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Trigonometric bases for matrix weighted L_p-spaces

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A R T I C L E I N F O

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A B S T R A C T

We give a complete characterization of 2π-periodic matrix weights W for which the vector-valued trigonometric system forms a Schauder basis for the matrix weighted space $L_p(\mathbb{T}; W)$. Then trigonometric quasi-greedy bases for $L_p(\mathbb{T}; W)$ are considered. Quasi-greedy bases are systems for which the simple thresholding approximation algorithm converges in norm. It is proved that such a trigonometric basis can be quasi-greedy only for $p = 2$, and whenever the system forms a quasi-greedy basis, the basis must actually be a Riesz basis.

1. Introduction

A (periodic) matrix weight is a function $W : \mathbb{T} \to \mathbb{C}^{N\times N}$ taking values in the set of strictly positive definite Hermitian forms. For technical reasons, we shall always assume that both W and W^{-1} are integrable. The associated weighted space $L_p(\mathbb{T}; W)$, $1 \leqslant p < \infty$, is the set of measurable (vector-)functions $\mathbf{f} : \mathbb{T} \to \mathbb{C}^N$ satisfying

$$\|\mathbf{f}\|^p_{L_p(\mathbb{T};W)} := \int_{\mathbb{T}} \left|W^{1/p}\mathbf{f}\right|^p dx < \infty, \tag{1.1}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^N$. One can easily verify that $L_p(\mathbb{T}; W)$ is a Banach space for $1 < p < \infty$, and a Hilbert space for $p = 2$ with norm induced by the inner product

$$\langle \mathbf{f}, \mathbf{g}\rangle_{L_2(\mathbb{T};W)} := \int_{\mathbb{T}} \langle W\mathbf{f}(t), \mathbf{g}(t)\rangle_{\ell_2(\mathbb{C}^N)}\, dt. \tag{1.2}$$

In this paper we study certain stability properties of a vector valued trigonometric system in $L_p(\mathbb{T}; W)$ in terms of properties of the weight W. The trigonometric system is defined in a straightforward way. Let $\{\mathbf{e}_j\}_{j=1}^N$ denote the standard basis for $\mathbb{C}^N$. Then we simply define $\mathbf{e}^j_k(t) := e^{-2\pi ikt}\mathbf{e}_j$, $t \in \mathbb{R}$, and let

$$\mathscr{T} := \left\{\mathbf{e}^j_k \mid j = 1, 2, \dots, N,\ k \in \mathbb{Z}\right\}. \tag{1.3}$$

For the trivial constant weight $W := \mathrm{Id}$, it is easy to verify that $\mathscr{T}$ forms a (Schauder) basis for $L_p(\mathbb{T}; \mathrm{Id})$, $1 < p < \infty$, and an orthonormal basis for $L_2(\mathbb{T}; \mathrm{Id})$, which can be deduced from the scalar case. It is also well known from the scalar unimodular case that the trigonometric system cannot be unconditional in $L_p(\mathbb{T})$ except in the Hilbert space case $p = 2$.

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This follows from Orlicz' Theorem, see [22]. However, when we consider $\mathscr{T}$ in a space with a matrix weight W with e.g. unbounded spectrum on $\mathbb{T}$, it is not so obvious what happens. In the scalar case, the seminal paper by Hunt, Muckenhoupt, and Wheeden [9] demonstrates that the trigonometric system is stable in weighted L_p spaces precisely when the weight satisfies a so-called Muckenhoupt A_p condition. An application of the results in [9] to Schauder bases problems related to Gabor systems was recently studied by Heil and Powell [8] for $p = 2$.

The main contribution of the present paper is to give a complete characterization of matrix weights W for which $\mathscr{T}$ forms a Schauder basis for $L_p(\mathbb{T}; W)$. The Schauder basis property turns out to be equivalent to W satisfying a so-called Muckenhoupt A_p matrix condition. Furthermore, we prove that $\mathscr{T}$ can only be a quasi-greedy basis for $L_p(\mathbb{T}; W)$ when $p = 2$. Quasi-greedy bases are systems for which the simple thresholding approximation algorithm converges in norm. Moreover, we also show that whenever $\mathscr{T}$ is quasi-greedy in $L_2(\mathbb{T}; W)$, then the basis must actually be a Riesz basis. In the reduced scalar case, the quasi-greedy property for the univariate trigonometric system was studied by the present author in [14].

To motivate the study of the vector valued system $\mathscr{T}$, let us mention one important application where problems on stability of $\mathscr{T}$ in $L_p(\mathbb{T}; W)$ arise naturally. For a finitely set of functions $\{f_k\}_{k=1}^N$ in $L_2(\mathbb{R})$, the associated shift invariant space S is given by

$$S := \overline{\operatorname{Span}\{f_k(\cdot - l) \mid l \in \mathbb{Z},\ k = 1, 2, \ldots, N\}} \subset L_2(\mathbb{R}).$$

A natural question to pose is whether $B := \operatorname{Span}\{f_k(\cdot - l) \mid l \in \mathbb{Z},\ k = 1, 2, \ldots, N\}$ forms a basis for S. Here basis can mean merely a Schauder basis, or a stronger condition such as unconditional or Riesz bases. The Schauder basis case was settled by Nielsen and Šikić [16] in the single generator case (i.e., $N = 1$). The finitely generated shift invariant case was settle by the present author in [13], where it is proved that the system B forms a Schauder basis for S precisely when $\mathscr{T}$ forms a basis for $L_2(\mathbb{T}; G)$, with G, the $N \times N$ Gram matrix, given by

$$G_{i,j} := \sum_{k\in\mathbb{Z}} \widehat{f}_i(\cdot - k)\overline{\widehat{f}_j(\cdot - k)},$$

where the Fourier transform is given by $\hat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}\,dt$. Moreover, $\mathscr{T}$ forms a basis for $L_2(\mathbb{T}; G)$ precisely when G is a Muckenhoupt A_2 matrix weight, a notion that will be discussed in details below.

