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On the Delta set and catenary degree of Krull monoids with infinite cyclic divisor class group

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ABSTRACT

Let M be a Krull monoid with divisor class group $\mathbb{Z}$, and let $S \subseteq \mathbb{Z}$ denote the set of divisor classes of M which contain prime divisors. We find conditions on S equivalent to the finiteness of both $\Delta(M)$, the Delta set of M, and $c(M)$, the catenary degree of M. In the finite case, we obtain explicit upper bounds on max $\Delta(M)$ and $c(M)$. Our methods generalize and complement a previous result concerning the elasticity of M.

1. Introduction

Because of their relevance to non-unique factorizations in algebraic number theory, Krull monoids M with finite divisor class group have been well studied. It is known in this case that the sets of lengths of M are almost-arithmetic multiprogressions (see [8] for a study of such sequences). When the divisor class group of M is not finite, we know considerably less about the structure of the length sets. In particular, if each divisor class of M contains a prime divisor, then a result of Kainrath [12] implies that each finite subset of $\{2, 3, 4, \ldots\}$ can be obtained as a set of lengths. In [11], Hassler gives conditions on this distribution of divisor classes with primes which yield "thin" sets of lengths. About 10 years ago, a series of papers appeared [2,3] which considered problems involving the elasticity of Krull monoids with divisor class group $\mathbb{Z}$. In such a Krull monoid, let S represent the set of divisor classes of M which contain prime divisors and write S as a disjoint union $S^+ \cup S^-$ where $S^+ = \{s \in S \mid s \geq 0\}$ and $S^- = \{s \in S \mid s < 0\}$. The main result of [2] (Theorems 2.1 and 2.3) implies that the elasticity, $\rho(M)$, is finite if and only if either S^+ or S^- is finite. In this note, we show that this result extends to the cardinality of the Delta set of M, $\Delta(M)$, as well as the catenary degree of M, $c(M)$. In particular, the results of the next two sections along with the result mentioned above from [2] will constitute a proof of the following.

Theorem 1.1. *Let M be a Krull monoid with divisor class group $\mathbb{Z}$ where $S = S^+ \cup S^-$ corresponds to the set of divisor classes which contain prime divisors. The following are equivalent:*

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(1) *$\rho(M)$ is finite,*
(2) *$\Delta(M)$ is finite,*
(3) *$c(M)$ is finite,*
(4) *Either S^+ or S^- is finite.*

By [10, Theorem 1.6.3], we have for all atomic monoids that (3) $\Rightarrow$ (2). Few other general relationships exist between (1), (2), and (3). For instance, Example 4.8.11 in [10] yields a monoid M with $\rho(M) < \infty$ and $|\Delta(M)| = \infty$ (other pathological type examples are found in [10] in Examples 1.6.11, 1.6.3.3, and Theorem 3.1.5).

We proceed with a brief summary of the necessary background and notation for the remainder of our work. For a commutative cancellative monoid M, let $M^\times$ denote its subgroup of units; M is called *reduced* if $M^\times = \{1\}$. Let $\mathcal{A}(M)$ denote the set of irreducible elements or *atoms* of M. If $x \in M$, an *atomic factorization* is an expression of the form $F : x = \alpha_1 \cdots \alpha_n$ where $\alpha_1, \ldots, \alpha_n \in \mathcal{A}(M)$; in this case we say that $|F| = n$ is the *length* of this atomic factorization. If $x \in M \setminus M^\times$ we define

$$\mathcal{L}(x) = \{\, n \mid x \text{ has an atomic factorization of length } n \,\}$$

and $\mathfrak{L}(M) = \{\, \mathcal{L}(x) \mid x \in M \setminus M^\times \,\}$. M is called *atomic* if every element has an atomic factorization. If M is atomic, $x \in M$, and $\alpha \in \mathcal{A}(M)$ divides x, then there is an atomic factorization of x in which α appears.

