On the Leibniz (co)homology of the Lie algebra of the Euclidean group

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1. Introduction

Recall that the Euclidean group $E(n)$ consists of all distance-preserving transformations of the Euclidean $n$-space; i.e. all transformations $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\varphi(v) = v_0 + Ov,$$

where $O$ is a $n \times n$ orthogonal matrix and $v_0$ is a fixed element of $\mathbb{R}^n$. For $n \geq 3$, let $h_n$ denote the Lie algebra of the affine orthogonal Lie group (the connected component of $E(n)$). Then $h_n$ is the Lie algebra of the Euclidean group.

The Leibniz (co)-homology of $h_n$ suggests several interesting questions in many fields in Physics. For instance, Schroek [14] has proven that the discovery of Leibniz homology opens new directions in Quantum Mechanics to unsolved problems related to the Poincaré group; more precisely, the determination of the Leibniz homology of $h_3$ provides a considerable amount of information to the Leibniz homology of the Lie algebra of the Poincaré group which is a double cover of $so(3)$; thus provides additional inputs for the resolution of these problems (cf. [14]).

The tools used to calculate the Leibniz algebra homology of $h_n$, i.e. $HL^*(h_n)$ include the Hochschild–Serre spectral sequence for Lie-algebra (co)homology, the Pirashvili spectral sequence for Leibniz homology, and the identification of certain orthogonal invariants of $h_n$ which are not detectable from the exterior algebra used in the computation of Lie-algebra homology. The method used to compute these invariants improves Lodder’s calculations in [11]. We prove in Section 5 that there is an isomorphism of graded vector spaces

$$HL^*(h_n) \cong (\mathbb{R} \otimes \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n),$$

where $\langle \tilde{\alpha}_n \rangle$ denotes a 1-dimensional vector space in degree $n$ on

$$\tilde{\alpha}_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \frac{\partial}{\partial x^{\sigma(1)}} \otimes \frac{\partial}{\partial x^{\sigma(2)}} \otimes \frac{\partial}{\partial x^{\sigma(3)}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\sigma(n)}}$$

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1 The author is very indebted to his advisor Pr. Jerry Lodder, and the results in this paper are included in his Ph.D. dissertation (2010).
and $T^*(\tilde{\gamma}_n)$ denotes the tensor algebra on the $(n - 1)$-degree generator
\[
\tilde{\gamma}_n = \sum_{1 \leq i < j \leq n, \sigma \in S_{n-2}} (-1)^{i+j+1} X_{ij} \otimes \left( \sum_{\alpha \in n} \frac{\partial}{\partial x^{\alpha(i)}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha(j)}} \right) \otimes X_{ij}.
\]

We show in Section 4 that $\tilde{\alpha}_n$ and $\tilde{\gamma}_n$ are $h_n$-invariant. Dually for cohomology, there is an isomorphism of dual Leibniz algebras
\[
HL^* (h_n) \cong (R \oplus \tilde{\alpha}_n^d) \otimes T^* (\tilde{\gamma}_d),
\]
where
\[
\tilde{\alpha}_n^d = \sum_{\sigma \in S_n} \text{sgn} (\sigma) dx^{\sigma(1)} \otimes dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)},
\]
and
\[
\tilde{\gamma}_n^d = \sum_{1 \leq i < j \leq n, \sigma \in S_{n-2}} (-1)^{i+j} X_{ij}^* \otimes dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(j)} \otimes \cdots \otimes dx^{\sigma(n)}.
\]

The new invariant $\tilde{\gamma}_n^d$ is different from Lodder's invariants in [11]. It can be viewed as factored from the volume element $\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$, where certain factors from $\mathcal{J}^n$ are replaced with their corresponding elements in $so(n)$ via an isomorphism $\mathcal{J}^n \xrightarrow{\sim} so(n)$ of $so(n)$-modules, and constitutes the core of this paper.

2. The Lie algebra of the Euclidean group

In this paper, we treat at once $h_n$ for the orthogonal Lie algebras of type $B_l$ and $D_l$.

Assume that $\mathbb{R}^n$ is given the coordinates $(x_1, x_2, \ldots, x_n)$, and let $\frac{\partial}{\partial x^i}$ be the unit vector fields parallel with the $x_i$ axes respectively. It is easy to show that the Lie algebra generated by the family $B_1$ below the vector fields (endowed with the bracket of vector fields) is isomorphic to the real orthogonal Lie algebra $so(n)$:
\[
B_1 = \left\{ X_{ij} := -x_i \frac{\partial}{\partial x^j} + x_j \frac{\partial}{\partial x^i} \; : \; 1 \leq i \neq j \leq n \right\}.
\]

Let $\mathcal{J}_n$ denote the Abelian Lie algebra with vector space basis $B_2 = \left\{ \frac{\partial}{\partial x^i} \; : \; 1 \leq i \leq n \right\}$. The Lie algebra $h_n$ has an $\mathbb{R}$-vector space basis $B_1 \cup B_2$ and there is a short exact sequence of Lie algebras [6, p. 203]
\[
0 \longrightarrow \mathcal{J}_n \longrightarrow h_n \longrightarrow so(n) \longrightarrow 0
\]
where $i$ is the inclusion map and $\pi$ is the projection
\[
h_n \longrightarrow (h_n/\mathcal{J}_n) \cong so(n).
\]

Note that $\mathcal{J}_n$ is an Abelian ideal of $h_n$ with $\mathcal{J}_n$ acting on $h_n$ via the bracket of vector fields. The bracket on $h_n \cong \mathcal{J}_n \oplus so(n)$ can be defined by
\[
[[m_1, x_1], (m_2, x_2)] = [[m_1, x_2] + [x_1, m_2], [x_1, x_2]] \quad [8].
\]