The structure of the paper is as follows. In Section 2, we impose an ordering on $\mathscr{T}$ and define associated partial sum operators. Then we study boundedness properties of these operators on $L_p(\mathbb{T}; W)$, and the main result of Section 2 shows that $\mathscr{T}$ forms a Schauder basis for $L_p(\mathbb{T}; W)$ precisely when the weight W satisfies a Muckenhoupt A_p matrix condition. In Section 3 we study conditions on the weight W that will make $\mathscr{T}$ a so-called quasi-greedy basis for $L_p(\mathbb{T}; W)$. Quasi-greedy bases are Schauder bases for which approximations obtained by thresholding converge in norm. It is proved in Section 3 that $\mathscr{T}$ can be quasi-greedy in $L_p(\mathbb{T}; W)$ only when $p = 2$, and in the affirmative case, the system is actually a Riesz basis. Section 4 concludes the paper with a selection of examples.

2. Trigonometric Schauder bases for $L_p(\mathbb{T}; W)$

In this section we characterize Schauder basis properties of $\mathscr{T}$ in $L_p(\mathbb{T}; W)$ in terms of properties of the matrix weight $W : \mathbb{T} \to \mathbb{C}^{N\times N}$. The main result is that $\mathscr{T}$ is a Schauder basis for $L_p(\mathbb{T}; W)$ precisely when W is a so-called Muckenhoupt A_p matrix weight. We thus obtain Schauder bases for a large class of weighted spaces in the vector valued setting. In the scalar case, the paper by Hunt, Muckenhoupt, and Wheeden [9] contains the first proof that the trigonometric system is stable in weighted L_p spaces precisely when the weight satisfies a suitable Muckenhoupt condition. The connection to Schauder bases was made precise by Heil and Powell [8] in the case of $p = 2$. The connection between Schauder bases for shift invariant spaces and the trigonometric system was established by Nielsen and Šikić [16].

Let us begin by making some observations about the system $\mathscr{T}$ in $L_p(\mathbb{T}; W)$. For $p = 2$, we notice that whenever $W, W^{-1} \in L_1$, then $(W^{-1}\mathbf{e}_k^j, \mathbf{e}_k^j)_{k,j}$ is a bi-orthogonal system in $L_2(\mathbb{T}; W)$ in the sense that

$$\langle W^{-1}\mathbf{e}_k^j, \mathbf{e}_{k'}^{j'}\rangle_{L_2(\mathbb{T};W)} := \int_{\mathbb{T}} \langle WW^{-1}\mathbf{e}_k^j, \mathbf{e}_{k'}^{j'}\rangle_{\ell_2(\mathbb{C}^N)}\,dt = \delta_{k,k'}\delta_{j,j'}.$$

This fact insures that we can define partial sum operators associated with a fixed ordering of $\mathscr{T}$ which we will describe now. First we choose the ordering $\rho : \mathbb{N} \to \mathbb{Z}$ of $\mathbb{Z}$ given by

$$\{0, -1, 1, -2, 2, \ldots\}.$$

Then we induce an ordering $\eta : \mathbb{N} \to \{1, 2, \ldots, N\} \times \mathbb{Z}$ as follows. For $m \in \mathbb{N}$ we write $m - 1 = kN + r$, $0 \leqslant r < N$, and define $\eta(m) := (r + 1, \rho(k + 1))$. Moreover, we let $\mathbf{e}(\eta(m)) := \mathbf{e}_{r+1}^{\rho(k+1)}$. With the ordering in place, we can consider the partial sum operators defined by

$$S_n(\mathbf{f}) := \sum_{s=1}^{n} \langle \mathbf{f}, W^{-1}\mathbf{e}(\eta(s))\rangle_{L_2(\mathbb{T};W)}\mathbf{e}(\eta(s)). \tag{2.1}$$

With the partial sum operators defined by (2.1), we can obviously study their boundedness properties in $L_2(\mathbb{T};W)$. Moreover, we claim that (2.1) can be made to make sense for $f \in L_p(\mathbb{T};W)$, $1<p<\infty$, with one additional assumption. The standing assumption $W \in L_1$ ensures that $\mathbf{e}_k^j \in L_p(\mathbb{T};W)$ for $1<p<\infty$, so we focus on the dual element $W^{-1}\mathbf{e}_k^j$. We recall that the dual space to $L_p(\mathbb{T};W)$, induced by the inner product given by (1.2), is $L_{p'}(\mathbb{T};W)$, with $1/p+1/p'=1$, see [6]. We have,

$$\|W^{-1}\mathbf{e}_k^j\|_{L_{p'}(\mathbb{T};W)} = \left(\int_{\mathbb{T}} \left|W^{1/p'-1}(t)\mathbf{e}_k^j(t)\right|^{p'}dt\right)^{1/p'} = \left(\int_{\mathbb{T}} \left|W^{-1/p}(t)\mathbf{e}_k^j(t)\right|^{p'}dt\right)^{1/p'}.$$

Hence, if we make the assumptions that $W, W^{-p'/p} \in L_1$, then (2.1) is well defined for $\mathbf{f} \in L_p(\mathbb{T};W)$. Notice that this assumption is compatible with the definition of the A_p class below, see Definition 2.1.

It turns out that boundedness of (2.1) in $L_p(\mathbb{T};W)$ is closely related to a study of certain singular integral operators on $L_p(\mathbb{T};W)$ related to the vector valued Hilbert transform. The vector-valued Hilbert transform was studied in the seminal paper by Treil and Volberg [20], and this was later generalized to other types of singular integral operators by Goldberg [6]. Treil and Volberg showed that the correct condition to impose on W to obtain boundedness is the so-called Muckenhoupt A_p condition.

The Muckenhoupt A_p-condition for matrix weights was introduced by Nazarov, Treĭl' and Volberg in [12,21] to study boundedness properties of the vector-valued Hilbert transform. Here we follow Roudenko [17] and give an equivalent and more direct definition of matrix A_p weights. It is proved in [17] that the following definition is equivalent to the A_p condition considered in [12,21]. We let $\mathscr{I}$ denote the family of all open intervals on $\mathbb{R}$.

Definition 2.1. Let $1<p<\infty$, and let p' denote the conjugate exponent to p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Let $W:\mathbb{T}\to\mathbb{C}^{N\times N}$ be a matrix weight. We say that W belongs to the matrix Muckenhoupt class A_p provided $W, W^{-p'/p}$ are integrable, and

$$A(p,W) := \sup_{I\in\mathscr{I}} \int_I \left(\int_I \left\|W^{1/p}(x)W^{-1/p}(t)\right\|^{p'}\frac{dt}{|I|}\right)^{p/p'}\frac{dx}{|I|} < \infty. \tag{2.2}$$

Remark 2.2. One can verify that $W \in A_p$ if and only if $W^{-p'/p} \in A_{p'}$, see [17].