For any $x \in M \setminus M^\times$ the ratio $\sup \mathcal{L}(x)/ \min \mathcal{L}(x)$ is called the *elasticity* of x, denoted $\rho(x)$. If M is not a group, we define the elasticity of the monoid M by

$$\rho(M) = \sup\{\, \rho(x) \mid x \in M \setminus M^\times \,\}.$$

A review of the main facts concerning the elasticity can be found in [10, Chapter 1.4]. Given $x \in M \setminus M^\times$, write its length set in the form $\mathcal{L}(x) = \{n_1, n_2, \ldots, n_k\}$ where $n_i < n_{i+1}$ for $1 \le i \le k - 1$. The *Delta set* of x is then defined by

$$\Delta(x) = \{\, n_{i+1} - n_i \mid 1 \le i < k \,\}$$

and the Delta set of M by

$$\Delta(M) = \bigcup_{x \in M \setminus M^\times} \Delta(x)$$

(see again [10, Chapter 1.4]). If $\Delta(M)$ is nonempty, $\min \Delta(M) = \gcd \Delta(M)$ (see [10, Proposition 1,4,4]). Computations of Delta sets for various types of monoids can be found in [4,5].

Suppose that M is reduced, $x \in M$ is not the identity, and that

$$F : x = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_s \quad \text{and} \quad F' : x = \alpha_1 \cdots \alpha_n \gamma_1 \cdots \gamma_t$$

are distinct atomic factorizations such that $\beta_i \neq \gamma_j$ for all i, j. With notation as above, we define $\gcd(F, F') = \alpha_1 \cdots \alpha_n$ and the *distance* between F and F' by $d(F, F') = \max\{s, t\}$. Extend d to all pairs of factorizations by $d(F, F) = 0$. The basic properties of the factorization distance function can be found in [10, Proposition 1.2.5].

An *N-chain of factorizations* from F to F' is a sequence $F_0, \ldots, F_k$ such that each F_i is a factorization of x, $F_0 = F$ and $F_k = F'$, and $d(F_i, F_{i+1}) \le N$ for all $i < k$. The catenary degree of x, denoted $c(x)$, is the least $N \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$ such that for any two factorizations F, F' of x there is an N-chain between F and F'. The catenary degree of the monoid M is defined as

$$c(M) = \sup\{\, c(x) \mid x \in M \setminus M^\times \,\}.$$

A review of the known facts concerning the catenary degree can be found in [10, Chapter 3]. An algorithm which computes the catenary degree of a finitely generated monoid can be found in [6] and a more specific version for numerical monoids in [7].

A monoid M is called a *Krull monoid* if there is an injective monoid homomorphism $\varphi : M \to D$ where D is a free abelian monoid and φ satisfies the following two conditions:

(1) If $a, b \in M$ and $\varphi(a) \mid \varphi(b)$ in D, then $a \mid b$ in M,
(2) For every $\alpha \in D$ there exists $a_1, \ldots, a_n \in M$ with $\alpha = \gcd\{\varphi(a_1), \ldots, \varphi(a_n)\}$.

The basis elements of D are called the *prime divisors* of M. The above properties guarantee that $\mathrm{Cl}(M) = D/\varphi(M)$ is an abelian group, which we call the *divisor class group* of M (see [10, Section 2.3]). Note that since any Krull monoid is isomorphic to a submonoid of a free abelian monoid, a Krull monoid is commutative, cancellative, and atomic.

Given an abelian group G (written additively), it is easy to explicitly construct a Krull monoid with divisor class group isomorphic to G. Let $\mathcal{F}(G)$ denote the free abelian monoid (written multiplicatively on the set G) and let $\theta : \mathcal{F}(G) \to G$ be the homomorphism $\prod_{g \in G} g^{a_g} \mapsto \sum_{g \in G} a_g \cdot g$. Then it is easy to check that the kernel of this homomorphism is a Krull monoid, where $D = \mathcal{F}(G)$ and φ is inclusion. We denote this monoid, called the *block monoid on G* by

$$\mathcal{B}(G) = \left\{ \prod_{g \in G} g^{a_g} \in \mathcal{F}(G) \;\middle|\; \sum_{g \in G} a_g \cdot g = 0 \right\}$$

and we refer to elements of $\mathcal{B}(G)$ as *blocks* or *zero sequences* from G. Note that every divisor class of $\mathcal{B}(G)$ contains a prime divisor: The divisor class corresponding to $g \in G$ contains the basis element g of $\mathcal{F}(G)$.