3. The Lie algebra homology of $h_n$

For any Lie algebra $g$ over a ring $k$ and $V$ any $g$-module, the Lie algebra homology of $g$ with coefficients in the module $V$, written $H^*_\text{Lie}(g; V)$, is the homology of the Chevalley–Eilenberg complex $V \otimes \wedge^*(g)$, namely
\[
V \leftarrow d \rightarrow V \otimes g^1 \leftarrow d \rightarrow V \otimes g^2 \leftarrow d \rightarrow \cdots \leftarrow V \otimes g^{n-1} \leftarrow d \rightarrow V \otimes g^n \leftarrow \cdots
\]
where $g^n$ is the $n$th exterior power of $g$ over $k$, and where
\[
d(v \otimes g_1 \wedge \cdots \wedge g_n) = \sum_{1 \leq i \leq n} (-1)^i [v, g_i] \otimes g_1 \wedge \cdots \hat{g}_i \cdots \wedge g_n + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} v \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \hat{g}_i \cdots \hat{g}_j \cdots \wedge g_n \quad [1]
\]
where \( \hat{g}_i \) means that the variable \( g_i \) is deleted. In particular if \( V = g \), we obtain the Lie algebra homology with coefficients in the adjoint representation, written \( H_{Lie}^n(g; g) \). Also taking \( V = k \) the trivial module, we identify \( 1 \otimes g_1 \wedge \cdots \wedge g_n \) with \( g_1 \wedge \cdots \wedge g_n \) and have

\[
\begin{array}{cccccccc}
1 & 0 & g^1 \wedge & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & d & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & d & \cdots & \cdots & \cdots & \cdots \\
& & & & d & \cdots & \cdots & \cdots \\
& & & & & d & \cdots & \cdots \\
& & & & & & \cdots & \cdots \\
\end{array}
\]

with

\[
d(g_1 \wedge \cdots \wedge g_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \cdots \wedge \hat{g}_j \cdots \wedge g_n.
\]

For each \( n \), we have the canonical projection \( g \otimes g^{\wedge n} \rightarrow g^{\wedge n+1} \). This gives a map of chain complexes \( g \otimes \wedge^*(g) \rightarrow \wedge^{*+1}(g) \) and thus induces a \( k \)-linear map on homology

\[
H_{Lie}^n(g; g) \rightarrow H_{Lie}^{n+1}(g; k).
\]

For a \( (\text{right}) \) \( g \)-module \( M \), the module of invariants \( M^g \) is defined as

\[M^g = \{ m \in M \mid g(m) = 0 \text{ for all } g \in g \} \]

The Lie algebra \( so(n) \) also acts on \( J_n \) and on \( h_n \) via the bracket of vector fields. This action is extended to \( J_n^\wedge k \) by

\[
[a_1 \wedge a_2 \wedge \cdots \wedge a_k, X] = \sum_{i=1}^k a_1 \wedge a_2 \wedge \cdots \wedge [a_i, X] \wedge \cdots \wedge a_k
\]

for \( a_i \in J_n \), \( X \in so(n) \), and the action of \( so(n) \) on \( h_n \otimes J_n^\wedge k \) is given by

\[
[h \otimes a_1 \wedge a_2 \wedge \cdots \wedge a_k, X] = [h, X] \otimes a_1 \wedge \cdots \wedge a_k + \sum_{i=1}^k h \otimes a_1 \wedge a_2 \wedge \cdots \wedge [a_i, X] \wedge \cdots \wedge a_k
\]

for \( h \in h_n \).

In this paper the calculations are done with \( k = R \). The following lemma is the main result of this section.

**Lemma 3.1.** There are natural vector space isomorphisms

\[
H_{Lie}^n(h_n; R) \cong H_{Lie}^n(so(n); R) \otimes [\wedge^*(J_n)]^{so(n)},
\]

\[
H_{Lie}^n(h_n, h_n) \cong H_{Lie}^n(so(n); R) \otimes H_{Lie}^n([h_n \otimes \wedge^*(J_n)]^{so(n)}; R).
\]

**Proof.** We use Lodder's procedure [11] which consists in applying the homological version of the Hochschild–Serre spectral sequence (for its cohomological version, see [4]). □

4. New invariants for the orthogonal Lie algebras

In this section, we prove some lemmas that determine the modules of invariants of \( \wedge^* J_n \) and \( h_n \otimes \wedge^* J_n \) under the action of \( so(n) \).

**Lemma 4.1.** There is a vector space isomorphism

\[
[\wedge^*(J_n)]^{so(n)} = R \oplus \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \right) \text{ for } n \geq 3
\]

**Proof.** It is enough to show that

\[
[\wedge_{\geq 0}^n J_n]^{so(n)} = R, \quad [\wedge^1_n J_n]^{so(n)} = \{0\}, \quad [\wedge_n^k J_n]^{so(n)} = \{0\} \text{ for } k \neq 0, 1, n \text{ and}
\]

\[
[\wedge_{\geq 1}^n J_n]^{so(n)} = \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \right).
\]

Indeed, that \( [\wedge_{\geq 0}^n J_n]^{so(n)} = R \) is clear. Also a straightforward verification shows that

\[
[\wedge_{\geq 1}^n J_n]^{so(n)} = \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \right).
\]

To show that \( [\wedge_{\geq 1}^n J_n]^{so(n)} = \{0\} \), let \( \omega_1 = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} \in J_n \) and assume without loss of generality that \( c_{i_0} \neq 0 \) for some \( i_0 \in \{1, 2, \ldots, n\} \). Then

\[
\left[ \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} - x_{i_0} \frac{\partial}{\partial x_n} + x_n \frac{\partial}{\partial x_{i_0}} \right] = -c_{i_0} \frac{\partial}{\partial x_n} + c_n \frac{\partial}{\partial x_{i_0}} \neq 0,
\]
Thus $\omega_1 \notin [\mathfrak{g}_n]^{so(n)}$. It follows that $[\mathfrak{g}_n]^{so(n)} = 0$. Next, we show by induction on $n$ that $[\mathfrak{g}_n]^{so(n)} = 0$ for $k \neq 0$, $1$, $n$. For $n = 3$, $\mathfrak{g}_3 = \left\{ \frac{1}{\sqrt{2}} x, \frac{1}{\sqrt{2}} y, \frac{1}{2} z \right\}$ and $so(3) = \langle X_{12}, X_{13}, X_{23} \rangle$. By direct calculation, $[\mathfrak{g}_3]^{\times 2, so(3)} = 0$. By the inductive hypothesis, suppose