We can now state the main result of this section characterizing boundedness of the partial sum operators $\{S_m\}_m$ given by (2.1) in terms of an A_p condition on the matrix weight. A variation of Proposition 2.3 valid in the simpler scalar valued case can be found in [16].

Proposition 2.3. *Let $1<p<\infty$, and let $W:\mathbb{T}\to\mathbb{C}^{N\times N}$ be a matrix weight with $W, W^{-\min(1,p'/p)} \in L_1$, where p' is the dual exponent to p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Then the partial sum operators $\{S_m\}_m$ given by (2.1) are uniformly bounded on $L_p(\mathbb{T};W)$ if and only if $W\in A_p$. Equivalently, the system $\mathscr{T}$ given by (1.3) is a Schauder basis for $L_p(\mathbb{T};W)$ if and only if $W\in A_p$.*

Before we can give the proof of Proposition 2.3, we need to state a few auxiliary results. The Hilbert transform H is defined on $L_2(\mathbb{T})$ by

$$H(f)(x) := \text{p.v.}\int_{\mathbb{T}} f(t)\cot\bigl(\pi(x-t)\bigr)\,dt.$$

We lift H to a linear operator on $L_2(\mathbb{T};W)$ for any $N\times N$ matrix weight W on $\mathbb{T}$ by letting it act coordinate-wise, i.e.,

$$(H\mathbf{f})_i := H(f_i), \quad i=1,\dots,N,\ f\in L_2(\mathbb{T};W).$$

The fundamental result by Treil and Volberg [20], see also [19], is the following.

Theorem 2.4. *(See [20].) Let $W:\mathbb{T}\to\mathbb{C}^{N\times N}$ be a matrix weight, and let $1<p<\infty$. Then the Hilbert transform extends to a bounded operator on $L_p(\mathbb{T};W)$ if and only if $W\in A_p$.*

We recall that the univariate Dirichlet kernel D_K is given by

$$D_K(t) = \frac{\sin 2\pi(K+1/2)t}{\sin \pi t}, \quad K \geqslant 1, \tag{2.3}$$

and for $f \in L_p(\mathbb{T})$, we have

$$f * D_K := \int_{\mathbb{T}} f(t) D_K(\cdot - t)\, dt = \sum_{k=-K}^{K} \hat{f}(k) e^{2\pi i k \cdot}.$$

We lift $f * D_K$ to an operator on $L_p(\mathbb{T}; W)$ by letting it act coordinate-wise.

We have the following immediate corollary to Theorem 2.4.

Corollary 2.5. *Let $1 < p < \infty$, and let $W : \mathbb{T} \to \mathbb{C}^{N\times N}$ be a matrix weight satisfying $W, W^{-\min(1,p'/p)} \in L_1$, with p' the dual exponent to p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Then the partial sum operators $\mathbf{f} \to \mathbf{f} * D_K$ are uniformly bounded on $L_p(\mathbb{T}; W)$ if and only if $W \in A_p$.*

Proof. Suppose $W \in A_p$. We let P_+ denote the Riesz projection onto H^2 for $f \in L_2(\mathbb{T}; W)$. Recall that $P_+ = \frac{1}{2}(I + iH + S_0)$, where S_0 is the orthogonal projection onto constant vectors. It follows that P_+ is bounded on $L_p(\mathbb{T}; W)$, since H is bounded on $L_p(\mathbb{T}; W)$ according to Theorem 2.4, and S_0 is bounded on $L_p(\mathbb{T}; W)$ according to [6, Proposition 2.1]. Notice that $\mathbf{f} \to \mathbf{f} e^{2\pi i M \cdot}$ is a unitary mapping on $L_p(\mathbb{T}; W)$, just as in the scalar case. Then we observe that

$$\mathbf{f} * D_K = e^{-2\pi i K \cdot} P_+\big(e^{2\pi i K \cdot} \mathbf{f}\big) - e^{2\pi i (K+1) \cdot} P_+\big(e^{-2\pi i (K+1) \cdot} \mathbf{f}\big),$$

and the uniform boundedness of $\mathbf{f} \to \mathbf{f} * D_K$ follows.

Now, suppose $\mathbf{f} \to \mathbf{f} * D_K$ are uniformly bounded on $L_p(\mathbb{T}; W)$. Then we notice that for $\mathbf{f}$ a vector of trigonometric polynomials,

$$P_+(\mathbf{f}) = e^{2\pi i K \cdot}\big[\big(e^{-2\pi i K \cdot} \mathbf{f}\big) * D_K\big],$$

for K large. It follows that P_+ extends to a bounded operator on $L_p(\mathbb{T}; W)$, and consequently, the Hilbert transform H is bounded on $L_p(\mathbb{T}; W)$ and $W \in A_p$ by Theorem 2.4. □

With Corollary 2.5 at our disposal, we can now prove Proposition 2.3.

Proof of Proposition 2.3. First, suppose the partial sum operators $\{S_m\}_m$ given by (2.1) are uniformly bounded on $L_p(\mathbb{T}; W)$. We notice that $S_{(2L+1)N}(\mathbf{f}) = \mathbf{f} * D_L$, so $\mathbf{f} \to \mathbf{f} * D_K$ are uniformly bounded on $L_p(\mathbb{T}; W)$ and $W \in A_p$ by Corollary 2.5.