If D is a free monoid, f is a basis element of D, and $X \in D$, we write $v_f(X)$ for the power of f appearing in X. We say that X is *supported on* some subset S of the basis of D if and only if $v_f(X) = 0$ for all $f \notin S$. For any $S \subseteq G$, the set

$$\mathcal{B}(G, S) = \{ B \in \mathcal{B}(G) \mid B \text{ is supported on } S \}$$

forms a submonoid of $\mathcal{B}(G)$ called *the restriction of $\mathcal{B}(G)$ to S*. $\mathcal{B}(G, S)$ is again a Krull monoid, and it is not difficult to check that the divisor class corresponding to some $g \in G$ contains a prime divisor if and only if $g \in S$.

Restricted block monoids play an important role in the theory of non-unique factorizations, as the following proposition indicates.

Proposition 1.2 ([9], Proposition 1 see also [10], Theorem 3.4.10(5)). *Suppose that M is a Krull monoid with divisor class group G and let $S \subseteq G$ denote the set of divisor classes which contain prime divisors. We have the equality*

$$\mathfrak{L}(M) = \mathfrak{L}\big(\mathcal{B}(G, S)\big)$$

and the inequality

$$c(\mathcal{B}(G, S)) \le c(M) \le \max\{c(\mathcal{B}(G, S)), 2\}.$$

In particular, $\Delta(M) = \Delta\big(\mathcal{B}(G, S)\big)$. Moreover, if $c(\mathcal{B}(G, S)) \ge 2$, then $c(M) = c\big(\mathcal{B}(G, S)\big)$ and if $c(\mathcal{B}(G, S)) = 0$, then $c(M) = 0$ or 2.

Due to Proposition 1.2 and [1, Lemma 3.3], to prove Theorem 1.1, it suffices to consider block monoids of the form $\mathcal{B}(\mathbb{Z}, S)$, where $S = S^+ \cup S^-$, and both S^+ and S^- are nonempty. Moreover, if $0 \in S^+$, then the irreducible block represented by 0 is prime in $\mathcal{B}(\mathbb{Z}, S)$ and does not effect any factorization properties, since it must appear in every factorization of a block that contains 0.

2. The unbounded case

Theorem 2.1. *If S^+ and S^- are infinite, then $\Delta\big(\mathcal{B}(\mathbb{Z}, S)\big)$ and $c\big(\mathcal{B}(\mathbb{Z}, S)\big)$ are infinite.*

Proof. Assume hereafter that S^+ and S^- are infinite.

- Fix an integer $j \ge 2$.
- Let $-m \in S^-$ (so $m > 0$).
- Since S^+ is infinite, there exists an infinite $S' \subseteq S^+$ all of whose elements lie in the same congruence class modulo m. Choose $n \in S'$ such that $n \ge m$ and let $e = \operatorname{lcm}\{n, m\}$.
- Fix $-M \in S^-$ such that $M > ej$.
- Finally, fix $N \in S'$ so that $N > 2eM$.
- Let α be the least positive integer such that $\alpha M \equiv bn \bmod m$ for some $b \in \mathbb{Z}$, and let β be the least such b which also satisfies $bn \ge \alpha M$. Define k to be the least nonnegative integer such that $(\beta + 1)n \le e(j + k)$.

We infer a number of immediate consequences from these definitions:

- α is the order of M in the group $(\mathbb{Z}/m\mathbb{Z})/(n \cdot \mathbb{Z}/m\mathbb{Z})$, so $\alpha \mid m$. In particular, $\alpha \le e$.
- $\beta > e/n$, since $\beta \ge \alpha M/n > \alpha ej/n \ge e/n$ and $\alpha, j \ge 1$.
- By the choice of k, $0 \le e(j + k)/n - \beta - 1 < e/n$.
- We have $\beta n - \alpha M < 2e$. Otherwise we could replace β with the smaller positive integer $\beta' = \beta - e/n$ which satisfies the same conditions.
- Because $N \equiv n \bmod m$ and $m \mid e$,

$$q = \frac{(N - eM) + e(j + k) - n}{m}$$

is an integer. In fact, q is a positive integer satisfying $q > eM/m$ and $q > j + k$.