$$[\mathfrak{g}_{n-1}]^{so(n-1)} = 0 \quad \text{for } k \neq 0, 1, n - 1$$

and let $z \in [\mathfrak{g}_n]^{so(n)}$ with $k \neq 0, 1, n$ fixed. Then $z = c_1 z_1 + c_2 z_2 \wedge \frac{\partial}{\partial x^1}$, where $z_1 \in \mathfrak{g}_n^{so(n-1)}$, $z_2 \in \mathfrak{g}_n^{so(n-1)}$, and $c_1, c_2 \in \mathbb{R} - \{0\}$. If $k = n - 1$, then $z_1 = \frac{d_n}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$ and $z_2 = \sum_{i=1}^{n-1} \frac{d_i}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$ for some $d_i$’s constants. Choose first $\mathfrak{x} = -x_1 \frac{\partial}{\partial x^1} + x_n \frac{\partial}{\partial x^n}$ for $i = 2, \ldots, n - 1$ to find $d_i = 0$ for all $i$’s and thus $z = 0$. If $k \neq n - 1$, let $\mathfrak{x} \in so(n-1) \subseteq so(n)$ as a Lie subalgebra, we have

$$0 = [z, \mathfrak{x}] = \left[ c_1 z_1 + c_2 z_2 \wedge \frac{\partial}{\partial x^1}, \mathfrak{x} \right] = c_1 [z_1, \mathfrak{x}] + c_2 \left[ z_2 \wedge \frac{\partial}{\partial x^1}, \mathfrak{x} \right]$$

$$= c_1 [z_1, \mathfrak{x}] + c_2 [z_2, \mathfrak{x}] \wedge \frac{\partial}{\partial x^1} + c_2 z_2 \wedge \frac{\partial}{\partial x^1} (\mathfrak{x}),$$

thus as $\frac{\partial}{\partial x^1} (\mathfrak{x}) = 0$, we have $c_1 [z_1, \mathfrak{x}] + c_2 [z_2, \mathfrak{x}] \wedge \frac{\partial}{\partial x^1} = 0$. If non-zero, the terms $[z_1, \mathfrak{x}]$ and $[z_2, \mathfrak{x}] \wedge \frac{\partial}{\partial x^1}$ are linearly independent since none of the terms of $[z_1, \mathfrak{x}]$ contains the vector field $\frac{\partial}{\partial x^1}$ in its expression, a contradiction. Thus $[z_1, \mathfrak{x}] = 0 = [z_2, \mathfrak{x}]$. This implies $z_1 \in [\mathfrak{g}_n]^{so(n-1)} = 0$, $z_2 \in [\mathfrak{g}_{n-1}]^{so(n-1)} = 0$; hence $z = 0$. \hfill \Box

**Lemma 4.2.** Setting $\sigma_{ij} := (i, j, \ldots, i, j, \ldots, n)$ the permutation, we have

$$[so(n) \otimes \mathfrak{g}_n^{so(n)}]^{so(n)} = \begin{cases} \langle \rho_n \rangle, & \text{if } k = 2 \\ \langle \gamma_n \rangle, & \text{if } k = n - 2 \\ 0, & \text{else} \end{cases}$$

with

$$\rho_n = \sum_{1 \leq i < j \leq n} X_{ij} \otimes \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad \gamma_n = \sum_{1 \leq i < j \leq n} sgn(\sigma_{ij}) X_{ij} \otimes \frac{\partial}{\partial x^i} \wedge \cdots \wedge \frac{\partial}{\partial x^i} \cdots \wedge \frac{\partial}{\partial x^n}.$$

**Proof.** Recall that the representations $\mathfrak{g}_n$ and $so(n)$ are respectively the standard $n$-dimensional representation and the adjoint representation of $so(n)$. Now $so(n)$ is isomorphic to $\mathfrak{g}_n^{so(n)}$ because $so(n)$ identifies with skew-symmetric $n$-by-$n$ matrices, which identifies with the second exterior power of $n$-dimensional vectors. Note also that as a representation of $so(n)$, $\mathfrak{g}_n$ is isomorphic to its dual because $so(n)$ preserves the dot product. So

$$[so(n) \otimes \mathfrak{g}_n^{so(n)}]^{so(n)} \cong [\mathfrak{g}_n^{so(n)}]^{so(n)} \cong [\mathfrak{g}_n^{\times 2}]^{\times 2} \otimes \mathfrak{g}_n^{so(n)} \cong [\text{Hom}_{\mathbb{R}}(\mathfrak{g}_n^{\times 2}, \mathfrak{g}_n^{so(n)})]^{so(n)} = \text{Hom}_{so(n)}(\mathfrak{g}_n^{\times 2}, \mathfrak{g}_n^{so(n)}) \quad [2, \text{p. 8}].$$

Now, it is known [3, Theorems 19.2 and 19.14] that the $\mathfrak{g}_n^{so(n)}$ are all irreducible representations of $so(n)$ (except for $k = \frac{n}{2}$ when $n$ is even) and they are nonisomorphic, except that $\mathfrak{g}_n^{so(n)}$ is isomorphic to $\mathfrak{g}_n^{so(n)}$. Moreover, the two irreducible summands of $\mathfrak{g}_n^{so(n)}$, in the case $n$ is even, are not isomorphic to any of the irreducible representations for other values of $k$, and are not isomorphic to each other. Consequently, we deduce that the space $\text{Hom}_{so(n)}(\mathfrak{g}_n^{\times 2}, \mathfrak{g}_n^{so(n)})$ is zero unless $k = 2$ or $k = n - 2$, in which case it is 1-dimensional, unless $n = 4$, in which case it is 2-dimensional.