Now, suppose $W \in A_p$. Given $m \in \mathbb{N}$, we choose K sufficiently large such that

$$S_m(\mathbf{f}) = \mathbf{f} * D_K + R(\mathbf{f}),$$

where $R(\mathbf{f})$ contains at most $2N - 1$ terms of the type

$$\big\langle \mathbf{f}, W^{-1} \mathbf{e}_k^j \big\rangle_{L_2(\mathbb{T}; W)} \mathbf{e}_j^k.$$

However, notice that[1]

$$\begin{aligned}
\big\| \big\langle \mathbf{f}, W^{-1} \mathbf{e}_k^j \big\rangle_{L_2(\mathbb{T}; W)} \mathbf{e}_j^k \big\|_{L_p(\mathbb{T}; W)} &\leqslant \big| \big\langle \mathbf{f}, W^{-1} \mathbf{e}_k^j \big\rangle_{L_2(\mathbb{T}; W)} \big| \big\| \mathbf{e}_j^k \big\|_{L_p(\mathbb{T}; W)} \\
&\leqslant \|\mathbf{f}\|_{L_p(\mathbb{T}; W)} \cdot \big\| W^{-1} \mathbf{e}_k^j \big\|_{L_q(\mathbb{R}; W)} \cdot \big\| \mathbf{e}_j^k \big\|_{L_p(\mathbb{T}; W)} \\
&\lesssim \|\mathbf{f}\|_{L_p(\mathbb{T}; W)}.
\end{aligned}$$

This, together with the uniform boundedness of $\mathbf{f} \to \mathbf{f} * D_K$ from Corollary 2.5, shows that the partial sum operators $\{S_m\}_m$ given by (2.1) are uniformly bounded on $L_p(\mathbb{T}; W)$.

Finally, we notice that uniform boundedness of $\{S_m\}_m$ on $L_p(\mathbb{T}; W)$ is equivalent to $\mathscr{T}$ being a Schauder basis for $L_p(\mathbb{T}; W)$ provided that $\mathscr{T}$ is complete in $L_p(\mathbb{T}; W)$, see [11]. Suppose $\mathscr{T}$ is not complete. Then the Hahn–Banach theorem provides a non-zero $\mathbf{g} \in L_q(\mathbb{T}; W)$, $1/p + 1/q = 1$, such that for all j, k,

$$\int_{\mathbb{T}} \big\langle W \mathbf{g}(t), \mathbf{e}_j^k(t) \big\rangle_{\ell_2(\mathbb{C}^N)}\, dt = 0.$$

However, one checks that $\mathbf{g} = W^{-1/q} \mathbf{f}$ for some $\mathbf{f} \in L_q(\mathbb{T}; \mathrm{Id})$, so $W\mathbf{g} = W^{1/p} \mathbf{f} \in L_p(\mathbb{T}; \mathrm{Id})$. Hence, each entry of the vector $W^{1/p}\mathbf{f}$ is in $L_p(\mathbb{T})$, and the completeness of the trigonometric system in $L_p(\mathbb{T})$ implies that $W^{1/p}\mathbf{f} = 0$ a.e. This implies $\mathbf{g} = 0$, which is a contradiction, and consequently $\mathscr{T}$ is complete in $L_p(\mathbb{T}; W)$. This completes the proof. □

[1] The notation $A \lesssim B$ means that there exists a constant c (independent of any significant parameters) such that $A \leqslant cB$. We use $A \asymp B$ whenever $A \lesssim B$ and $B \lesssim A$.

3. Beyond Schauder bases: Quasi-greedy bases

Proposition 2.3 tells us that $\mathscr{T}$ defined by (1.3) is a Schauder basis for $L_p(\mathbb{T}; W)$ if and only if $W \in A_p$. In this section, we explore what happens when we impose slightly stronger conditions on $\mathscr{T}$. The condition we have in mind is quasi-greediness. A quasi-greedy basis is a quasi-normalized Schauder basis for which the thresholding approximation algorithm converges, see Definition 3.1 below.

The main result of this paper stated as Theorem 3.8 below is that $\mathscr{T}$ can be quasi-greedy in $L_p(\mathbb{T}; W)$ only for $p = 2$, and we completely characterize the weights $W \in A_2(\mathbb{T})$ for which $\mathscr{T}$ is quasi-greedy in $L_2(\mathbb{T}; W)$. It turns out that $\mathscr{T}$ is quasi-greedy in $L_2(\mathbb{T}; W)$ precisely when the spectrum of W is bounded and bounded away from zero on $\mathbb{T}$ so, in particular, a quasi-greedy system $\mathscr{T}$ in $L_2(\mathbb{T}; W)$ is always a Riesz-basis.

We begin by introducing some notation and recalling some necessary results on quasi-greedy bases. First we give the precise definition of a quasi-greedy system in a Banach space X. A bi-orthogonal system is a family $(x_n, x_n^*)_{n\in\mathbb{N}} \subset X \times X^*$ such that $x_n^*(x_n) = \delta_{n,m}$. We fix a bi-orthogonal system $(x_n, x_n^*)_{n\in\mathbb{N}}$ with $\operatorname{span}_n(x_n)$ dense in X. We assume that the system is quasi-normalized, i.e., $\inf_n \|x_n\|_X > 0$ and $\sup_n \|x_n^*\|_{X^*} < \infty$. For each $x \in X$ and $m \in \mathbb{N}$, we define

$$\mathscr{G}_m(x) := \sum_{n\in A} x_n^*(x) x_n,$$

where A is a set of cardinality m satisfying $|x_n^*(x)| \geqslant |x_k^*(x)|$ whenever $n \in A$ and $k \notin A$. Whenever A is not uniquely defined, we arbitrarily pick any such set. The definition of $\mathscr{G}_m$ leads directly to the definition of a quasi-greedy system, see [10].

Definition 3.1. A quasi-normalized bi-orthogonal system $(x_n, x_n^*)_{n\in\mathbb{N}} \subset X \times X^*$, with $\operatorname{span}_n(x_n)$ dense in X, is called a quasi-greedy system provided

$$\big\|\mathscr{G}_m(x)\big\|_X \lesssim \|x\|_X, \quad \forall x \in X. \tag{3.1}$$

If the system is also a Schauder basis for X, we will use the term quasi-greedy basis.

Remark 3.2. It is straightforward to check that an unconditional basis is also quasi-greedy, but the converse statement is false. There exist conditional quasi-greedy bases, see [10].

Remark 3.3. In was proved by Wojtaszczyk in [23] that a system is quasi-greedy if and only if for each $x \in X$, the sequence $\mathscr{G}_m(x)$ converges to x in norm. In particular, if a Schauder basis is not quasi-greedy, then there exists $x_0 \in X$ such that the $\mathscr{G}_m(x_0)$ fails to converge to x_0. Put another way, approximations obtained by decreasing rearrangements are not norm convergent.

