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Sums and products with smooth numbers

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ABSTRACT

We estimate the sizes of the sumset $\mathcal{A}+\mathcal{A}$ and the productset $\mathcal{A}\cdot\mathcal{A}$ in the special case that $\mathcal{A}=\mathcal{S}(x,y)$, the set of positive integers $n \leqslant x$ free of prime factors exceeding y.

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1. Background

For any nonempty subset $\mathcal{A}$ of a ring, the *sumset* and *productset* of $\mathcal{A}$ are defined as

$$\mathcal{A}+\mathcal{A}=\{a+a' \colon a,a' \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}\cdot\mathcal{A}=\{a\cdot a' \colon a,a' \in \mathcal{A}\},$$

respectively. A famous problem of Erdős and Szemerédi [6] asks one to show that the sumset and productset of a finite set of integers cannot both be small.

Conjecture *(Erdős–Szemerédi). For any fixed $\delta > 0$ the lower bound*

$$\max\{|\mathcal{A}+\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\} \gg_{\delta} |\mathcal{A}|^{2-\delta}$$

holds for all finite sets $\mathcal{A} \subset \mathbb{Z}$.

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Erdős and Szemerédi [6] took the first step towards this conjecture by showing that for some $\epsilon > 0$, one has a lower bound of the form

$$\max\{|\mathcal{A}+\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\} \geqslant c(\epsilon)|\mathcal{A}|^{1+\epsilon} \tag{1}$$

for all finite sets $\mathcal{A} \subset \mathbb{Z}$. Nathanson [10] gave the first explicit bound by showing that one can take $\epsilon = \frac{1}{31}$ and $c(\epsilon) = 0.00028\ldots$ in this inequality, and later, Ford [8] showed that $\epsilon = \frac{1}{15}$ is acceptable. Establishing an important connection between the sum–product problem and geometric incidence theory, Elekes [3] showed that one can take $\epsilon = \frac{1}{4}$ via a clever application of the Szemerédi–Trotter incidence theorem (which counts incidences between points and lines in the plane); moreover, his argument readily extends to finite sets of real numbers. Further improvements, including the best known bound to date, have been given by Solymosi [12,13]; he has shown that (1) holds with any $\epsilon < \frac{1}{3}$ for all finite sets $\mathcal{A} \subset \mathbb{R}$.

Although the Erdős–Szemerédi conjecture remains open, it is known that the productset must be large whenever the sumset is sufficiently small. In fact, Nathanson and Tenenbaum [11] have shown that

$$|\mathcal{A}\cdot\mathcal{A}| \geqslant \frac{c|\mathcal{A}|^2}{\log|\mathcal{A}|} \quad \text{if } |\mathcal{A}+\mathcal{A}| \leqslant 3|\mathcal{A}| - 4. \tag{2}$$

The aforementioned best known bound to date, given by Solymosi [13], follows from his more general inequality

$$|\mathcal{A}+\mathcal{A}|^2|\mathcal{A}\cdot\mathcal{A}| \geqslant \frac{|\mathcal{A}|^4}{4\lceil\log|\mathcal{A}|\rceil}. \tag{3}$$

Note that (3) provides a quantitive generalization of the Nathanson–Tenenbaum result (2) (see also the results in [3,4,12]); it implies that $|\mathcal{A}\cdot\mathcal{A}| \geqslant |\mathcal{A}|^{2-\delta_\epsilon}$ whenever $|\mathcal{A}+\mathcal{A}| < |\mathcal{A}|^{1+\epsilon}$, where $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

In the opposite direction, Chang [2] has shown that the sumset must be large whenever the productset is sufficiently small. More precisely, she has shown that

$$|\mathcal{A}+\mathcal{A}| > 36^{-\alpha}|\mathcal{A}|^2 \quad \text{if } |\mathcal{A}\cdot\mathcal{A}| < \alpha|\mathcal{A}| \text{ for some constant } \alpha. \tag{4}$$

A great deal of attention has also been given to the sum–product problem in other rings, including (but not limited to) finite fields, polynomial rings, and matrix rings. For a thorough account of the subject, we refer the reader to [14] and the references contained therein.