The generators can be made explicit by induction on $n$. It is easy to check the result for $n = 3$ and $n = 4$, by direct calculation. Suppose the result true at the level $n - 1$. Since $so(n)$ is a simple Lie algebra, we have $[so(n)]^{so(n)} = 0$. Now let $B_1$ be the vector space basis of $so(n - 1)$, then it is clear that $B_1 \cup \{ X_{ii}, i = 1, \ldots, n - 1 \}$ is the vector space basis of $so(n)$. Let $S' = \{ x^1, x^2, \ldots, x^n \}$ and $S'' = \{ x^1, x^2, \ldots, x^{n-1} \}$. A vector space basis of $(so(n) \otimes \mathfrak{g}_n^{so(n)}) / (so(n - 1) \otimes \mathfrak{g}_n^{so(n - 1)})$ is given by the families of elements:

1. $e \otimes \frac{\partial}{\partial x^i} \wedge \cdots \wedge \frac{\partial}{\partial x^{i-1}}, \ e \in B_1, \ z^i \in S''$
2. $X_{ii} \otimes \frac{\partial}{\partial x^i} \wedge \cdots \wedge \frac{\partial}{\partial x^{i-1}}, \ z^i \in S''$

Given $\omega \in [so(n) \otimes \mathfrak{g}_n^{so(n)}]^{so(n)}$, let $\omega = u + v$ where

$$u \in (so(n - 1) \otimes \mathfrak{g}_n^{so(n - 1)}), \quad v \in (so(n) \otimes \mathfrak{g}_n^{so(n)}) / (so(n - 1) \otimes \mathfrak{g}_n^{so(n - 1)}).$$

Write $v = S_1 + S_2$ with

$$S_1 = \sum_{z^1, \ldots, z^{k-1} \in S''} c_{1, \ldots, k-1} e \otimes \frac{\partial}{\partial x^i} \wedge \cdots \wedge \frac{\partial}{\partial x^{i-1}}.$$
and
\[ S_2 = \sum_{z^1, \ldots, z^{k-1} \in S, \; 1 \leq j < n} c_{2, \ast} (X_{in}) \otimes \frac{\partial}{\partial z^1} \land \frac{\partial}{\partial z^2} \land \cdots \land \frac{\partial}{\partial z^k}. \]

For all \( \mathcal{X} \in \mathfrak{so}(n-1) \subseteq \mathfrak{so}(n) \), as a Lie subalgebra, we have
\[ 0 = [\mathcal{X}, \; \omega] = [\mathcal{X}, \; u] + [\mathcal{X}, \; v]. \]
Clearly \([\mathcal{X}, \; u]\) and \([\mathcal{X}, \; v]\) are zero; otherwise both are non-zero and not linearly independent; a contradiction because \([\mathcal{X}, \; v]\) contains the vector field \( \frac{\partial}{\partial x^j} \) in its expression whereas \([\mathcal{X}, \; u]\) does not. Thus \([\mathcal{X}, \; u] = 0\) and \( u \in [\mathfrak{so}(n-1) \otimes \mathfrak{g}_{n-1}]^{e_{\mathcal{X}} - \mathcal{X}} \).

If \( k = 2, \; u = c \rho_{n-1} \) for some constant \( c \in \mathbb{R} \). We have
\[ S_1 = \sum_{z \in S'} c_{1, \ast} e \otimes \frac{\partial}{\partial x^\nu} \land \frac{\partial}{\partial z} \quad \text{and} \quad S_2 = \sum_{z^1, z^2 \in S, \; 1 \leq j < n} c_{2, \ast} X_{jn} \otimes \frac{\partial}{\partial z^1} \land \frac{\partial}{\partial z^2}, \]
and since \( 0 = [\mathcal{X}, \; v] = [\mathcal{X}, \; S_1 + S_2] = [\mathcal{X}, \; S_1] + [\mathcal{X}, \; S_2] \), we must have \( [\mathcal{X}, \; S_1] = [\mathcal{X}, \; S_2] = 0 \). Now
\[ 0 = [\mathcal{X}, \; S_1] = - \sum_{z \in S'} c_{1, \ast} \left[ [\mathcal{X}, \; e] \otimes \frac{\partial}{\partial z} \right] \land \frac{\partial}{\partial x^\nu}, \]
thus
\[ \sum_{z \in S'} c_{1, \ast} \left[ [\mathcal{X}, \; e] \otimes \frac{\partial}{\partial z} \right] = 0. \]
And thus \( \sum_{z \in S'} c_{1, \ast} e \otimes \frac{\partial}{\partial x^\nu} \in [\mathfrak{so}(n-1) \otimes \mathfrak{g}_{n-1}]^{e_{\mathcal{X}} - \mathcal{X}} = \{ 0 \}, \) therefore \( S_1 = 0 \).

In particular for \( \mathcal{X} = X_{1n} := -x_1 \frac{\partial}{\partial x^1} + x_n \frac{\partial}{\partial x^n} \in \mathfrak{so}(n) \); we have
\[ 0 = [\mathcal{X}, \; \omega] = [\mathcal{X}, \; u] + [\mathcal{X}, \; S_2] = \sum_{2 \leq j \leq n-1} c X_{in} \otimes \frac{\partial}{\partial x^j} \land \frac{\partial}{\partial x^j} + \sum_{2 \leq j \leq n-1} c X_{ij} \otimes \frac{\partial}{\partial x^j} \land \frac{\partial}{\partial x^j} \]
\[ + \sum_{z^1, z^2 \in S, \; 1 \leq j \leq n-1} c_{2, \ast} X_{ijn} \otimes \frac{\partial}{\partial z^1} \land \frac{\partial}{\partial z^2} \quad \text{and} \quad \sum_{z^1, z^2 \in S, \; 1 \leq j \leq n-1} c_{2, \ast} X_{jn} \otimes \left[ X_{in}, \frac{\partial}{\partial z^1} \land \frac{\partial}{\partial z^2} \right] \]
\[ + \sum_{z \in S', \; 1 \leq j \leq n-1} c_{3, \ast} X_{ijn} \otimes \frac{\partial}{\partial z} \land \frac{\partial}{\partial x^\nu} + \sum_{z \in S', \; 1 \leq j \leq n-1} c_{3, \ast} X_{jn} \otimes \left[ X_{in}, \frac{\partial}{\partial z} \land \frac{\partial}{\partial x^\nu} \right] = 0. \]