Let us explain our strategy to obtain information about the weight W. We are going to probe the weight with very special vector functions induced by the Dirichlet kernel and its translates. The Dirichlet kernel has two important properties that we will use extensively: It has unimodular coefficients relative to the trigonometric system, and its powers (properly normalized) form approximations to the identity.

The first result we state is due to Wojtaszczyk [23], see also [5]. It shows that quasi-greedy bases are unconditional for constant coefficients. Below in Lemma 3.7 we will use this fact to estimate the norm of vector functions induced by, e.g., the Dirichlet kernel.

Lemma 3.4. *Suppose $\{b_k\}_{k\in\mathbb{N}}$ is a quasi-greedy basis in a Banach space X. Then there exist constants $0 < c_1 \leqslant c_2 < \infty$ such that for every choice of signs $\varepsilon_k = \pm 1$ and any finite subset $A \subset \mathbb{N}$, we have*

$$c_1 \Big\|\sum_{k\in A} b_k\Big\|_X \leqslant \Big\|\sum_{k\in A} \varepsilon_k b_k\Big\|_X \leqslant c_2 \Big\|\sum_{k\in A} b_k\Big\|_X. \tag{3.2}$$

For our purpose, Lemma 3.4 is not quite enough. When we consider translates of the Dirichlet kernel, we need to be able to handle arbitrary unimodular complex coefficients and not only ± 1 as covered by Lemma 3.4. For that purpose, we need to use some facts about the Banach space geometry of $L_p(\mathbb{T}; W)$. For the definition of type and cotype of a Banach space we refer to, e.g., [22, Chap. III]. We also recall that the scalar valued $L_p(\mathbb{T})$ has type 2 for $2 \leqslant p < \infty$ and cotype 2 for $1 < p \leqslant 2$, see [22, Chap. III]. These properties are inherited by $L_p(\mathbb{T}; W)$ as the following lemma shows.

Lemma 3.5. *Let $W : \mathbb{T} \to \mathbb{C}^{N\times N}$ be an integrable matrix weight. The Banach space $L_p(\mathbb{T}; W)$ has type 2 for $2 \leqslant p < \infty$, and cotype 2 for $1 < p \leqslant 2$.*

Proof. We focus on the type claim, so we assume that $2 \leqslant p < \infty$. Let $r_1, r_2, \ldots$ be the Rademacher functions on $[0, 1)$ defined by $r_k(t) = \operatorname{sign}(\sin(2^k \pi t))$, and let $\{\mathbf{f}_k\}_{k=1}^{\infty}$ be an arbitrary sequence of functions from $L_p(\mathbb{T}; W)$. We use $[\mathbf{f}]_i$ to denote the ith entry of the vector $\mathbf{f}$. Then by repeated use of the fact that any two norms on $\mathbb{R}^N$ are equivalent, and the fact that $L_p(\mathbb{T})$ has type 2, we obtain for any $K \in \mathbb{N}$,

$$\begin{aligned}
\int_0^1 \left\| \sum_{k=1}^K r_k(t)\mathbf{f}_k \right\|_{L_p(\mathbb{T};W)} dt &= \int_0^1 \left(\int_{\mathbb{T}} \left| \sum_{k=1}^K r_k(t) W^{1/p}(x)\mathbf{f}_k(x) \right|^p dx \right)^{1/p} dt \\
&\asymp \int_0^1 \left(\sum_{i=1}^N \int_{\mathbb{T}} \left| \sum_{k=1}^K r_k(t) \left[W^{1/p}(x)\mathbf{f}_k(x) \right]_i \right|^p dx \right)^{1/p} dt \\
&\asymp \sum_{i=1}^N \int_0^1 \left(\int_{\mathbb{T}} \left| \sum_{k=1}^K r_k(t) \left[W^{1/p}(x)\mathbf{f}_k(x) \right]_i \right|^p dx \right)^{1/p} dt \\
&\lesssim \sum_{i=1}^N \left[\sum_{k=1}^K \left(\int_{\mathbb{T}} \left| \left[W^{1/p}(x)\mathbf{f}_k(x) \right]_i \right|^p dx \right)^{2/p} \right]^{1/2} \\
&\asymp \left[\sum_{i=1}^N \sum_{k=1}^K \left(\int_{\mathbb{T}} \left| \left[W^{1/p}(x)\mathbf{f}_k(x) \right]_i \right|^p dx \right)^{2/p} \right]^{1/2} \\
&\asymp \left[\sum_{k=1}^K \left(\sum_{i=1}^N \int_{\mathbb{T}} \left| \left[W^{1/p}(x)\mathbf{f}_k(x) \right]_i \right|^p dx \right)^{2/p} \right]^{1/2} \\
&\asymp \left(\sum_{k=1}^K \|\mathbf{f}_k\|_{L_p(\mathbb{T};W)}^2 \right)^{1/2}.
\end{aligned}$$

The claim about cotype follows from a slight modification of the above estimate. We leave the details for the reader. □

The next lemma gives a simple embedding result for the weighted L_p-spaces.

Lemma 3.6. *Suppose $W \in L_1$ is a matrix weight. Then for $1 \leqslant p < q < \infty$ we have the continuous embedding $L_q(\mathbb{T}; W) \hookrightarrow L_p(\mathbb{T}; W)$.*

Proof. First notice that for $0 < \alpha < 1$, $\|W(t)^\alpha\|^{1/\alpha} = \|W(t)\|$, so $\int_{\mathbb{T}} \|W(t)^\alpha\|^{1/\alpha}\, dt < \infty$. Now suppose $1 \leqslant p < q < \infty$ and $\mathbf{f} \in L_p(\mathbb{T}; W)$. Then, using Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{T}} \left| W^{1/p}(t)\mathbf{f}(t) \right|^p dt &= \int_{\mathbb{T}} \left| W^{1/p-1/q} W^{1/q}(t)\mathbf{f}(t) \right|^p dt \leqslant \int_{\mathbb{T}} \left\| W^{1/p-1/q}(t) \right\|^p \cdot \left| W^{1/q}(t)\mathbf{f}(t) \right|^p dt \\
&\leqslant \left(\int_{\mathbb{T}} \left\| W^{1/p-1/q}(t) \right\|^{qp/(q-p)} dt \right)^{(q-p)/q} \left(\int_{\mathbb{T}} \left| W^{1/q}(t)\mathbf{f}(t) \right|^q dt \right)^{p/q} \\
&\lesssim \left(\int_{\mathbb{T}} \left| W^{1/q}(t)\mathbf{f}(t) \right|^q dt \right)^{p/q}.
\end{aligned}$$

This proves the claim. □

We now combine the previous lemmata to prove the following result, which can be use to estimate the $L_p(\mathbb{T}; W)$-norm of expansions with unimodular coefficients relative to $\mathscr{T}$ provided $\mathscr{T}$ is quasi-greedy.

