \), of sizes 55 and 165. For \( B = B_1 \cup B_2 \) in the orbit of size 55, \( (A_{11})_B = S_6 \cdot (A_B)^B \), so \( G_0 \) has two orbits on the 1-spaces in the blocks in this orbit, both of size \( 55 \times 1080 \). For \( B \) in the orbit of size 165, \( (A_{11})_B = (A_6 \times A_5) \cdot 2 \). So \( ((A_{11})_B)^B = S_8 < S_6 = (A_B)^B \). As \( S_8 \) cannot be transitive on the 1080 points of \( B_1 \) or \( B_2 \), we deduce that \( G_0 \) has at least four orbits on the 1-spaces in blocks in this orbit. Thus \( G_0 \) has at least six orbits on \( \Phi_3 \cup \Phi_4 \).

Next consider the orbits \( \Phi_5 \) and \( \Phi_6 \). From 3.5, \( \Phi_5 \cup \Phi_6 \) consists of the 1-spaces lying in the 462 blocks in the system \( \Sigma_2 \). For \( C \in \Sigma_2 \), \( A_C = (S_6 \wr S_3) \cap A \), so \( (A_{11})_C = (A_5 \times A_5) \cdot 2 \). This group fixes the vectors \( y \) and \( -y \) (where \( y = (1^6, 0^6) \) as in 3.5), and so fixes both blocks \( C_1, C_2 \) in the block system \( \{C_1, C_2\} \) for \( A_C \) on \( C \). Consequently, \( G_0 \) has at least four orbits on \( \Phi_5 \cup \Phi_6 \).

Finally, consider \( \Phi_7 \cup \Phi_8 \), the 1-spaces in the 9240 blocks in \( \Sigma_3 \). For \( D \in \Sigma_3 \), \( A_D = ((S_5 \wr S_5) \times S_3) \cap A \), so \( A_{11} \) has two orbits on \( \Sigma_3 \), both of size 4620. Each of these orbits leads to at least two \( A_{11} \)-orbits on the corresponding set of 1-spaces, so \( G_0 \) has at least four orbits on \( \Phi_7 \cup \Phi_8 \).

We have now shown that \( G_0 \) has at least 17 orbits on the 1-spaces of \( V = V_{16}(3) \), giving (a).

(b) If \( c = 11 \) then \( G_0 = 2.A_{11} < SL_{16}(3) \) by 3.4, so \( G_0 \) has at most 16 orbits on non-zero vectors, by 2.2(b). This contradicts part (a). □

LEMMA 3.8. (a) The group \( G_0 = A_9 < SL_{20}(2) \) (as in 3.4) has more than 20 orbits on non-zero vectors.

(b) We have \( c \leq 7 \).

PROOF. (a) Let \( H \) be a subgroup \( A_8 \) of \( G_0 \). From [14] we see that \( V \downarrow H \) is irreducible, and is isomorphic to \((V_4 \otimes V_6)/V_4^* \), where \( V_4 \) and \( V_6 \) are irreducible \( F_2H \)-modules of dimension 4 and 6 respectively. Pick permutations \( h, k \in H \) of cycle shapes \( 2^4 \) and \( 4^2 \) respectively (so that \( h \) is in class 2A in [7, p. 22], and \( k \) in class 4A). We work out lower bounds for the number of fixed points of \( h \) and \( k \) on \( V \). Observe that for \( g \in H \), \( \dim C_V(g) \geq \dim C_{V_4 \otimes V_6}(g) - \dim C_{V_4}(g) \).

From [7, p. 22], we see that \( h \) fixes 7 points in \( V_4 \setminus \{0\} \), so \( h \) acts as a transvection on \( V_4 \). As \( V_6 = A^2V_4 \), we may take

\[
h^{V_4} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad h^{V_6} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

We calculate from this that \( \dim C_{V_4 \otimes V_6}(h) = 14 \), and hence \( \dim C_V(h) \geq 11 \).
Next, \( k \) fixes 3 points in \( V \setminus \{0\} \), so we may take
\[
k^V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad k^{V_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.
\]
Calculation shows that \( \dim C_{V \otimes V_2}(k) = 8 \), so \( \dim C_V(k) \geq 6 \).