2. Statement of results

Let Ω be any infinite collection of finite sets within a given ring. We shall say that Ω has the *Erdős–Szemerédi property* if

$$\max\{|\mathcal{A}+\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\} = |\mathcal{A}|^{2+o(1)} \quad \text{as } |\mathcal{A}| \to \infty \text{ with } \mathcal{A} \in \Omega.$$

Then, the Erdős–Szemerédi conjecture is the assertion that the collection consisting of *all* finite sets of integers has the Erdős–Szemerédi property.

In this paper, we study the Erdős–Szemerédi property with collections of sets of *smooth numbers*, i.e., sets of the form

$$\mathcal{S}(x,y) = \{n \leqslant x:\ P^+(n) \leqslant y\} \quad (x \geqslant y \geqslant 2),$$

where $P^+(n)$ denotes the largest prime factor of an integer $n \geqslant 2$, and $P^+(1) = 1$. These sets are well known in analytic number theory; for a background on integers free of large prime factors, we refer the reader to [15, Chapter III.5] (see also the survey [9]).

Theorem 1. *There is an absolute constant $c > 0$ for which the collection*

$$\Omega = \{\mathcal{S}(x, y):\ 2 \leqslant y \leqslant c \log x\}$$

has the Erdős–Szemerédi property.

Remarks. In Theorem 4 we show that for values of y of size $o(\log x)$, the productset of $\mathcal{A} = S(x, y)$ has size $|\mathcal{A}|^{1+o(1)}$; thus, only the sumset is large in this region. Using only Theorem 4 and Chang's result (4), one can show that the smaller collection

$$\Omega = \{\mathcal{S}(x, y):\ 2 \leqslant y \leqslant C(\log\log\log x)(\log\log\log\log x)\}$$

has the Erdős–Szemerédi property for any constant $C < 1/\log 2$.

Theorem 2. *Let f be an arbitrary real-valued function such that $f(x) \to \infty$ as $x \to \infty$. Then, the collection*

$$\Omega = \{\mathcal{S}(x, y):\ f(x)\log x \leqslant y \leqslant x\}$$

has the Erdős–Szemerédi property.

Remark. For slightly larger values of y exceeding $(\log x)^{f(x)}$ we show that the sumset of $\mathcal{A} = \mathcal{S}(x, y)$ has size $|\mathcal{A}|^{1+o(1)}$ (see Theorem 5), and hence only the productset is large in this region.

Since each set $\mathcal{S}(x, y)$ is multiplicatively defined, it is quite difficult to estimate the size of the sumset $\mathcal{S}(x, y) + \mathcal{S}(x, y)$ for values of y close to $\log x$. It is reasonable to expect that for every fixed $\kappa > 0$ one has

$$\left|\mathcal{S}(x, y) + \mathcal{S}(x, y)\right| = \left|\mathcal{S}(x, y)\right|^{2+o(1)} \quad (x \to \infty,\ y = \kappa \log x).$$

In view of (12), the Erdős–Szemerédi conjecture implies that this is true. A partial result in this direction is provided by (13). We also expect that for any fixed $A > 1$ one has

$$\left|\mathcal{S}(x, y) + \mathcal{S}(x, y)\right| = \left|\mathcal{S}(x, y)\right|^{\beta_A+o(1)} \quad \left(x \to \infty,\ y = (\log x)^A\right)$$

for some constant β_A in the open interval $(1, 2)$. For $A > 2$, a partial result in this direction is provided by Theorem 8.

3. Preliminaries

As before, we write

$$\mathcal{S}(x, y) = \{n \leqslant x:\ P^+(n) \leqslant y\} \quad (x \geqslant y \geqslant 2),$$

and we now set

$$\Psi(x, y) = \left|\mathcal{S}(x, y)\right| \quad (x \geqslant y \geqslant 2).$$

We also put

$$G(t) = \log(1+t) + t\log\bigl(1+t^{-1}\bigr) \quad (t > 0).$$

From this definition we immediately derive the crude estimates

$$G(t) = \log t\left\{1 + O\left(\frac{1}{\log t}\right)\right\} \quad (t \geqslant 2) \tag{5}$$

and

$$G(t) = t\log t^{-1}\left\{1 + O\left(\frac{1}{\log t^{-1}}\right)\right\} \quad (0 < t \leqslant 1/2). \tag{6}$$

The following result is due to de Bruijn [1].