Proposition 3.7. *Let $W \in L_1$ be a matrix weight. Suppose that $\mathscr{T} := \{\mathbf{e}_k^j\}$ defined by (1.3) is a quasi-greedy system in $L_p(\mathbb{T}; W)$ for some $1 < p < \infty$. Then there exist constants $0 < c_1 \leqslant c_2 < \infty$ such that for every finite unimodular sequence $\{\alpha_k^j\}_{(j,k)\in F} \subset \mathbb{C}$, $F \subset \{1, \ldots, N\} \times \mathbb{Z}$, and every vector $\mathbf{v} \in \mathbb{C}^N$, we have*

$$c_1 \Bigl(\max_j |v_j| \Bigr) L^{1/2} \leqslant \left\| \sum_{(j,k)\in F} v_j \alpha_k^j \mathbf{e}_k^j \right\|_{L_p(\mathbb{T};W)} \leqslant c_2 \Bigl(\max_j |v_j| \Bigr) (\#F)^{1/2}, \tag{3.3}$$

where $L := \min_j \#\{k \in \mathbb{Z}\colon \alpha_k^j \neq 0\}$.

Proof. We begin by proving the upper estimate. An easy application of the triangle inequality yields

$$\Big\| \sum_{(j,k)\in F} v_j \alpha_k^j \mathbf{e}_k^j \Big\|_{L_p(\mathbb{T};W)} \leqslant \Big(\max_j |v_j| \Big) \sum_j \Big\| \sum_{k:\ (j,k)\in F} \alpha_k^j \mathbf{e}_k^j \Big\|_{L_p(\mathbb{T};W)}. \tag{3.4}$$

For technical reasons we define a new scalar sequence $\{\beta_k^j\}$ by

$$\beta_k^j = \begin{cases} \alpha_k^j, & (j,k) \in F, \\ 1, & (j,k) \in (\{1,\dots,N\}\times\mathbb{Z})\backslash F. \end{cases}$$

Now observe that $\{\beta_k^j \mathbf{e}_k^j\}_{j\in\{1,\dots,N\},k\in\mathbb{Z}}$ is also a quasi-greedy system in $L_p(\mathbb{T};W)$. In fact, the greedy approximation operator $\tilde{\mathscr{G}}_m$ for $\{\beta_k^j \mathbf{e}_k^j\}_{j\in\{1,\dots,N\},k\in\mathbb{Z}}$ is identical to the approximation operator $\mathscr{G}_m$ for $\{\mathbf{e}_k^j\}_{j\in\{1,\dots,N\},k\in\mathbb{Z}}$. This follows from the trivial observation that if $\mathbf{f}_k^\psi$ is the dual element to $\mathbf{e}_k^j$, then $\beta_k^j \mathbf{f}_k^j$ is the dual element to $\beta_k^j \mathbf{e}_k^j$, since $|\beta_k^\psi| = 1$.

According to Lemma 3.5, $L_p(\mathbb{T};W)$ has Rademacher cotype 2 for $1 < p \leqslant 2$ and type 2 for $2 \leqslant p < \infty$. Let us first suppose $2 \leqslant p < \infty$. Then, for any sequence $\{\mathbf{f}_\ell\}_{\ell\in\mathbb{N}} \subset L_p(\mathbb{T};W)$, we have the uniform estimate

$$\int_0^1 \Big\| \sum_{\ell=1}^n r_\ell(t) \mathbf{f}_\ell \Big\|_{L_p(\mathbb{T};W)} dt = \mathrm{Avg}_{\varepsilon_\ell=\pm1} \Big\| \sum_{\ell=1}^n \varepsilon_\ell \mathbf{f}_\ell \Big\|_{L_p(\mathbb{T};W)} \lesssim C \Big(\sum_{\ell=1}^n \|\mathbf{f}_\ell\|^2_{L_p(\mathbb{T};W)} \Big)^{1/2}, \tag{3.5}$$

where $\{r_\ell\}_{\ell\in\mathbb{N}}$ is the sequence of Rademacher functions on $[0,1)$, see Lemma 3.5. We use (3.5), together with Lemma 3.4 applied to the quasi-greedy system $\{\beta_k^j \mathbf{e}_k^j\}$, to obtain

$$\Big\| \sum_{k:\ (j,k)\in F} \alpha_k^j \mathbf{e}_k^j \Big\|_{L_p(\mathbb{T};W)} \asymp \mathrm{Avg}_{\varepsilon_k^j=\pm1} \Big\| \sum_{k:\ (j,k)\in F} \varepsilon_k^j [\beta_k^j \mathbf{e}_k^j] \Big\|_{L_p(\mathbb{T};W)} \lesssim \Big(\sum_{k:\ (j,k)\in F} \|\beta_k^j \mathbf{e}_k^j\|^2_{L_p(\mathbb{T};W)} \Big)^{1/2} \lesssim \sqrt{\#F}. \tag{3.6}$$

The estimates (3.6) and (3.4) prove the upper estimate in (3.3).