By this last remark,

$$B = [N][-M]^e[n]^{e(j+k)/n-1}[-m]^q$$

is an element of $\mathcal{B}(\mathbb{Z}, S)$. We will show that $\mathcal{L}(B) = \{2, j + k\}$. The following observations will be used repeatedly in the remainder of the proof:

(i) $P = [n]^{e/n}[-m]^{e/m}$ is the unique atom of $\mathcal{B}(\mathbb{Z}, S)$ supported on $\{n, -m\}$. Any element of $\mathcal{B}(\mathbb{Z}, S)$ which is supported on $\{n, -m\}$ factors uniquely as a power of P.

(ii) Suppose that $X \in \mathcal{B}(\mathbb{Z}, S)$ is supported on $\{n, -M, -m\}$. We claim the following:

If $v_{[-M]}(X) > 0$ then $v_{[n]}(X) \ge \beta$.

To prove this, write $X = [n]^x[-M]^y[-m]^z$ where x, y, z are nonnegative integers satisfying $xn = yM + zm$. Then $xn \equiv yM \bmod m$, so since y is positive, $y \ge \alpha$ by the choice of α. Since we also have $xn \ge \alpha M$, it follows that $x \ge \beta$, as desired.

Now, let F be an atomic factorization of B as an element of $\mathcal{B}(\mathbb{Z}, S)$. Since $v_{[N]}(B) = 1$, there is exactly one atom A of $\mathcal{B}(\mathbb{Z}, S)$ appearing in F which satisfies $v_{[N]}(A) = 1$. F now falls into one of two cases, depending on the value of $v_{[-M]}(A)$.

CASE 1: Suppose that $v_{[-M]}(A) = e$. Then B/A is supported on $\{n, -m\}$ so by (i), B/A factors uniquely as P^r.

Let us compute r. Write $A = [N][-M]^e[n]^f[-m]^g$ where f, g are nonnegative integers and $N + fn = eM + gm$. Since $N \equiv n \bmod m$ and $m \mid e$, we have $(f + 1)n \equiv 0 \bmod m$. Hence $e \mid (f + 1)n$, so $f \geq e/n - 1$. We claim that equality holds: $f \geq e/n$ and $N > 2eM$ together would imply $gm = (N - eM) + fn > e$. But then $g > e/m$, so P would be a proper divisor of A, contradicting the hypothesis that A is irreducible. Thus $v_{[n]}(A) = e/n - 1$ and

$$r = \frac{v_{[n]}(B/A)}{e/n} = \frac{e(j+k)/n - 1 - (e/n - 1)}{e/n} = j + k - 1.$$

So in this case, $F : B = AP^{j+k-1}$ and $|F| = j + k$.

To show that $j + k \in \mathcal{L}(B)$, it remains to show that such a factorization F actually exists. That is, we must show that there is an atomic factor A_1 of B which satisfies the conditions of this case. By the preceding arguments, the only candidate is $A_1 = [N][-M]^e[n]^{e/n-1}[-m]^{q-(j+k-1)}$. Since $q > j + k$ by an earlier remark, $A_1 \in \mathcal{B}(\mathbb{Z}, S)$. Suppose that $X \in \mathcal{B}(\mathbb{Z}, S)$ is a divisor of A_1 such that $v_{[N]}(X) = 0$. Since $v_{[n]}(A_1) < e/n \leq \beta$, observation (ii) guarantees that $v_{[-M]}(X) = 0$. Thus by (i), $X = P^d$ for some $d \geq 0$, but $v_{[n]}(X) < e/n$, so $d = 0$, $X = 1$, and A_1 is irreducible.