Now \( G_0 = A_9 \) contains 945 conjugates of \( h \) and 11340 conjugates of \( k \). Hence
\[
d = \text{number of orbits of } G_0 \text{ on } V \setminus \{0\} = \frac{1}{|G_0|} \sum_{g \in G_0} |\text{fix}_{V \setminus \{0\}}(g)| \geq (2^{20} - 1 + 945(2^{11} - 1) + 11340(2^6 - 1))/181440 > 20,
\]
giving (a).

(b) Suppose that \( c > 7 \). Then \( c = 9 \) by 3.4–3.7. By 3.4, we have \( G_0 = A_9 \triangleleft SL_{20}(2) = SL(V) \). Then, by (a), \( G_0 \) has more than 20 orbits on non-zero vectors, contradicting 2.2(b).

We are grateful to Dr van Bon for his assistance in the proof of the next lemma.

**Lemma 3.9.** (a) The group \( G_0 = F^+_1 \circ 2A_7 < GL_4(11) \), acting irreducibly on \( V = V_4(11) \), has precisely three orbits on \( V \setminus \{0\} \), of sizes 1200, 5040 and 8400.

(b) \( c \) is not 7.

**Proof.** (a) Write \( \Omega \) for the set of 1464 1-spaces in \( V \). Denoting elements in the conjugacy classes of \( A_7 \) by \( 1A, 2A \), etc. as in [7, p. 10], it is not hard to see that the numbers of fixed points on \( \Omega \) of elements of \( A_7 \), are as follows:
\[
\begin{array}{c|cccccccc}
g & 1A & 2A & 3A & 3B & 4A & 5A & 7A, B \\
|\text{fix}_G(g)| & 1464 & 0 & 0 & 12 & 0 & 4 & 1 \\
\end{array}
\]
(The only difficulty occurs with the classes \( 3A, 3B \); by [7, p. 10] these have traces \(-2, 1\) on \( V \), respectively, so their fixed point spaces have dimensions 0, 2.) Consequently,
\[
\text{no. of orbits of } A_7 \text{ on } \Omega = (1464 + 280 \cdot 12 + 504 \cdot 4 + 720)/2520 = 3.
\]
To determine the orbit sizes, observe first that a subgroup 7.3 of \( A_7 \) must fix a 1-space, so there is an orbit size dividing 2520/21 = 120. The stabiliser of a point in this orbit is either \( 7.3 \) or \( L_2(7) \), so the size is 120 or 15; it is not 15, since otherwise the remaining 1449 points would fall into just two \( A_7 \)-orbits, an arithmetical impossibility. Hence the orbit size is 120. The group \( A_7 \) has exactly two orbits on the remaining 1344 points, and these orbits must therefore have size 840 and 504. Thus the orbit sizes of \( G_0 \) on \( V \setminus \{0\} \) are 1200, 5040 and 8400, as in (a).

(b) Suppose \( c = 7 \), so that, by 3.4, \( (n, q) \) is \((6, 7), (4, 7)\) or \((4, 11)\). In the first case, \( G_0 = A_7 \) (not \( S_7 \)) by [14], so the inequality of 3.2 is \((7^7 - 7^6 - 1)/6 \), which is false. And in the second case, \( d = 2 \) by [11, 4.4], contrary to assumption.

Thus \( q = 11 \) and \( G_0 = F^+_1 \circ 2A_7 < GL_4(11) = GL(V) \). The orbit sizes of \( G_0 \) on \( V \) are given by part (a). Let \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) be the orbits of \( G_0 \) of sizes 1200, 5040 and 8400 respectively, and pick \( v_i \in \Delta_i \) (\( i = 1, 2, 3 \)); then \( (G_0)v_i \) is \( 7.3, 5 \) and \( 3 \), respectively. In particular, any 2-element of \( G_0 \) acts fixed-point-freely on \( V \setminus \{0\} \). The character of the representation of \( G_0 \) on \( V \) can be read off from [7, p. 7].
Let \((x, y)\) be a subgroup \(7.3\) of \(G_0\), with \(x\) of order 7 and \(y\) of order 3. Then \(C_g(y)\) is a 2-space, and \(N_{G_0}(y)\) has two non-trivial orbits on this 2-space, both of length 60. One of these orbits is contained in \(\Delta_1\), and the other in \(\Delta_3\). Hence there are two vectors in \(\Delta_1\), which differ by a vector in \(\Delta_3\), so we cannot have \(I_{\ell_1}(0) = \Delta_1\), \(I_{\ell_2}(0) = \Delta_2\).