We turn to the lower estimate in (3.3). Pick an index j' such that $|v_{j'}| = \max_j |v_j|$. Now we use the fact that the system $\{\beta_k^j e_k^j\}$ is quasi-greedy to conclude that for every $\varepsilon > 0$,

$$(1+\varepsilon)|v_{j'}| \Big\| \sum_{k:\ (j',k)\in F} \alpha_k^{j'} \mathbf{e}_k^{j'} \Big\|_{L_p(\mathbb{T};W)} \leqslant Q \Big\| (1+\varepsilon) v_{j'} \sum_{k:\ (j',k)\in F} \alpha_k^{j'} \mathbf{e}_k^{j'} + \sum_{j,\, j\neq j'} v_j \sum_{k\in\mathbb{Z}} \alpha_k^j \mathbf{e}_k^j \Big\|_{L_p(\mathbb{T};W)}, \tag{3.7}$$

where Q is the quasi-greedy constant in $L_p(\mathbb{T};W)$ for $\{\beta_k^j \mathbf{e}_k^j\}_{j\in\{1,\dots,N\},k\in\mathbb{Z}}$. We let $\varepsilon \to 0^+$ to conclude that

$$|v_{j'}| \Big\| \sum_{k:\ (j',k)\in F} \alpha_k^{j'} \mathbf{e}_k^{j'} \Big\|_{L_p(\mathbb{T};W)} \leqslant Q \Big\| \sum_{(j,k)\in F} v_j \alpha_k^j \mathbf{e}_k^j \Big\|_{L_p(\mathbb{T};W)}. \tag{3.8}$$

Let $L := \min_{j\in\{1,\dots,N\}} \#\{k \in \mathbb{Z}\colon\ \alpha_k^j \neq 0\}$. Then, since $L_2(\mathbb{T};W)$ has both type and cotype 2, and $L_p(\mathbb{T};W) \hookrightarrow L_2(\mathbb{T};W)$ by Lemma 3.6,

$$\begin{aligned} \sqrt{L} &\asymp \mathrm{Avg}_{\varepsilon_k^{j'}=\pm1} \Big\| \sum_{k:\ (j',k)\in F} \varepsilon_k^{j'} [\beta_k^{j'} \mathbf{e}_k^{j'}] \Big\|_{L_2(\mathbb{T};W)} \lesssim \mathrm{Avg}_{\varepsilon_k^{j'}=\pm1} \Big\| \sum_{k:\ (j',k)\in F} \varepsilon_k^{j'} [\beta_k^{j'} \mathbf{e}_k^{j'}] \Big\|_{L_p(\mathbb{T};W)} \\ &\asymp \Big\| \sum_{k:\ (j',k)\in F} \alpha_k^{j'} \mathbf{e}_k^{j'} \Big\|_{L_p(\mathbb{T};W)}. \end{aligned} \tag{3.9}$$

This together with (3.8) shows that

$$\Big(\max_j |v_j| \Big) L^{1/2} \lesssim \Big\| \sum_{(j,k)\in F} v_j \alpha_k^j \mathbf{e}_k^j \Big\|_{L_p(\mathbb{T};W)},$$

which is the lower estimate in (3.3).

Let us sketch the modifications needed to prove the result in the case $1 < p < 2$. The lower estimate in (3.3) now follows from (3.8) and the fact that $L_p(\mathbb{T};W)$ has cotype 2 (which reverses the estimates in (3.6)). For the upper estimate, we use (3.4) and modify (3.9) using $L_2(\mathbb{T};W) \hookrightarrow L_p(\mathbb{T};W)$. □

With Proposition 3.7 at our disposal, we can now state and prove the main result of this paper. Theorem 3.8 shows that $\mathscr{T}$ fails to be quasi-greedy in $L_p(\mathbb{T};W)$, $p \neq 2$. This generalizes what is known about the trigonometric system in $L_p(\mathbb{T})$.

Temlyakov [18] proved that $\mathscr{T}$ fails to be a quasi-greedy basis for $L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, $p \neq 2$. This negative result was also proved independently by Córdoba and Fernández [3] and by Wojtaszczyk [23].

We need the following notation. For a positive matrix W, we let $\lambda(W)$ and $\Lambda(W)$ denote the smallest, resp. largest, eigenvalue of W.

Theorem 3.8. *Let $W : \mathbb{T} \to \mathbb{C}^{N\times N}$ be a weight with $W, W^{-1} \in L_1$. Suppose $\mathscr{T}$ is a quasi-greedy basis for $L^p(\mathbb{T}; W)$ for some $1 < p < \infty$. Then $p = 2$, $W \in A_2$, and there exists a positive constant C such that*

$$C^{-1} \leqslant \lambda\big(W(\cdot)\big) \leqslant \Lambda\big(W(\cdot)\big) \leqslant C, \quad \text{a.e.} \tag{3.10}$$

Proof. Suppose $\mathscr{T}$ is a quasi-greedy basis for $L^p(\mathbb{T}; W)$ for some $1 < p < \infty$. Then, in particular, $\mathscr{T}$ is a Schauder basis for $L^p(\mathbb{T}; W)$ and $W \in A_p$ by Proposition 2.3.

We now turn to a proof of (3.10) and the fact that we must have $p = 2$. Let $\mathscr{L} \subseteq \mathbb{T}$ denote the common set of Lebesgue points for the entries in $W^{1/p}$. We notice that $\mathscr{L}$ has full measure. Pick $u \in \mathscr{L}$, and let $\mathbf{v} := \mathbf{v}(u) \in \mathbb{C}^N$ be an ℓ_2-normalized eigenvector corresponding to the smallest eigenvalue of $W^{1/p}(u)$. We use the Dirichlet kernel

$$D_K(t) := \sum_{k\in\mathbb{Z}:\ |k|\leqslant K} e^{2\pi ikt}, \quad K \in \mathbb{N},$$

to create the vector functions

$$\tau_K(\cdot) := D_K(u - \cdot)\mathbf{v}, \quad K \in \mathbb{N}.$$

Notice that $D_K(u - \cdot)$ is a trigonometric polynomial with exactly $2K + 1$ non-zero *unimodular* coefficients. Moreover, one easily checks that uniformly in K, $\|D_K\|^p_{L_p(\mathbb{T})} \asymp K^{p-1}$.