Lemma 1. *Uniformly for $x \geqslant y \geqslant 2$ we have*

$$\log\Psi(x, y) = \frac{\log x}{\log y} G\left(\frac{y}{\log x}\right)\left\{1 + O\left(\frac{1}{\log y} + \frac{1}{\log\log 2x}\right)\right\}.$$

For smaller values of y, we need the following result of Ennola [5].

Lemma 2. *Uniformly for $2 \leqslant y \leqslant \sqrt{\log x \log\log x}$ we have*

$$\Psi(x, y) = \frac{1}{\pi(y)!}\prod_{p\leqslant y}\frac{\log x}{\log p}\left\{1 + O\left(\frac{y^2}{\log x \log y}\right)\right\},$$

where $\pi(y) = |\{p \leqslant y\}|$.

For any finite set of primes S, let $\mathcal{O}_S^*$ denote the group of S-units in $\mathbb{Q}^*$; that is,

$$\mathcal{O}_S^* = \bigl\{a/b \in \mathbb{Q}^*:\ p \mid ab \Rightarrow p \in S\bigr\}.$$

The next statement is a special case of a more general result of Evertse on solutions to S-unit equations (see [7, Theorem 3]).

Lemma 3. *Given $a_1 \cdots a_n \in \mathbb{Q}^*$ and a finite set of primes S of cardinality $|S| = s$, the S-unit equation*

$$a_1u_1 + \cdots + a_nu_n = 1 \quad \bigl(u_1, \ldots, u_n \in \mathcal{O}_S^*\bigr)$$

has at most $(2^{35}n^2)^{n^3s}$ solutions $(u_1, \ldots, u_n)$ with $\sum_{j\in\mathcal{J}} a_ju_j \neq 0$ for every nonempty subset $\mathcal{J} \subseteq \{1, \ldots, n\}$.

To get a better handle on productsets of smooth numbers, we shall apply the following technical lemma.

Lemma 4. *We have*

$$\Psi\bigl(x^2/y, y\bigr) \leqslant \bigl|\mathcal{S}(x, y)\cdot\mathcal{S}(x, y)\bigr| \leqslant \Psi\bigl(x^2, y\bigr) \quad (x \geqslant y \geqslant 2).$$

Proof. It is easy to see that $\mathcal{S}(x,y)\cdot\mathcal{S}(x,y)\subseteq\mathcal{S}(x^2,y)$, which yields the second inequality. For the first inequality, it suffices to show that $\mathcal{S}(x^2/y,y)$ is contained in the productset $\mathcal{S}(x,y)\cdot\mathcal{S}(x,y)$. To this end, let $n\in\mathcal{S}(x^2/y,y)$, and let d be the largest divisor of n that does not exceed x. Note that $\max\{P^+(d),P^+(n/d)\}\leqslant y$. There are three possibilities for the number d:

(i) $d>x/y$;
(ii) $d=n\leqslant x/y$;
(iii) $d\leqslant x/y$ and $d<n$.

In case (i) we have $n/d\leqslant x$, hence we can write $n=d\cdot(n/d)$ where d and n/d both lie in $\mathcal{S}(x,y)$; this shows that $n\in\mathcal{S}(x,y)\cdot\mathcal{S}(x,y)$ as required. In case (ii) the number n lies in the set $\mathcal{S}(x/y,y)$, which is a subset of $\mathcal{S}(x,y)\cdot\mathcal{S}(x,y)$. To finish the proof, we need only show that the case (iii) is not possible. Indeed, suppose $d\leqslant x/y$ and $d<n$, and let p be any prime factor of n/d; then $p\leqslant P^+(n/d)\leqslant y$, $dp\mid n$, and $dp\leqslant x$, which contradicts the maximal property of d. □

4. Small values of y

Theorem 3. *There is an absolute constant $c>0$ such that the estimate*

$$\left|\mathcal{S}(x,y)+\mathcal{S}(x,y)\right|\sim\frac{1}{2}\Psi(x,y)^2\quad(x\to\infty)$$

holds uniformly for $2\leqslant y\leqslant c\log x$.