It now follows from 2.1 that either

(i) \((k_1, k_2, k_3) = (1200, 8400, 5040)\), or
(ii) \((k_1, k_2, k_3) = (5040, 8400, 1200)\).

We now rule out these possibilities. Consider a subgroup \(H = 10 \cdot A_5\) of \(G_0\), where the \(A_5\) fixes two points in the natural action of \(A_7\) of degree 7. Let \(N = N_{G_0}(H) = 10 \cdot S_5\). The group \(H\) fixes two 2-spaces \(U, W\) with \(V = U \oplus W\), and elements of \(N \cap H\) interchange \(U \oplus W\) (this follows from [7, pp. 2, 10]). Moreover, \(H\) acts transitively on the non-zero vectors of \(V\) and also those of \(W\), and these vectors are all in \(\Delta_2\) (as they are fixed by elements of order 5). Since any vector in \(V\) is a sum \(u + w\) with \(u \in U\), \(w \in W\), we deduce that possibility (ii) above cannot hold, and hence (i) holds.

Let \(X = V \setminus (U \cup W)\). Then \(|X| = 14400\), and \(N = 10 \cdot S_5\) acts on \(X\) with 12 regular orbits, each of size 1200 (since the 3-elements of \(N\) act fixed-point-freely). Thus \(\Delta_1\) is one of these orbits. Since (i) holds, \(I_{\ell_2}(0) \subseteq X\), no two distinct vectors of \(\Delta_1\) can differ by a vector in \(U \cup W\), and so \(\Delta_1\) forms an empty subgraph (coclique) of the Hamming graph on \(V\) defined by joining distinct vectors \(u_1 + w_1\) and \(u_2 + w_2\) \((u_i \in U\), \(w_i \in W)\) iff \(u_1 = u_2\) or \(w_1 = w_2\). But the maximum number of vertices in any coclique of this Hamming graph is 120, which is a contradiction.

\[\text{Lemmma 3.10.} \] \(c\) is not 6.

\[\text{Proof.}\] Suppose \(c = 6\) so, by 3.4, \((n, q)\) is \((3, 4)\), \((3, 9)\) or \((4, 9)\) with \(q \in \{5, 7, 11, 13\}\).

When \((n, q) = (3, 4)\), it can be seen from [7, p. 231] that \(d = 2\). Now assume \((n, q) = (3, 9)\). Then \((G_0)' = \Omega_4(9)\), which has three orbits on the 1-spaces of \(V\), of sizes 10, 36 and 45. By 2.3, the group \(F^*_q\) of scalars is not contained in \(G_0\) (since none of the conclusions of 2.3 can hold when \((n, q) = (3, 9)\)). Thus \(G_0 \cap F^*_q < Z_4\). But then \(G_0 \cong Z_4 \cdot (\Omega_4(9) \cdot 2^2) < GO_4^*(3)\) (see [10, §4.3]), and 2.3 again applies to give a contradiction.

Hence \(n = 4, q = 5, 7, 11\) or 13 and \(G_0 \cong PS_{P^4}(q)\). 2 (see 3.4). Suppose \(q = 5\). By the groups \(A_7\) and \(S_5\) both have two orbits on 1-spaces, of sizes 120 and 240; the respective point stabilisers in \(A_7\) are \(7.3\) and \(3.2\), from which we deduce that \(A_6\) has 4 orbits, of sizes 40, 120, 120 and 120. Also \(S_6\) has 3 orbits (see [11, 4.4]), of sizes 40, 120 and 240.

Write \(I_{\ell_1} = \langle \langle u \rangle \mid u \in I_{\ell_1}(0) \rangle\). If \(G_0 = A_6\) then, as \(S_6\) fixes the orbit of size 40, we have \(|I_{\ell_1}| = 120\); but then \(k_1 = k_2\), contradicting 2.1(b). Hence \(G_0 = S_6\). Since \(S_7\) fixes the orbit of size 120, we have \(|I_{\ell_1}| = 40\).

