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Non-embeddable quasi-residual designs

Dedicated to N. G. de Bruijn on the occasion of his 60th birthday

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ABSTRACT

We present a new non-embeddable quasi-residual design which has the same parameters as Bhattacharya's design but which is much easier to describe. Furthermore we give the first example of a non-trivial non-embeddable design on less than 16 points.

1. INTRODUCTION

We assume that the reader is familiar with the definition and general theory of block designs (cf. [5]). However, we recall a few definitions and well-known facts. We shall use the notation $BD(v, k; b, r, \lambda)$ for a block design $\mathscr{D}$ consisting of b blocks of size k (not necessarily distinct) from a set $\{P_1, P_2, \ldots, P_v\}$ of v points. Here r is the number of blocks containing a given point P_i and λ is the number of blocks containing a given pair of points $\{P_i, P_j\}$. The parameters satisfy the relations

$$(1.1) \qquad bk = vr, \ \lambda(v-1) = r(k-1), \ b \geqslant v.$$

If $v = b$ (and hence $k = r$) the design is called symmetric. In this case we refer to the design as a (v, k, λ)-design. For such a design any two distinct blocks intersect in λ points. A *residual design* $\mathscr{D}'$ of a symmetric design $\mathscr{D}$ is obtained by removing one of the blocks of $\mathscr{D}$ and all the points of this block. One then obtains a $BD(v-k, k-\lambda; v-1, k, \lambda)$. A *derived design*

of $\mathscr{D}$ is obtained by removing one of the blocks of $\mathscr{D}$ and all the points not in this block. One obtains in this way a $BD(k, \lambda; v-1, k-1, \lambda-1)$.

A block design $\mathscr{D}'$ for which the parameters are such that $\mathscr{D}'$ could be the residual of some symmetric design $\mathscr{D}$ is called a *quasi-residual* design. If a corresponding symmetric design $\mathscr{D}$ indeed exists, then $\mathscr{D}'$ is called *embeddable* (i.e. $\mathscr{D}'$ is a residual design); otherwise $\mathscr{D}'$ is called *non-embeddable* (also: non-extensible). From (1.1) and the definition it follows that for a quasi-residual design $\mathscr{D}' = BD(v, k; b, r, \lambda)$

(1.2) $\quad r = k+\lambda,$

(1.3) $\quad v = k(k+\lambda-1)/\lambda, \; b = (k+\lambda)(k+\lambda-1)/\lambda.$

If $\mathscr{D}'$ is embeddable in $\mathscr{D}$, then $\mathscr{D}$ is a $(b+1, r, \lambda)$-design.

A quasi-residual design with $\lambda = 1$ is an affine plane which is known to be embeddable in a projective plane. It was shown by M. Hall and W. S. Connor (cf. [5], [6]) that a quasi-residual design with $\lambda = 2$ is embeddable. For $\lambda > 2$ the situation is not understood. It is easy to give trivial examples of non-embeddable quasi-residual designs. We mention one infinite class. Let $k \equiv 0$ or $1 \pmod 4$, k not a square. Let $\mathscr{D}'$ be the complete design of all k-subsets of a $(k+2)$-set. If $\mathscr{D}'$ were embeddable in a (V, K, Λ)-design $\mathscr{D}$ then $V = 1 + \binom{k+2}{2}$, $K = \binom{k+1}{2}$, $\Lambda = \binom{k}{2}$. Since V is even and $K - \Lambda = k$ is not a square, such a design $\mathscr{D}$ does not exist by the Bruck-Ryser-Chowla Theorem (cf. [5] Theorem 10.3.1). For other examples see [9], [11]. Recently it was shown by N. M. Singhi and S. S. Shrikhande (cf. [2], [10]) that for any fixed λ a quasi-residual $BD(v, k; b, r, \lambda)$ is embeddable for all sufficiently large k. In this paper we consider small values of k. The smallest known non-trivial non-embeddable quasi-residual design is a $BD(16, 6; 24, 9, 3)$ constructed by K. N. Bhattacharya [1] (caution: (16.1.19) in [5] contains an error). The fact that this design is non-embeddable is obvious since it has a pair of blocks which meet in four points. An example of a non-embeddable design with the same parameters but no pair of blocks meeting in four points was given by R. B. Brown [3] and other examples, again with the same parameters were found by J. F. Lawless [7]. In fact all known non-trivial examples of non-embeddable designs have the same parameters as Bhattacharya's design. Furthermore none of these examples is easy to describe. In section 2 we shall give an example of a $BD(16, 6; 24, 9, 3)$ which is very easy to describe and which is obviously non-embeddable. In section 3 we shall describe a new non-embeddable design with $v = 12$. If we consider only designs with $k \leqslant \frac{1}{2}v$ then this example is the smallest possible non-trivial non-embeddable design. If we allow $k > \frac{1}{2}v$ there is one smaller possibility, namely $v = 11$. We study this case in section 4, without offering a solution however. Since our main interest is in small designs we consider in section 5 all possible parameter sets with $v \leqslant 16$ for which a quasi-residual design with $\lambda > 2$ can exist and survey what is now known about these designs.