CASE 2: Suppose instead that $v_{[-M]}(A) < e$. Then B/A has an irreducible factor Y such that $v_{[-M]}(Y) > 0$. Since we also have $v_{[N]}(Y) = 0$, (ii) implies that $v_{[n]}(AY) \geq \beta$, so

$$v_{[n]}(B/AY) \leq \frac{e(j+k)}{n} - 1 - \beta < e/n.$$

Moreover, $v_{[N]}(B/AY) = 0$, so (ii) guarantees that $v_{[-M]}(B/AY) = 0$, and hence by (i), $B/AY = P^d$ for some nonnegative integer d. However, since $v_{[n]}(B/AY) < e/n$, we must have $d = 0$. So $B/A = Y$ is irreducible, $F : B = A(B/A)$ and any factorization obtained in this case has length 2.

It remains to show that such a factorization exists. Let $r = (\beta n - \alpha M)/m$. By the definitions of α and β, r is a positive integer, so $Y' = [-M]^\alpha[n]^\beta[-m]^r$ is an element of $\mathcal{B}(\mathbb{Z}, S)$. By earlier remarks, $\alpha \leq e$, $\beta \leq e(j + k)/n - 1$, and $r < 2e/m \leq eM/m < q$. It follows that Y' divides B, so there is an atom A_2 dividing B/Y' such that $v_{[N]}(A_2) = 1$. Since $\alpha > 0$, we have $v_{[-M]}(A_2) < e$, so $A = A_2$ satisfies the conditions of this case and it follows that B has a factorization of length 2.

By Case 1 and Case 2, we have $\mathcal{L}(B) \subseteq \{2, j+k\}$. Conversely, we have shown that appropriate factorizations exist in both cases, so $\mathcal{L}(B) = \{2, j + k\}$. From the length set, we calculate $\Delta(B) = \{j + k - 2\}$ and, since distinct factorizations of length 2 and $j + k \geq 2$ cannot share any common factors, $c(B) = j + k$. Since j may be taken arbitrarily large, the result follows. □

3. When either S^+ or S^- is finite

The main theorem of this section is the following:

Theorem 3.1. *Suppose that $S' = \{-m_r, \ldots, -m_1, n_1, \ldots, n_k\} \subseteq S$ where $m_i, n_i > 0$ for all i and $-m_r = \min S'$. If B is an element of $\mathcal{B}(\mathbb{Z}, S)$ supported on S', then*

$$\max \Delta(B) \leq m_r(m_r + r^2) - 2$$

(if $\Delta(B)$ is nonempty) and

$$c(B) \leq m_r(m_r + r^2).$$

The above is sufficient to complete the proof of Theorem 1.1. If S is bounded from either above or below, we may assume that S^- is finite, possibly after replacing S with $-S$. From the above it follows then that if $\Delta\big(\mathcal{B}(\mathbb{Z}, S)\big)$ is nonempty, then it is bounded above by $M(M + |S^-|^2) - 2$ where $-M = \min(S^-)$. Similarly, we obtain $c\big(\mathcal{B}(\mathbb{Z}, S)\big) \leq M(M + |S^-|^2)$.

The proof is based on a result of Lambert, which we now present adapted to the language of block monoids. Since the proof makes use of elements of $\mathcal{F}(\mathbb{Z})$ outside $\mathcal{B}(\mathbb{Z})$, we define some convenient notation. As before, we let $\theta : \mathcal{F}(\mathbb{Z}) \to \mathbb{Z}$ be the obvious homomorphism (so that $\mathcal{B}(\mathbb{Z}) = \ker \theta$). We define another homomorphism $\varphi : \mathcal{F}(\mathbb{Z}) \to \mathbb{Z}_{\geq 0}$ by

$$\varphi : X \mapsto \sum_{n \in S^+} v_{[n]}(X)$$

If $X, Y \in \mathcal{F}(\mathbb{Z})$, then we write $X \leq Y$ whenever $v_{[z]}(X) \leq v_{[z]}(Y)$ for all $z \in \mathbb{Z}$.

Theorem 3.2 (*Lambert [13]*). *If S^- is finite with $\min(S^-) = -M$, then $\varphi(A) \leq M$ for all atoms A of $\mathcal{B}(\mathbb{Z}, S)$.*

Proof. Let $A = [n_1]^{a_1} \cdots [n_k]^{a_k}[-m_1]^{b_1} \cdots [-m_r]^{b_r}$ be an atom of $\mathcal{B}(\mathbb{Z}, S)$. An element X of $\mathcal{F}(\mathbb{Z})$ is called a *positive subsequence* of A if $X \le A$ and $\theta(X) \ge 0$.