The action of \(G_0\) on \(I_{\ell_1}\) has a block system \(\Sigma = \{B_1, \ldots, B_{10}\}\) of 10 blocks of size 4. The stabiliser in \(G_0\) of a point in \(B_i\) contains a group \(T_i \approx 3^3\), which is a Sylow 3-subgroup of \(PS_{P^4}(7)\), and fixes precisely four 1-spaces, say \(\langle v_{i1} \rangle, \ldots, \langle v_{i4} \rangle\); moreover, \(V = \langle v_{i1} \rangle \oplus \ldots \oplus \langle v_{i4} \rangle\). Thus \(B_i = \{\langle v_{i1} \rangle, \ldots, \langle v_{i4} \rangle\}\). The group \(T_i\) acts regularly on the 9 blocks in \(\Sigma \setminus \{B_i\}\). Consequently, any 1-space of the form \(\langle \lambda v_{ki} + \mu v_{li} \rangle (\lambda, \mu \neq 0, k \neq l)\) lies in \(I_{\ell_2}\). Write \(\langle w \rangle = \langle \lambda v_{ki} + \mu v_{li} \rangle\). Now \(\langle w \rangle\) is fixed.
by a subgroup of order 3 in $T_i$, which acts semiregularly on $\Sigma \setminus \{B_i\}$. So the number $x_*$ of triples $(m, n, j)$ with $m \neq n$, such that $\langle w \rangle = \langle \alpha v_{m_i} + \beta v_{n_j} \rangle$ for some $\alpha, \beta \neq 0$, is 1, 4 or 7. As $G_0$ is transitive on $I_2^*$, every 1-space in $I_2^*$ is of this form for some $m, n, j$, and the number $x = x_*$ is independent of $\langle w \rangle \in I_2^*$. Hence, counting pairs $(B, \langle w \rangle)$ with $\langle w \rangle = \langle \alpha v_{m_i} + \beta v_{n_j} \rangle$ for some $m \neq n$, $\alpha, \beta \neq 0$, we have

$$|I_2^*| x = 10 \times 36.$$  

As $x$ is 1, 4 or 7, this forces $|I_2^*| = 360$ or 90, a contradiction.

Next, consider the case in which $q = 11$. Arguing with fixed points as in 3.9, we see that $A_6$ has 7 orbits on 1-spaces here, while $S_6$ has only 4. Hence $\tilde{G}_0 = S_6$ by 2.2(b). The orbit sizes of $\tilde{G}_0$ on 1-spaces are 144, 240, 360 and 720. Again write $\tilde{I}_i = \{ \langle v \rangle \mid v \in I_i \}$. and let $\tilde{k}_i = |\tilde{I}_i|$. By 2.1(b), $\tilde{k}_1 = 144$ or 240. If $\tilde{k}_1 = 240$ then $\tilde{k}_4 = 144$ by 2.1(b), so $(\tilde{k}_2, \tilde{k}_3) = (720, 360)$ or (360, 720). In the first case, $b_2/c_3 = 2 < b_3/c_4 = 5/2$, and in the second, $b_1/c_2 = 3/2 < b_2/c_3 = 2$, both contradictions by 2.1(a). Thus $\tilde{k}_1 = 144$. If $\tilde{k}_2 = 720$ then $\tilde{k}_3 = 360$, $\tilde{k}_4 = 240$, so $b_2/c_3 = 1/2 < b_3/c_4 = 2/3$; if $\tilde{k}_2 = 240$ then $b_1/c_2 = 5/3 < 9/4 < 3b_2/2c_3$; and if $\tilde{k}_2 = 360$ then $\tilde{k}_3 = 720$ and $b_1/c_2 = 5/2 < 3 = 3b_2/2c_3$. Thus 2.1(d) gives a contradiction in all cases.

Finally, suppose $q = 13$. Here $V$ has 2380 1-spaces and $\tilde{G}_0 = S_6$ (not $A_6$, by 2.2(c)). As $d \leq 4$ by 2.2(b), $G_0$ must have 3 orbits of size 720 on the 1-spaces; but then $G_0$ cannot be transitive on the remaining 220 1-spaces.  