Since $\mathbf{v}$ is normalized in ℓ_2, we have $\max_i |v_i| \geqslant 1/\sqrt{N}$. We use this fact together with Proposition 3.7 to obtain the estimate,

$$\|\tau_K\|_{L_p(\mathbb{T};W)} \asymp K^{1/2},$$

uniformly in K and $u \in \mathbb{T}^d$. Next we observe that

$$\frac{|D_K(u-\cdot)|^p}{\|D_K\|^p_{L_p(\mathbb{T})}}$$

is an approximation to the identity at u, which implies that

$$\int_{\mathbb{T}} \big|W^{1/p}(t)\mathbf{v}\big|^p \frac{|D_K(u-t)|^p}{\|D_K\|^p_{L_p(\mathbb{T})}}\, dt \to \big|W^{1/p}(u)\mathbf{v}\big|^p = \lambda\big(W^{1/p}(u)\big)^p = \lambda\big(W(u)\big), \tag{3.11}$$

as $K \to \infty$, where we used that $\mathbf{v} \in \mathbb{C}^N$ is ℓ_2-normalized. At the same time,

$$\frac{1}{\|D_K\|^p_{L_p(\mathbb{T})}} \int_{\mathbb{T}} \big|W^{1/p}(t)\mathbf{v}\big|^p \big|D_K(u-t)\big|^p\, dt = \frac{\|\tau_K\|^p_{L_p(\mathbb{T};W)}}{\|D_K\|^p_{L_p(\mathbb{T})}} \asymp \frac{K^{p/2}}{K^{p-1}}, \tag{3.12}$$

uniformly in K and $u \in \mathbb{T}^d$. Now, let us consider the possible values of p. If $2 < p < \infty$, then obviously $K^{1-p/2} \to 0$ as $K \to \infty$, and we deduce from (3.11) and (3.12) that $\lambda(W(t)) = 0$ a.e., which is a contradiction (recall, W is strictly positive a.e.). Also, if $1 < p < 2$, then $K^{1-p/2} \to \infty$ as $K \to \infty$, and Eqs. (3.11) and (3.12) show that W has unbounded spectrum a.e. which is another contradiction.

Thus the only possible value is $p = 2$, and it follows from (3.11) and (3.12) that there exists a constant $c > 0$ such that $c \leqslant \sigma_{\min}(W(t))$ a.e. To get the upper estimate for $\Lambda(W(t))$, we repeat the argument with $\mathbf{w} \in \mathbb{C}^N$, a normalized eigenvector corresponding to the largest eigenvalue of $W(u)$. Hence, the estimate (3.10) holds true. □

We conclude this section by the following straightforward corollary to Theorem 3.8.

Corollary 3.9. *Let $1 < p < \infty$, and let $W : \mathbb{T} \to \mathbb{C}^{N\times N}$ be a weight with $W, W^{-p'/p} \in L_1$, $1/p + 1/p' = 1$. Suppose $\mathscr{T}$ is a quasi-greedy basis for $L^p(\mathbb{T}; W)$. Then $\mathscr{T}$ is a Riesz basis for $L_2(\mathbb{T}; W)$.*

Proof. This follows directly from (3.10), see e.g. [4,15]. □

4. Some examples

Let us conclude this paper by considering some examples.

Example 4.1. We consider

$$W(t) := \begin{bmatrix} |t|^{\alpha_1} & 0 \\ 0 & |t|^{\alpha_2} \end{bmatrix}, \quad t \in [-1/2, 1/2).$$

It is easy to check, using the scalar A_p-condition, that $W \in A_p$ provided $-1 < \alpha_j < p-1$, $j = 1, 2$, see e.g. [7, Chap. 9]. For example, pick $0 < \alpha_j < p-1$, then $\mathscr{T}$ forms a conditional Schauder basis for $L_p(\mathbb{T}; W)$. The basis is only conditional since the spectrum of W is not bounded from below. This can be fixed by considering $W' = W + \varepsilon \mathrm{Id}$, $\varepsilon > 0$. $\mathscr{T}$ forms a Riesz basis for $L_2(\mathbb{T}; W')$.

A more intricate example is the following.

Example 4.2. Let us consider the following matrix weight introduced by Bownik [2]. For $t \in [-1/2, 1/2)$ we define

$$W(t) = U(t)^* \begin{bmatrix} 1 & 0 \\ 0 & b(t) \end{bmatrix} U(t), \qquad U(t) = \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix},$$

where $\alpha(t) = \mathrm{sign}(t)|t|^{\delta}, b(t) = |t|^{\varepsilon}$, with $-1 < \varepsilon < 1$, and δ satisfying $-2\delta \leqslant \varepsilon \leqslant 2\delta$. Then $W(t) \in A_2$, see [2]. Thus, $\mathscr{T}$ forms a Schauder basis for $L_2(\mathbb{T}; W)$, but we also notice that the spectrum of $W(t)$ is not bounded away from zero, so according to Theorem 3.8, $\mathscr{T}$ is not quasi-greedy in $L_2(\mathbb{T}; W)$. We also mention that it is a general fact that $A_2 \subseteq A_p$ for $2 \leqslant p < \infty$, see [6], so in this case $\mathscr{T}$ forms a (conditional) Schauder basis for $L_p(\mathbb{T}; W)$, for $2 \leqslant p < \infty$.

We conclude with the following variation on Example 4.2.

Example 4.3. Let W be defined as in Example 4.2. Let us try to "pull back" $\mathscr{T}$ to the unweighted setting. We notice that any $\mathbf{g} \in L_2(\mathbb{T}; W)$ can be written $\mathbf{g} = W^{-1/2}\mathbf{f}$, with $\mathbf{f} \in L_2(\mathbb{T}; \mathrm{Id})$. Hence, the expansion of $\mathbf{g} \in L_2(\mathbb{T}; W)$ can be re-written

$$S_m(\mathbf{g}) = \sum_{n=1}^{m} \int_{\mathbb{T}} \big\langle W(t)\mathbf{g}(t), W^{-1}(t)\mathbf{e}\big(\eta(n)\big)(t)\big\rangle_{\ell_2}\, dt\, \mathbf{e}\big(\eta(n)\big) = \sum_{n=1}^{m} \int_{\mathbb{T}} \big\langle \mathbf{f}(t), W^{-1/2}(t)\mathbf{e}\big(\eta(n)\big)(t)\big\rangle_{\ell_2}\, dt\, \mathbf{e}\big(\eta(n)\big),$$

which shows that $W^{1/2}S_m(\mathbf{g}) \to \mathbf{f}$ in $L_2(\mathbb{T}; \mathrm{Id})$. It follows that $\{W^{1/2}\mathbf{e}(\eta(n))\}_{n=1}^{\infty}$ is a conditional Schauder basis for $L_2(\mathbb{T}; \mathrm{Id})$ with corresponding bi-orthogonal system $\{W^{-1/2}\mathbf{e}(\eta(n))\}_{n=1}^{\infty}$. This example can be considered a vector-valued analog to the example of Babenko of a conditional Schauder basis for $L_2(\mathbb{T})$, see [1].

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