We construct a strictly increasing chain $1 = X_0 < \cdots < X_{\varphi(A)} = A$ of positive subsequences of A. Given X_i when $0 < i < \varphi(A)$, consider all positive subsequences X of A such that $\varphi(X) = \varphi(X_i)+1$. Such an X must exist, since $\varphi(X_i) < \varphi(A)$ implies that there is some j with $X_i[n_j] \le A$, and $\varphi(X_i[n_j]) = \varphi(X_i)+1$. Let X_{i+1} be such an X with $\theta(X_{i+1})$ as small as possible.

We claim that $\theta(X_i) < M$ for all i. This is clear when $i = 0$ or $i = \varphi(A)$. When $0 < i < \varphi(A)$, then since X_i is a positive subsequence, there exists j such that $X_i[-m_j] \le A$. If we had $\theta(X_i) \ge M$, then $X_i[-m_j]$ would be a positive subsequence of A, but then $0 \le \theta(X_i[-m_j]) < \theta(X_i)$, which contradicts the choice of X_i.

If $0 < i < \varphi(A)$, then $\theta(X_i) > 0$ as otherwise X_i would be a proper nontrivial divisor of A in $\mathcal{B}(\mathbb{Z}, S)$ which contradicts the hypothesis that A is irreducible. Thus the set $\{\theta(X_1), \dots, \theta(X_{\varphi(A)-1})\}$ is a set of size $\varphi(A) - 1$ contained in $\{1, \dots, M-1\}$. Hence, if $\varphi(A) > M$ there would exist distinct i, j with $i < j$ such that $\theta(X_i) = \theta(X_j)$. But then $A' = X_j/X_i \in \mathcal{B}(\mathbb{Z}, S)$ and A' is a proper nontrivial divisor of A, which is impossible. Thus $\varphi(A) \le M$ as desired. □

Proof of Theorem 3.1. We proceed by induction on $\max \mathcal{L}(B)$. In the base case B is irreducible, so $\Delta(B)$ is empty and $c(B) = 0$.

Suppose then that B is not irreducible, let $L = \max \mathcal{L}(B) \ge 2$ and let $\ell \in \mathcal{L}(B)$. Write $B = [n_1]^{d_1} \cdots [n_k]^{d_k} [-m_1]^{e_1} \cdots [-m_r]^{e_r}$. Fix an atomic factorization $F^* : B = A_1 \cdots A_L$ of maximal length, and let $F_* : B = C_1 \cdots C_\ell$ be any atomic factorization of B. For i, j, set $e_{ij} = v_{[m_j]}(A_i)$.

Claim: There is an index f and a subset $I \subseteq \{1, \dots, \ell\}$ of cardinality at most $m_r + r^2$ such that A_f divides $\prod_{i \in I} C_i$.

Proof of the Claim. If $r > \ell$, then by taking any index f and $I = \{1, \dots, \ell\}$ satisfies the claim. Hence, let us assume $r \le \ell$. We will find f with $1 \le f \le L$ and $e_{fj}/e_j \le r/L$ for all j. Note that

$$\sum_{i=1}^{L} \max_j \frac{e_{ij}}{e_j} \le \sum_{i=1}^{L} \sum_{j=1}^{r} \frac{e_{ij}}{e_j} = \sum_{j=1}^{r} \left(\frac{1}{e_j} \sum_{i=1}^{L} e_{ij} \right) = r$$

since $\sum_i e_{ij} = e_j$. If for all i we had $\max_j (e_{ij}/e_j) > r/L$, then

$$\sum_{i=1}^{L} \max_j \frac{e_{ij}}{e_j} > \sum_{i=1}^{L} \frac{r}{L} = r$$

which is a contradiction, so a desired f must exist. After reordering, we may assume $f = L$.