NOTATION: In the following a block design is usually described by its b by v (0, 1) incidence matrix. In fact we generally identify the design and its incidence matrix. We denote by $0_{n,m}$ respectively $J_{n,m}$ the n by m matrix with all entries equal to 0 resp. 1. If the dimensions are obvious we omit the indices.

2. A NON-EMBEDDABLE $BD(16, 6; 24, 9, 3)$

Let A be the 9 by 12 incidence matrix of $AG(2, 3)$, the affine plane of order 3. The columns of A represent the lines of the plane. Using A we construct a 6 by 12 matrix C in which the rows represent the six possible pairs of parallel classes of lines in $AG(2, 3)$. Finally the 9 by 4 matrix B is obtained by repeating each of the rows (1100), (1010), (1001) three times. We define

$$M := \begin{pmatrix} C & 0 \\ A & B \\ A & J-B \end{pmatrix}.$$

It is easily seen that M is the 24 by 16 incidence matrix of a $BD(16, 6; 24, 9, 3)$. Since the matrix A is repeated inside M and A has rowsums equal to 4, there are nine pairs of blocks in M which meet in 4 points. Therefore M is not embeddable in a (25, 9, 3)-design.

3. A NON-EMBEDDABLE $BD(12, 6; 22, 11, 5)$

In fig. 1 we possibly have part of the incidence matrix of a (23, 11, 5)-design $\mathscr{D}$. It is not difficult to check that the blocks B_1 to B_{22} on the points 1, 2, ..., 12 form a $BD(12, 6; 22, 11, 5)$ denoted by $\mathscr{D}'$. This design consists of two disjoint blocks B_1 and B_2, a $BD(6, 3; 20, 5, 4)$ formed by B_3 to B_{22} on the points 1, 2, ..., 6 (which is in fact the complete 3-design on these points), and finally a second design with these parameters on the points 7, 8, ..., 12. This second design is the union of two $BD(6, 3; 10, 5, 2)$. All these facts are easily checked by inspection of fig. 1.

The blocks B_3 and B_{11} of $\mathscr{D}'$ have five points in common (this is a unique pair). This implies that if $\mathscr{D}'$ is embeddable in a (23, 11, 5)-design $\mathscr{D}$ then the blocks B_1, B_2, B_3, B_{11} of $\mathscr{D}$ can be taken as in fig. 1 (w.l.o.g.). For any i we can now count the number of incidences of B_i and the pair B_3, B_{11}. This uniquely determines column 13 in fig. 1. Now, let column 14 be as in fig. 1, where each symbol denotes 0 or 1. We make the convention $x := x_1 + x_2 + x_3$, etc. We calculate the inner product of column 14 with columns 1, 2 and 3 and combine this with the fact that $\mathscr{D}$ has $r = 11$. We find

$$\begin{aligned} a+x+y+z+u+v+w &= 7 \\ y+z+u &= 3 \\ x+\quad z+\quad v &= 3 \\ x+y+\quad w &= 3 \end{aligned} \tag{3.1}$$