Clearly, all the coefficients except \( c_{3, \ast}^{1, \ast} \) appear in this summation and the only basis vectors repeated are \( X_{jn} \otimes \frac{\partial}{\partial x^j} \land \frac{\partial}{\partial x^j} \) and \( X_{ij} \otimes \frac{\partial}{\partial x^j} \land \frac{\partial}{\partial x^j} \) for \( 2 \leq j \leq n-1 \). It follows that all the \( c_{2, \ast} \) are zero and all the \( c_{3, \ast} \) are zero except \( c_{3, \ast}^l \) with \( j \neq 1 \) which satisfy \( c_{3, \ast}^l - c = 0 \). So
\[ \omega = c \rho_{n-1} + c \sum_{z \in S', \; 2 \leq j \leq n-1} X_{jn} \otimes \frac{\partial}{\partial x^j} \land \frac{\partial}{\partial x^j} + c_{3, \ast}^{1, \ast} X_{1n} \otimes \frac{\partial}{\partial x^1} \land \frac{\partial}{\partial x^1}. \]
Applying finally \( \mathcal{X} = X_{2n} \) to the condition \([\mathcal{X}, \; \omega] = 0\) gives \( c_{3, \ast}^{1, \ast} = c \). Hence \( \omega = c \rho_{n} \).

If \( k = n - 2, \; u = 0 \) by inductive hypothesis. Since
\[ 0 = [\mathcal{X}, \; v] = [\mathcal{X}, \; S_1 + S_2] = [\mathcal{X}, \; S_1] + [\mathcal{X}, \; S_2], \]
we must have \( [\mathcal{X}, \; S_1] = [\mathcal{X}, \; S_2] = 0 \), otherwise they are linear dependant; a contradiction.
\[ 0 = [\mathcal{X}, \; S_1] = \sum_{z^1, \ldots, z^{n-3} \in S'} c_{1, \ast} (-1)^{n-3} \left[ [\mathcal{X}, \; e] \otimes \frac{\partial}{\partial z^1} \land \cdots \land \frac{\partial}{\partial z^{n-3}} \right] \land \frac{\partial}{\partial x^n}, \]
thus
\[ \sum_{z^1, \ldots, z^{n-3} \in S'} c_{1, \ast} (-1)^{n-3} \left[ [\mathcal{X}, \; e] \otimes \frac{\partial}{\partial z^1} \land \cdots \land \frac{\partial}{\partial z^{n-3}} \right] = 0 \]
and thus we have
\[ \sum_{z^1, \ldots, z^n \in \mathbb{S}} c_{1,*} (-1)^{n-3} e \otimes \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-3}} \in \left[ \mathfrak{so}(n-1) \otimes \mathcal{J}_{n-1}^{\wedge^{n-3} \mathfrak{so}^{n-1}} \right] = \{ \gamma_{n-1} \}; \]

and thus \( S_1 = c \gamma_{n-1} \wedge \frac{\partial}{\partial x^m} \).

Now choose \( \mathcal{X} = \mathcal{X}_{in} \). Then

\[
0 = [\mathcal{X}, \nu] = \sum_{z^1, \ldots, z^n \in \mathbb{S}, \, 2 \leq i \leq n-1} c_{2,*} X_{ii} \otimes \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-2}}
\]

\[
+ \sum_{z^1, \ldots, z^n \in \mathbb{S}, \, 2 \leq i \leq n-1} c_{2,*} X_{in} \otimes \left( - \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-2}} \right)
\]

\[
+ \sum_{z^1, \ldots, z^n \in \mathbb{S}, \, 2 \leq i \leq n-1} c_{2,*} X_{in} \otimes \left( \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial x^i} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-2}} \right)
\]

\[
+ c \sum_{1 \leq i \leq n-1} (-1)^{j+i+1} [X_{ij}, X_{ij}] \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^i} \cdots \wedge \frac{\partial}{\partial x^j} \cdots \wedge \frac{\partial}{\partial x^{n-2}} \otimes \frac{\partial}{\partial x^i}. \]

All coefficients \( c_{2,*} \) are zero in this summation except the coefficients \( c_{2, in} \) of basis vectors \( X_{in} \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^i} \wedge \cdots \wedge \frac{\partial}{\partial x^j} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-2}} \) with \( i \neq j \).

Similarly we apply other choices of \( \mathcal{X} = \mathcal{X}_{in} \in \mathfrak{so}(n) \) to the conditions \([\mathcal{X}, \nu] = 0\) to find

\[
S_2 = c \sum_{1 \leq i \leq n-1} (-1)^{j+i+1} X_{ij} \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^i} \cdots \wedge \frac{\partial}{\partial x^j} \cdots \wedge \frac{\partial}{\partial x^{n-1}}. \]

Thus \( \omega = c \sum_{1 \leq i \leq n} (-1)^{j+i+1} X_{ij} \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^i} \cdots \wedge \frac{\partial}{\partial x^j} \cdots \wedge \frac{\partial}{\partial x^{n-1}} \wedge \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \cdots \wedge \frac{\partial}{\partial x^m} \wedge \frac{\partial}{\partial x^n} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}}. \)

with \( \text{sgn}(\sigma_{ij}) = (-1)^{i+j+1} \).

**Lemma 4.3.**

\[ [\mathcal{J}_n \otimes \mathcal{J}_n^{\wedge^k}]^{\mathfrak{so}(n)} = \{ 0 \} \quad \text{for } k \notin \{ 1, n - 1 \}. \]

**Proof.** For the same reasons as in the proof of **Lemma 4.2**, we have

\[ \text{dim}[\mathcal{J}_n \otimes \mathcal{J}_n^{\wedge^k}]^{\mathfrak{so}(n)} = \text{dimHom}_{\mathfrak{so}(n)}(\mathcal{J}_n, \mathcal{J}_n^{\wedge^k}). \]

Now since the \( \mathcal{J}_n^{\wedge^k} \) are all nonisomorphic irreducible representations of \( \mathfrak{so}(n) \) except when \( k = \frac{n}{2} \) with \( n \) even. We deduce that the space \( \text{Hom}_{\mathfrak{so}(n)}(\mathcal{J}_n; \mathcal{J}_n^{\wedge^k}) \) is zero unless \( k = 1 \) or \( k = n - 1 \), in which case it is one-dimensional.