By Lemmas 3.1–3.10, we must have $c = 5$. Then Lemma 3.4 gives $(n, q) = (2, q)$ ($q < 119$), (3, 9), (4, 5), (4, 7) or (6, 3).

Suppose first that $n = 2$. As the representation of $G_0^{(n)}$ on $V$ is not realised over any proper subfield of $F_q$ (see part (iv) of Theorem 1.1 in the Introduction), we have $q = p$ if $p = \pm 1 \pmod{5}$ or $p = 5$, and $q = p^2$ otherwise. If $q = p$ then $d \leq 2$ by 2.2(b), contrary to assumption. So $q = p^2$ and $p = 2$ or 3 ($\pmod{5}$). As $q < 119$, it follows that $q = 4, 9$ or 49. When $q = 4$, $G_0$ is transitive on $V \setminus \{0\}$; and when $q = 9$, $\tilde{G}_0$ is transitive on the 10 1-spaces of $V$, so as $d > 2$, $G_0$ has 4 orbits of size 20 on $V \setminus \{0\}$, which is impossible by 2.1(b). Thus $q = 49$. By [11], $G_0 = A_5$ has 2 orbits on the 1-spaces of $V$, of sizes 20 and 30. Thus the group $3 \times 2A_5$ (which lies in $G_0$ by 2.2(a)) has 8 orbits of size $20 \times 6$ and 4 orbits of size $30 \times 12$ on $V \setminus \{0\}$. As $d \leq 4$ by 2.2(b), we deduce that $G_0 = 24 \times 2A_5$, with 2 orbits of size $20 \times 24$ and 2 of size $30 \times 24$ on $V \setminus \{0\}$. It follows from 2.1(b) that $k_1 = |I_1(0)| = 20 \times 24$. If $v \in I_1(0)$ then \{kv $|$ $\lambda$ non-zero square$\} \subseteq \langle k \rangle(0)$ and \{kv $|$ $\mu$ non-square$\}$ is contained in the other orbit of size $20 \times 24$. Pick a non-zero square $\lambda$ such that $1 + \lambda$ is non-square. Then $v + \lambda v \in I_2(0)$, so $k_1 = k_2 = 20 \times 24$. This conflicts with 2.1(b).

Now assume $(n, q) = (3, 9)$. Here $\tilde{G}_0 = A_5$ or $S_5 \leq S_6 = \Omega_5(9)$. By 2.3, the group of scalars $F_q^*$ is not contained in $G_0$. Hence $G_0 \leq Z_4 \cdot (\Omega_5(9). 2 \leq GO_5^+(3)$ and 2.3 again gives a contradiction.

Next consider $(n, q) = (4, 5)$. Here $\tilde{G}_0 < S_6 < PSp_4(5). 2$, and by [11, 4.3], $S_6$ has just 2 orbits on 1-spaces, of sizes 36 and 120. It follows easily that $\tilde{G}_0$ has at least 5 orbits, contrary to 2.2(b).

Now let $(n, q) = (4, 7)$. Here $\tilde{G}_0' = A_5 < A_7 < PSp_4(7)$. The calculation of [11, 4.4] gives that the number of $A_5$-orbits on 1-spaces is $(400 + 20 \cdot 10)/60 = 10$, whence $d = 5$, contrary to 2.2(b).

Finally, consider $(n, q) = (6, 3)$. Here $\tilde{G}_0 \leq S_5 < PSp_6(3). 2$. The fixed points of elements of $S_5$ on 1-spaces can be calculated using [7, p. 113]:

<table>
<thead>
<tr>
<th>Element of $S_5$</th>
<th>$1A$</th>
<th>$2A$</th>
<th>$2B$</th>
<th>$3A$</th>
<th>$4A$</th>
<th>$5A$</th>
<th>$6A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\text{fix}(g)</td>
<td>$</td>
<td>364</td>
<td>0</td>
<td>26</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
Hence the number of orbits of $S_5$ on 1-spaces is $(364 + 260 + 80 + 96 + 40)/120 = 7$, which once more conflicts with 2.2(b).

This final contradiction shows that $c \neq 5$, and completes the proof of the Theorem.

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