Blocks	Points 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
B_1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0
B_2	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
B_3	1	1	1	0	0	0	1	1	1	0	0	0	0	1	1	0	0	1	1	1	0	0	0
B_4	0	0	0	1	1	1	1	0	1	0	1	0	0	a									
B_5	0	1	1	1	0	0	0	0	1	1	1	0	1	x_1									
B_6	0	1	1	0	1	0	0	1	0	0	1	1	0	x_2									
B_7	0	1	1	0	0	1	0	0	1	1	0	1	0	x_3									
B_8	1	0	1	1	0	0	0	0	1	0	1	1	1	y_1									
B_9	1	0	1	0	1	0	1	0	0	1	1	0	0	y_2									
B_{10}	1	0	1	0	0	1	0	1	0	1	1	0	0	y_3									
B_{11}	1	1	0	1	0	0	1	1	1	0	0	0	0	0	0	1	1	0	0	0	1	1	1
B_{12}	1	1	0	0	1	0	1	0	0	0	1	1	1	z_2									
B_{13}	1	1	0	0	0	1	1	0	0	1	0	1	1	z_3									
B_{14}	1	0	0	0	1	1	0	1	1	1	0	0	1	u_1									
B_{15}	1	0	0	1	0	1	0	1	0	0	1	1	0	u_2									
B_{16}	1	0	0	1	1	0	0	0	1	1	0	1	0	u_3									
B_{17}	0	1	0	0	1	1	0	1	1	0	1	0	1	v_1									
B_{18}	0	1	0	1	0	1	1	0	0	1	1	0	0	v_2									
B_{19}	0	1	0	1	1	0	0	1	0	1	0	1	0	v_3									
B_{20}	0	0	1	0	1	1	1	0	1	0	0	1	0	w_1									
B_{21}	0	0	1	1	0	1	1	1	0	0	0	1	1	w_2									
B_{22}	0	0	1	1	1	0	1	1	0	1	0	0	1	w_3									
B_{23}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1

Fig. 1.

From (3.1) we obtain

$$(3.2)\qquad \begin{aligned} &x+y+z=a+2\\ &u=x-a+1,\ v=y-a+1,\ w=z-a+1 \end{aligned}$$

We do the same thing for columns 4 to 6. After substitution of (3.2) this yields

$$(3.3)\qquad (x_i+y_i+z_i)-(u_i+v_i+w_i)=a-1\quad (i=1,2,3).$$

The following equations are then found by calculating the inner product of column 14 with the columns 7, 9, 10, 12 and 13 (where in (3.6) we have used (3.4), and in all equations (3.2) and (3.3) have been substituted):

$$(3.4)\qquad y_2+v_2 = -2z+2$$

$$(3.5)\qquad 2(x_1+y_1)+x_3+u_3 = 2$$

$$(3.6)\qquad 2(x_3+y_3+z_3)+x_1+u_1 = 2z+a+1$$

$$(3.7) \qquad x_1+y_2+y_3+u_1+v_1+v_2+w_3=-a+3$$

$$(3.8) \qquad x_1+y_1+w_2+w_3 \qquad =1$$

$$(3.9) \qquad u_1+v_1 \qquad =-z+1.$$

Observe that the same equations hold for column 15. If both column 14 and column 15 had $z=0$ then by (3.4) both would have $y_2=v_2=1$ and then these columns would have inner product $\geqslant 6$. Hence for column 14 we may assume $z=1$. Then subtracting (3.4) and (3.9) from (3.7) yields

$$(3.10) \qquad x_1+y_3+w_3=3z-a=3-a.$$

Since $x_1+w_3\leqslant 1$ by (3.8) it follows that $a=1$ and furthermore $y_3=1$, $x_1+w_3=1$. Then (3.8) yields $y_1=w_2=0$. From (3.9) we find $u_1=v_1=0$, from (3.4) $y_2=v_2=0$. Then (3.6) implies $x_1=0$, i.e. $w_3=1$. By (3.5) we have $x_3=u_3=1$. Again using (3.6) we find $z_3=0$ (i.e. $z_2=1$) and then (3.3) implies $v_3=0$.

So $y=1$, $v=0$, contradicting (3.2). We have therefore proved that the design $\mathscr{D}'$ is non-embeddable.

4. QUASI-RESIDUAL DESIGNS $BD(11, 6; 22, 12, 6)$

In a search for non-embeddable designs with the parameters of the title it is possible to introduce a new idea which might make the problem easier, namely letting $\mathscr{D}'$ have a repeated block. Consider for example

$$\mathscr{D}'=\begin{pmatrix} J_{2,6} & 0_{2,6} \\ A & B \\ J-A & B \end{pmatrix}$$

where A is the incidence matrix of the unique $BD(6, 3; 10, 5, 2)$ and B is the incidence matrix of the complete design $BD(5, 3; 10, 6, 3)$. We can order the blocks of B in any way we like and thus obtain a $BD(11, 6; 22, 12, 6)$. This was done in a number of ways but we were always able to embed the design in a (23, 12, 6)-design. Some of the symmetric designs found in this way are probably new.