If $J \subseteq \{1, \dots \ell\}$ we set $C_J = \prod_{i \in J} C_i$. By the pigeonhole principle and the fact that $r \le \ell$, for each $j = 1, \dots, r$ there is a subset $I_j \subseteq \{1, \dots, \ell\}$ with $|I_j| \le r$ such that $v_{[-m_j]}(C_{I_j}) \ge re_j/\ell$. But since $\ell \le L$, this means that $v_{[-m_j]}(C_{I_j}) \ge e_{fj}$. Furthermore, we know by Theorem 3.2 that $\varphi(A_f) \le m_r$, so we may choose $I_0 \subseteq \{1, \dots, \ell\}$ such that $|I_0| \le m_r$ and $v_{[n]}(A_f) \le v_{[n]}(C_{I_0})$ for all $n \in S^+$. Finally, take $I = I_0 \cup I_1 \cup \cdots \cup I_r$. Then by construction, C_I is divisible by A_f. Moreover, $|I| \le |I_0| + \sum_j |I_j| \le m_r + r^2$. This completes the proof of the claim and after reordering the C_i's, we may assume $I = \{1, \dots, q\}$ for some

$$q \le m_r + r^2. \tag{$*$}$$

We have

$$B = A_1 \cdots A_L = C_1 \cdots C_\ell = (A_L D_1 \cdots D_t)(C_{q+1} \cdots C_\ell).$$

where $A_L D_1 \cdots D_t$ is a factorization of $C_1 \cdots C_q$ in which A_L appears. We denote the factorization $A_L D_1 \cdots D_t C_{q+1} \cdots C_\ell$ by F.

Applying Theorem 3.2 to the identity $\varphi(A_L D_1 \cdots D_t) = \varphi(C_1 \cdots C_q)$, we find

$$t + 1 \le \sum_{i=1}^{q} \varphi(C_i) \le q m_r \le m_r(m_r + r^2) \tag{$**$}$$

and $q \le \varphi(A_L) + \sum_{i=1}^{t} \varphi(D_i) \le (t+1)m_r$.

We now demonstrate the desired bounds on the Delta set. Note that if $q = 1$ or $t + 1 = 1$ then $F = F_*$. Otherwise, $|\,|F| - |F_*|\,| = |(t+1) - q| \le \max\{t-1, q-2\} \le m_r(m_r + r^2) - 2$ (that $t - 1 \le m_r(m_r + r^2) - 2$ follows from ($**$), similarly $q - 2 \le m_r(m_r + r^2) - 2$ by ($*$)). Let $\tilde{F}$ and $\tilde{F}^*$ denote the factorizations of B/A_L obtained from F and F^* (the long factorization of B) by removing the irreducible factor A_L. By induction, if $\Delta(B/A_L)$ is nonempty, $\max \Delta(B/A_L) \le m_r(m_r + r^2) - 2$. Hence, there is an increasing sequence $|\tilde{F}| = \ell_0, \dots, \ell_s = |\tilde{F}^*|$ of lengths of atomic factorizations of B/A_L such that $\ell_{i+1} - \ell_i \le m_r(m_r + r^2) - 2$ for all $i < s$ (if $\Delta(B/A_L)$ is empty, it follows that F and F^* have the same length). By concatenating the corresponding factorizations with A_L, we find $\{\ell, |F| = \ell_0 + 1, \dots, \ell_s + 1\} \subseteq \mathcal{L}(B)$. As listed, the consecutive terms differ by no more than $m_r(m_r + r^2)$, with the first term being $|F_*| = \ell$ and the last being $|F^*| = \ell_s + 1$. Hence if $\Delta(B)$ is nonempty, the arbitrary choice of F_* has shown $\max \Delta(B) \le m_r(m_r + r^2) - 2$.

To show the bound on the catenary degree, we again pass to factorizations of B/A_L and use the induction hypothesis. Thus we have an $m_r(m_r + r^2)$-chain from $\tilde{F}$ to $\tilde{F}^*$, and this lifts to a $m_r(m_r + r^2)$-chain from F to F^* after multiplying every term in the chain by A_L. Finally, $d(F_*, F) \le t + 1 \le m_r(m_r + r^2)$, so appending F_* to this chain proves $c(B) \le m_r(m_r + r^2)$. □

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