**Lemma 4.4.**

\[ [\mathcal{J}_n \otimes \mathcal{J}_n]^{\mathfrak{so}(n)} = \left( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2} + \cdots + \frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial x^n} \right) \quad \text{for } n \geq 3 \]

**Proof.** We proceed by induction on \( n \). By direct calculation, we easily check that

\[ [\mathcal{J}_3 \otimes \mathcal{J}_3]^{\mathfrak{so}(3)} = \left( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \otimes \frac{\partial}{\partial x^3} \right). \]

Now assume that

\[ [\mathcal{J}_{n-1} \otimes \mathcal{J}_{n-1}]^{\mathfrak{so}(n-1)} = \left( \sum_{i=1}^{n-1} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i} \right) \]

and let \( \omega \in [\mathcal{J}_n \otimes \mathcal{J}_n]^{\mathfrak{so}(n)} \), then \( \omega = u_1 + u_2 \) where

\[ u_1 \in (\mathcal{J}_{n-1} \otimes \mathcal{J}_{n-1}), \quad u_2 \in (\mathcal{J}_n \otimes \mathcal{J}_n)/(\mathcal{J}_{n-1} \otimes \mathcal{J}_{n-1}). \]

A vector space basis of \( (\mathcal{J}_n \otimes \mathcal{J}_n)/(\mathcal{J}_{n-1} \otimes \mathcal{J}_{n-1}) \) has \( 2n - 1 \) elements and is given by the families of elements

1. \( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1}, z^1 \in \{ x^1, \ldots, x^n \} \)
2. \( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^n}, z^1 \in \{ x^1, x^2, \ldots, x^n \} \).
So
\[ u_2 = \sum_{1 \leq i \leq n} c_{2,i} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial z^i} + \sum_{1 \leq j < n} c_{3,j} \frac{\partial}{\partial z^j} \otimes \frac{\partial}{\partial x^n}. \]

Now for \( \mathcal{X} \in \mathfrak{so}(n-1) \subseteq \mathfrak{so}(n) \) as a Lie subalgebra, we have
\[ 0 = [\omega, \mathcal{X}] = [u_1, \mathcal{X}] + [u_2, \mathcal{X}]. \]

So \( [u_1, \mathcal{X}] \) and \( [u_2, \mathcal{X}] \) are linearly dependant if they are non-zero; a contradiction. Thus \( [u_1, \mathcal{X}] = 0 \) i.e. \( u_1 \in [\mathcal{J}_{n-1} \otimes \mathcal{J}_{n-1}]^{\mathfrak{so}(n-1)} \), hence \( u_1 = c_1 \sum_{i=1}^{n-1} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i} \) for some constant \( c_1 \). Let \( \mathcal{X}_k = -x_k \frac{\partial}{\partial x^k} + x_n \frac{\partial}{\partial x^n} \) with \( k \neq n \).

\[ 0 = [u_2 + u_1, \mathcal{X}_k] = \sum_{1 \leq i \leq n} c_{2,i} \left[ \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial z^i}, -x_k \frac{\partial}{\partial x^k} + x_n \frac{\partial}{\partial x^n} \right] \]
\[ + \sum_{1 \leq j < n} c_{3,j} \left[ \frac{\partial}{\partial z^j} \otimes \frac{\partial}{\partial z^j}, -x_k \frac{\partial}{\partial x^k} + x_n \frac{\partial}{\partial x^n} \right] + c_1 \sum_{i=1}^{n-1} \left[ \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i}, -x_k \frac{\partial}{\partial x^k} + x_n \frac{\partial}{\partial x^n} \right] \]
\[ = \sum_{1 \leq i \leq n, i \neq k} c_{2,i} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial z^i} + \sum_{1 \leq j < n, j \neq k} c_{3,j} \frac{\partial}{\partial z^j} \otimes \frac{\partial}{\partial z^j} + (c_{2,k} + c_{3,k}) \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial z^k} - (c_{2,k} + c_{3,k}) \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^k} - c_1 \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^k} - c_1 \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^k}. \]

So \( c_{2,i} = 0 \), for \( 1 \leq i \leq n-1 \), \( i \neq k \); \( c_{3,j} = 0 \) for \( 1 \leq j < n \), \( j \neq k \); \( c_{2,k} + c_{3,k} = 0 \) and \( c_1 - c_{2,n} = 0 \). So \( u_2 = c_1 \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \), therefore \( \omega = u_1 + u_2 = c_1 \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i} \). \( \square \)

**Lemma 4.5.**
\[ [\mathcal{J}_n \otimes \wedge^{n-1}(\mathcal{J}_n)]^{\mathfrak{so}(n)} = \left( \sum_{m=1}^{n} (-1)^{m-1} \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^m} \wedge \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^m} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \right) \]
for \( n \geq 3 \)**

**Proof.** We apply induction on \( n \). It is easy to check by direct calculations that
\[ [\mathcal{J}_3 \otimes \mathcal{J}_3]^{\mathfrak{so}(3)} = \left( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \otimes \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^2} \right) \].