A second approach is to replace $J-A$ by some other $BD(6, 3; 10, 5, 2)$ and to reorder the rows of the two copies of B. This can indeed be done in such a way that one obtains a $BD(11, 6; 22, 12, 6)$. The examples which we constructed were too difficult to analyze without the aid of a computer. The ones which were all turned out to be embeddable. A simple way to construct several designs with the required parameters is to take the union of two (11, 6, 3)-designs.

We considered a number of non-isomorphic designs, all with a repeated block. Again, all our examples turned out to be embeddable. The search in this area is being continued.

5. SMALL QUASI-RESIDUAL DESIGNS

Using (1.2) and (1.3) we can find the parameters of all the quasi-residual designs with $v \leqslant 16$. Since we are interested in non-embeddable designs we make the restriction $\lambda > 2$. This leads to the following list.

No.	v	k	b	r	λ
1	4	3	12	9	6
2	5	3	10	6	3
3	6	4	15	10	6
4	7	4	14	8	4
5	7	5	21	15	10
6	8	4	14	7	3
7	9	5	18	10	5
8	10	5	18	9	4
9	11	6	22	12	6
10	12	6	22	11	5
11	13	7	26	14	7
12	14	7	26	13	6
13	16	6	24	9	3
14	16	8	30	15	7

At present the following is known about the designs with these parameters.

No. 1 to 3 : Trivial complete designs. They are all embeddable and in fact for 1 and 2 even the corresponding symmetric designs are unique.

No. 4 : The complement of $BD(7, 3; 14, 6, 2)$ which is the union of two projective planes of order 2. It is well-known that there are four such designs. All the non-isomorphic (15, 8, 4)-designs are known (cf. [4]). By inspection one finds that a $BD(7, 4; 14, 8, 4)$ is embeddable.

No. 5 : Again a trivial complete design but non-embeddable, namely the smallest k of the trivial infinite sequence described in section 1.

No. 6 : There are four non-isomorphic designs with these parameters and these are embeddable (cf. [4]).

No. 7 and 8: : In an earlier attempt to find a smaller non-embeddable quasi-residual design than Bhattacharya's example it turned out that the designs with these parameters are all embeddable (cf. [8]).

No. 9 : The smallest v for which a non-trivial non-embeddable design might exist. The situation is unresolved. See section 4.

No. 10 : In section 3 the smallest known non-embeddable design is described.

No. 11, 12, 14: Nothing seems to be known concerning these parameters.

No. 13 : These are the parameters of the Bhattacharya design and the simple example of section 2.

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REFERENCES

1. Bhattacharya, K. N. – A new balanced incomplete block design, Sci. Cult. 9, 508 (1944).
2. Bose, R. C., S. S. Shrikhande and N. M. Singhi – Edge regular multigraphs and partial geometric designs with an application to the embedding of quasi-residual designs, Teorie Combinatorie I, Acad. Naz. Lincei, 49–81 (1976).
3. Brown, R. B. – A new non-extensible (16, 24, 9, 6, 3)-design, J. Comb. Theory (A) 19, 115–116 (1975).
4. Gibbons, P. B. – Computing techniques for the construction and analysis of block designs, Thesis Univ. of Toronto, 1976.
5. Hall, Jr., M. – Combinatorial Theory, Blaisdell Publ. Co., Waltham, Mass. 1967.
6. Hall, M. and W. S. Connor – An embedding theorem for balanced incomplete block designs, Can. J. Math. 6, 35–41 (1953).
7. Lawless, J. F. – An investigation of Bhattacharya-type designs, J. Comb. Theory 11, 139–147 (1971).
8. Lint, J. H. van, H. C. A. van Tilborg and J. R. Wiekema – Block designs with $v=10$, $k=5$, $\lambda=4$, J. Comb. Theory (A) 23, 105–115 (1977).
9. Parker, E. T. – A result in balanced incomplete block designs, J. Comb. Theory 3, 283–285 (1967).
10. Singhi, N. M. and S. S. Shrikhande – Embedding of quasi-residual designs with $\lambda=3$, Utilitas Math. 4, 35–53 (1973).
11. Stanton, R. G. – Interconnections of related BIBDS, J. Comb. Theory 6, 387–391 (1969).