Proof. We have

$$\Psi(x,y)^2=\left|\mathcal{S}(x,y)\right|^2=\sum_{n\in\mathcal{S}(x,y)+\mathcal{S}(x,y)}\ \sum_{\substack{m_1,m_2\in\mathcal{S}(x,y)\\ m_1+m_2=n}}1.$$

Using the Cauchy inequality it follows that

$$\Psi(x,y)^4\leqslant\left|\mathcal{S}(x,y)+\mathcal{S}(x,y)\right|\cdot|\mathcal{T}|,$$

where $\mathcal{T}$ is the set of quadruples (m_1,m_2,m_3,m_4) with entries in $\mathcal{S}(x,y)$ such that $m_1+m_2=m_3+m_4$. It is easy to see that there are precisely $2\Psi(x,y)^2-\Psi(x,y)$ quadruples in $\mathcal{T}$ for which $m_1=m_3$ or $m_1=m_4$. Let $\mathcal{T}^*$ be the set of quadruples in $\mathcal{T}$ with $m_1\neq m_3$ and $m_1\neq m_4$ (thus, $m_2\neq m_3$ and $m_2\neq m_4$ as well). If we put $a_1=a_2=1$ and $a_3=-1$, the equation $m_1+m_2=m_3+m_4$ becomes

$$a_1u_1+a_2u_2+a_3u_3=1, \tag{7}$$

where

$$u_1=\frac{m_1}{m_4},\qquad u_2=\frac{m_2}{m_4}\quad\text{and}\quad u_3=\frac{m_3}{m_4}. \tag{8}$$

Let S be the set of primes $p\leqslant y$, and let $\mathcal{O}_S^*$ be the group of S-units in $\mathbb{Q}^*$. According to Lemma 3, there are at most $(2^{35}9)^{27\pi(y)}$ solutions to the S-unit equation (7) with $u_j\in\mathcal{O}_S^*$, $j=1,2,3$, and $\sum_{j\in\mathcal{J}}a_ju_j\neq0$ for each nonempty subset $\mathcal{J}\subseteq\{1,2,3\}$. On the other hand, for every fixed solution (u_1,u_2,u_3) to (7) there are at most $\Psi(x,y)$ quadruples (m_1,m_2,m_3,m_4) in $\mathcal{T}^*$ for which (8) holds (since each choice of $m_4\in\mathcal{S}(x,y)$ determines m_1, m_2, m_3 uniquely). Putting everything together, it follows that the bound

$$\Psi(x,y)^4\leqslant\left|\mathcal{S}(x,y)+\mathcal{S}(x,y)\right|\cdot\bigl(2\Psi(x,y)^2-\Psi(x,y)+\exp(c_1y/\log y)\Psi(x,y)\bigr)$$

holds with some absolute constant $c_1 > 0$. Taking into account the trivial upper bound

$$\left|\mathcal{S}(x,y) + \mathcal{S}(x,y)\right| \leqslant \frac{1}{2}\big(\Psi(x,y)^2 + \Psi(x,y)\big),$$

it suffices to show that there is an absolute constant $c > 0$ such that for all sufficiently large x, we have

$$\exp(c_1 y/\log y) \leqslant \Psi(x,y)^{1/2} \quad (2 \leqslant y \leqslant c\log x). \tag{9}$$