Now assume that
\[ [\mathcal{J}_{n-1} \otimes \wedge^{n-2}(\mathcal{J}_{n-1})]^{\mathfrak{so}(n-1)} = \left( \sum_{m=1}^{n-1} (-1)^{m-1} \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^m} \wedge \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^m} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}} \right) \]
and let \( \omega \in [\mathcal{J}_n \otimes \wedge^{n-1}(\mathcal{J}_n)]^{\mathfrak{so}(n)} \), then \( \omega = u_1 + u_2 \) where
\[ u_1 \in [\mathcal{J}_{n-1} \otimes \wedge^{n-1}(\mathcal{J}_{n-1})], \quad u_2 \in ([\mathcal{J}_n \otimes \wedge^{n-1}(\mathcal{J}_n)] / ([\mathcal{J}_{n-1} \otimes \wedge^{n-1}(\mathcal{J}_{n-1}))]. \]

A vector space basis of \( ([\mathcal{J}_n \otimes \wedge^{n-1}(\mathcal{J}_n)] / ([\mathcal{J}_{n-1} \otimes \wedge^{n-1}(\mathcal{J}_{n-1})) \) has exactly the \( n^2 - n + 1 \) elements given by the families:
\begin{enumerate}
  \item \( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}} \)
  \item \( \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}} \)
  \item \( \frac{\partial}{\partial x^3} \otimes \frac{\partial}{\partial x^3} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}} \)
\end{enumerate}
where for each family, the \( z^i \)'s are elements of \( R = \{ x^1, x^2, \ldots, x^{n-1} \} \). So
\[ u_2 = c_1 \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}} + S_2 + S_3 \]
where
\[ S_2 = \sum_{z^1,\ldots,z^{n-2} \in R} c_{2,*} \frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-2}} \]
and
\[ S_3 = \sum_{z^1,\ldots,z^{n-1} \in R} c_{3,*} \frac{\partial}{\partial z^2} \otimes \frac{\partial}{\partial z^2} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-1}}. \]
Now for \( \mathcal{X} \in \text{so}(n - 1) \subseteq \text{so}(n) \) as a Lie subalgebra, we have on one hand

\[
0 = [\omega, \mathcal{X}] = [u_1, \mathcal{X}] + [u_2, \mathcal{X}].
\]

So \([u_1, \mathcal{X}] = 0 = [u_2, \mathcal{X}]\); otherwise they will be linearly dependant; a contradiction. But as \(u_1 \in S_{n-1} \otimes S_{n-1}^{(n-1)}\), it follows that

\[
u_1 = \sum_{i=1}^{n-1} c_{0,i} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}}
\]

for some constants \(c_{0,i}\). So for \(\mathcal{X} = x_j \frac{\partial}{\partial x^{n-1}} - x_i \frac{\partial}{\partial x^1} \) with \(1 \leq j \leq n - 2\), we have

\[
0 = [u_1, \mathcal{X}] = \sum_{i=1}^{n-1} c_{0,i} \left( \left( - \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} + \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^j} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \right) \right)
\]

Thus \(c_{0,j} = 0 = c_{0,n-1}\). Repeating for \(0 \leq j \leq n - 2\) yields \(c_{0,i} = 0\) for all \(i = 1, 2, \ldots, n - 1\); hence \(u_1 = 0\). On the other hand, for all \(\mathcal{X} \in \text{so}(n - 1) \subseteq \text{so}(n)\) as a Lie subalgebra, we also have

\[
0 = [u_2, \mathcal{X}] = [S_2, \mathcal{X}] + [S_3, \mathcal{X}]
\]

(note that a simple verification shows that \([c_1 \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}}, \mathcal{X}] = 0\) and since \([S_2, \mathcal{X}] \) and \([S_3, \mathcal{X}] \) are linearly independent, they are both zero. Thus

\[
0 = [S_2, \mathcal{X}] = \sum_{j=1}^{n-2} c_{2,j} \frac{\partial}{\partial z^j} \wedge \ldots \wedge \frac{\partial}{\partial z^{n-2}} \otimes \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}}, \mathcal{X},
\]

and thus

\[
\sum_{j=1}^{n-2} c_{2,j} \frac{\partial}{\partial z^j} \wedge \ldots \wedge \frac{\partial}{\partial z^{n-2}} \in [S_{n-1} \otimes S_{n-1}^{(n-1)}] = \{0\}.
\]

by Lemma 4.1. Therefore \(S_2 = 0\). Similarly we have

\[
0 = [S_3, \mathcal{X}] = \sum_{j=1}^{n-1} c_{3,j} \frac{\partial}{\partial z^j} \wedge \ldots \wedge \frac{\partial}{\partial z^{n-1}} \otimes \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}}, \mathcal{X},
\]

thus

\[
\sum_{j=1}^{n-1} c_{3,j} \frac{\partial}{\partial z^j} \wedge \ldots \wedge \frac{\partial}{\partial z^{n-1}} \in [S_{n-1} \otimes S_{n-1}^{(n-2)}] = \{0\}.
\]

So

\[
S_3 = (-1)^n c \sum_{m=1}^{n-1} (-1)^m \frac{\partial}{\partial x^m} \wedge \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \otimes \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \wedge \frac{\partial}{\partial x^1}
\]

by inductive hypothesis. Now taking in particular \(\mathcal{X} = -x_1 \frac{\partial}{\partial x^1} + x_n \frac{\partial}{\partial x^1}\), we have

\[
0 = [\omega, \mathcal{X}] = [u_2, \mathcal{X}] = c_1 \left[ \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}}, \mathcal{X} \right] + (-1)^n \sum_{m=1}^{n-1} (-1)^m \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \otimes \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}}, \mathcal{X}
\]

\[
= (c_1 + c) \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} - (c_1 + c) \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} .
\]

thus \(c_1 + c = 0\) i.e. \(c_1 = -c\). Hence

\[
\omega = u_2 = (-1)^n c \sum_{m=1}^{n-1} (-1)^m \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \wedge \frac{\partial}{\partial x^m} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \wedge \frac{\partial}{\partial x^m} .
\]

\(\square\)
Lemma 4.6. Let \( \bar{\gamma}_n = \bar{\gamma}_n + \bar{\gamma}_n \) with
\[
\bar{\gamma}_n = \frac{1}{n!} \sum_{1 \leq i < j \leq n, \sigma \in S_{n-2}} \text{sgn}(\sigma) \partial X_{ij} \otimes \frac{\partial}{\partial x^{(i)}} \otimes \cdots \otimes \frac{\partial}{\partial x^{(j)}} \otimes \frac{\partial}{\partial x^{(n)}}
\]
and
\[
\bar{\gamma}'_n = \frac{(-1)^{n+1}}{n!} \sum_{1 \leq i < j \leq n, \sigma \in S_{n-2}} \text{sgn}(\sigma) \partial X_{ij} \otimes \frac{\partial}{\partial x^{(i)}} \otimes \cdots \otimes \frac{\partial}{\partial x^{(j)}} \otimes \frac{\partial}{\partial x^{(n)}} \otimes X_{ij}.
\]
Then
- \( \bar{\gamma}_n \) is an \( h_n \)-invariant
- \( \pi_3^\ast(\bar{\gamma}_n) = \pi_3^\ast(\bar{\gamma}_n) = [\gamma_n] \) in \( H^4_{n-2} \), where \( \pi_3 : b_{n-1} \rightarrow h_n \otimes h^{n-2} \) is the projection.