For every sufficiently large integer N, Lemma 1 implies that:

$$\log\Psi(x,y) \geqslant \frac{1}{2}\frac{\log x}{\log y} G\left(\frac{y}{\log x}\right) \quad (x \geqslant y > N)$$

if x is sufficiently large. Let $N \geqslant 2$ be fixed with this property. For every sufficiently small constant $c > 0$ we also have by (6):

$$G(t) \geqslant \frac{1}{2} t \log t^{-1} \quad (0 < t \leqslant c).$$

Let $0 < c \leqslant e^{-8c_1}$ be fixed with this property. Combining the two bounds, we see that

$$\log\Psi(x,y) \geqslant \frac{\log(1/c)}{4}\frac{y}{\log y} \geqslant 2c_1\frac{y}{\log y} \quad (N < y \leqslant c\log x)$$

if x is large enough; this implies (9) in the range $N < y \leqslant c\log x$. For the smaller values of y in the range $2 \leqslant y \leqslant N$, we simply observe that $\exp(c_1 y/\log y) = O(1)$, whereas

$$\Psi(x,y) \geqslant \Psi(x,2) = 1 + \left\lfloor \frac{\log x}{\log 2} \right\rfloor \to \infty \quad \text{as } x \to \infty.$$

Hence, (9) also holds for these values of y if x is sufficiently large. This completes the proof. □

Theorem 4. *Suppose that $y \geqslant 2$ and $y = o(\log x)$. Then*

$$\left|\mathcal{S}(x,y) \cdot \mathcal{S}(x,y)\right| = \Psi(x,y)^{1+o(1)}.$$

Proof. By Lemma 4 we have

$$\Psi(x,y) \leqslant \Psi\big(x^2/y, y\big) \leqslant \left|\mathcal{S}(x,y) \cdot \mathcal{S}(x,y)\right| \leqslant \Psi\big(x^2, y\big),$$

hence it suffices to show that $\Psi(x^2,y) = \Psi(x,y)^{1+o(1)}$ as $x \to \infty$.

First, suppose that $2 \leqslant y \leqslant \sqrt{\log x}$. By Lemma 2 we have

$$\Psi(x,y) \sim \frac{1}{\pi(y)!}\prod_{p \leqslant y}\frac{\log x}{\log p} \quad (x \to \infty)$$

and

$$\Psi\big(x^2,y\big) \sim \frac{1}{\pi(y)!}\prod_{p \leqslant y}\frac{\log x^2}{\log p} \sim 2^{\pi(y)}\Psi(x,y) \quad (x \to \infty).$$

Since the inequality $\pi(y)! \leqslant y^{\pi(y)}$ implies

$$\Psi(x,y) \geqslant \big(1+o(1)\big)\left(\frac{\log x}{y\log y}\right)^{\pi(y)} \geqslant \big(1+o(1)\big)\left(\frac{2\sqrt{\log x}}{\log\log x}\right)^{\pi(y)},$$

it follows that $2^{\pi(y)} = \Psi(x,y)^{o(1)}$; thus, $\Psi(x^2,y) = \Psi(x,y)^{1+o(1)}$ as required.

Next, suppose that $y > \sqrt{\log x}$ and $y = o(\log x)$ as $x \to \infty$. Using Lemma 1 together with (6) we see that the estimate

$$\log \Psi(z,y) = \frac{y}{\log y}\log\left(\frac{\log z}{y}\right)\left\{1 + O\left(\frac{1}{\log((\log x)/y)}\right)\right\}$$

holds uniformly for all z in the range $x \leqslant z \leqslant x^2$. Applying this estimate with $z = x$ and with $z = x^2$, we derive that $\Psi(x^2,y) = \Psi(x,y)^{1+o(1)}$ in this case as well. □

5. Large values of *y*

For values of y exceeding any fixed power of $\log x$, we have:

Theorem 5. *Suppose that* $(\log y)/\log\log x \to \infty$. *Then,*

$$\big|\mathcal{S}(x,y) + \mathcal{S}(x,y)\big| = \Psi(x,y)^{1+o(1)} \quad (x \to \infty).$$

Proof. Using Lemma 1 and (5) we see that

$$\log \Psi(x,y) \sim \frac{\log x}{\log y} G\left(\frac{y}{\log x}\right) \sim \frac{\log x}{\log y}(\log y - \log\log x) \sim \log x \quad (x \to \infty),$$

since $(\log\log x)/\log y \to 0$; that is,

$$\Psi(x,y) = x^{1+o(1)} \quad (x \to \infty).$$

Using the trivial bounds

$$\Psi(x,y) \leqslant \big|\mathcal{S}(x,y) + \mathcal{S}(x,y)\big| \leqslant 2x$$

together with the previous estimate, we obtain the desired result. □

Theorem 6. *Let* $y/\log x \to \infty$. *Then,*

$$\big|\mathcal{S}(x,y) \cdot \mathcal{S}(x,y)\big| = \Psi(x,y)^{2+o(1)} \quad (x \to \infty). \tag{10}$$

Proof. In the case that $(\log y)/\log\log x \to \infty$, we can apply Theorem 5 together with (3) to obtain (10) immediately. Thus, we can assume that $\log y \asymp \log\log x$. Since $y/\log x \to \infty$, we derive from Lemma 1 and (5) the estimate

$$\log \Psi(x,y) = \frac{\log x}{\log y}\log\left(\frac{y}{\log x}\right)\big\{1 + o(1)\big\}, \tag{11}$$

whereas both $\log \Psi(x^2/y, y)$ and $\log \Psi(x^2, y)$ are of the size

$$\frac{\log x}{\log y} \log\left(\frac{y}{\log x}\right)\left\{2 + o(1)\right\}.$$

Therefore,

$$\Psi\left(x^2/y, y\right) = \Psi(x, y)^{2+o(1)} \quad \text{and} \quad \Psi\left(x^2, y\right) = \Psi(x, y)^{2+o(1)},$$

and the estimate (10) follows from Lemma 4. $\square$

6. Intermediate values of *y*

Theorem 7. *Suppose that $y = \kappa \log x$, where $\kappa > 0$ is fixed. Then,*

$$\left|\mathcal{S}(x, y) \cdot \mathcal{S}(x, y)\right| = \Psi(x, y)^{\alpha_k + o(1)} \tag{12}$$

and

$$\left|\mathcal{S}(x, y) + \mathcal{S}(x, y)\right| \geqslant \Psi(x, y)^{(4-\alpha_\kappa)/2+o(1)}, \tag{13}$$

where

$$\alpha_\kappa = \frac{2\log(1+\kappa/2) + \kappa\log(1+2/\kappa)}{\log(1+\kappa) + \kappa\log(1+1/k)}.$$

Remark. For every positive real number κ we have $1 < \alpha_\kappa < 2$. Also, $\alpha_\kappa \to 1$ as $\kappa \to 0^+$ and $\alpha_\kappa \to 2$ as $\kappa \to \infty$.

Proof. First note that (13) follows from combining (12) and (3). It remains to prove (12). By Lemma 1 we have

$$\log \Psi(x, y) = \left(G(\kappa) + o(1)\right)\frac{\log x}{\log\log x} \quad (x \to \infty)$$

and

$$\log \Psi\left(x^2, y\right) = \left(2G(\kappa/2) + o(1)\right)\frac{\log x}{\log\log x} \quad (x \to \infty),$$

where the functions implied by $o(1)$ depend only on κ. Since G is continuous it is also easy to see that

$$\log \Psi\left(x^2/y, y\right) = \left(2G(\kappa/2) + o(1)\right)\frac{\log x}{\log\log x} \quad (x \to \infty).$$

Using Lemma 4, the above estimates, and the fact that $\alpha_\kappa = 2G(\kappa/2)/G(\kappa)$, the result follows. $\square$

Theorem 8. *Suppose that $y \asymp (\log x)^A$, where $A > 2$ is fixed. Then,*

$$\left|\mathcal{S}(x, y) + \mathcal{S}(x, y)\right| \leqslant \Psi(x, y)^{\frac{A}{A-1}+o(1)} \quad (x \to \infty).$$

Proof. If $y \asymp (\log x)^A$ for some $A > 1$, then the estimate $\Psi(x, y) = x^{\frac{A-1}{A}+o(1)}$ follows immediately from (11). Taking into account the trivial bound $|\mathcal{S}(x, y) + \mathcal{S}(x, y)| \leqslant 2x$, we obtain the stated result (which is nontrivial in the range $A > 2$). $\square$

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