**Proof.** As \( \bar{\gamma}_n \) and \( \bar{\gamma}'_n \) are \( o(n) \)-invariant, so is \( \bar{\gamma}_n \). Also since \( \left[ \frac{\partial}{\partial x^i}, \bar{\gamma}_n \right] = -\left[ \frac{\partial}{\partial x^i}, \bar{\gamma}_n \right] \) for all \( i = 1 \ldots n \), it follows that \( \bar{\gamma}_n \) is an \( J_n \)-invariant. For the second assertion, it is clear that
\[
\pi_3(\bar{\gamma}_n) = \partial \left( \sum_{1 \leq i_1 < \cdots < i_n \leq n} \text{sgn}(\sigma) \frac{\partial}{\partial x^{(i_1)}} \otimes \frac{\partial}{\partial x^{(i_2)}} \wedge \cdots \wedge \frac{\partial}{\partial x^{(i_n)}} \wedge \cdots \wedge \frac{\partial}{\partial x^{(n)}} \otimes X_{i_1 i_2} \wedge \cdots \wedge X_{i_n i_1} \right)
\]
with the permutation \( \sigma_n = (i_1, i_2, \ldots, i_l, \ldots, i_n, i_r, i_l) \). \( \square \)

**Remark 4.7.** Since as an \( o(n) \)-module \( h_n \cong J_n \oplus o(n) \), it follows that
\[
[h_n \wedge^\ast (J_n)]_{\text{rel}} \cong [J_n \wedge^\ast (J_n)]_{\text{rel}} + [o(n) \wedge^\ast (J_n)]_{\text{rel}}.
\]
Therefore the four last lemmas combined completely give the homology groups \( H^4_{\text{Lie}}(h_n; \mathbb{R}) \) and \( H^4_{\text{Lie}}(h_n; \mathbb{R}) \). It is known for the cohomology of the orthogonal Lie group (viewed as a manifold), the Hopf algebras
\[
H^4_{\text{dr}}(\mathfrak{so}(2k); \mathbb{R}) \cong \wedge^\ast(u_3, u_7, \ldots, u_{4k-5}, u_{2k-1})
\]
and
\[
H^4_{\text{dr}}(\mathfrak{so}(2k-1); \mathbb{R}) \cong \wedge^\ast(u_3, u_7, \ldots, u_{4k-5}),
\]
where the \( u_i \)'s are primitive generators of odd degrees \( i \) and \( H^4_{\text{dr}}(\mathfrak{so}(n); \mathbb{R}) \) denotes the de Rham cohomology [5, p. 1742].

Also as vector spaces
\[
H^4_{\text{Lie}}(\mathfrak{so}(n); \mathbb{R}) \cong H^4_{\text{Lie}}(\mathfrak{so}(n); \mathbb{R}).
\]

5. The Leibniz homology of \( h_n \)

For any Leibniz algebra \( \mathfrak{g} \) (thus for Lie algebras in particular) over a ring \( k \), the Leibniz homology of \( \mathfrak{g} \) with coefficients in \( k \) denoted \( H_*^{\text{Lie}}(\mathfrak{g}; k) \), is the homology of the Leibniz complex \( T^\ast(\mathfrak{g}) \), namely
\[
k \xleftarrow{\mathfrak{g}} \xleftarrow{\mathfrak{g}^2} \xleftarrow{d} \cdots \xleftarrow{d} \mathfrak{g}^n \leftarrow \mathfrak{g}^n \xleftarrow{d} \cdots \xleftarrow{d} \mathfrak{g}^0 \xleftarrow{\mathfrak{g}^0} k
\]
where \( \mathfrak{g}^n \) is the \( n \)-th tensor power of \( \mathfrak{g} \) over \( k \), and where
\[
d(g_1 \otimes g_2 \otimes \cdots \otimes g_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j} g_1 \otimes g_2 \otimes \cdots \otimes \hat{g}_i \otimes \cdots \otimes \hat{g}_j \otimes \cdots \otimes g_n \ [7].
\]
The canonical projection \( \mathfrak{g}^n \xrightarrow{\pi_1} \mathfrak{g}^n \xrightarrow{\pi_2} \mathfrak{g}^{n+1} \), \( n \geq 0 \), is a map of chain complexes, \( T^\ast(\mathfrak{g}) \rightarrow \wedge^\ast(\mathfrak{g}) \), and induces the following \( k \)-linear map on homology \( H_*^{\text{Lie}}(\mathfrak{g}; k) \rightarrow H^\ast_{\text{Lie}}(\mathfrak{g}; k) \). Considering
\[
(ker \pi_1)_n = ker[g^{(n+2)} \rightarrow \mathfrak{g}^{(n+2)}], \quad n \geq 0.
\]
The relative theory \( H^\ast_{\text{rel}}(\mathfrak{g}) \) was defined by Pirashvili [13] as the homology of the complex
\[
C^\ast_n(\mathfrak{g}) = (ker \pi_1)_n.
\]
Also, the projection \( \mathfrak{g} \otimes \mathfrak{g}^n \xrightarrow{\pi_2} \mathfrak{g}^{n+1} \), \( n \geq 0 \), is a map of chain complexes,
\[
\pi_2 : \mathfrak{g} \otimes \wedge^\ast(\mathfrak{g}) \rightarrow \wedge^{\ast+1}(\mathfrak{g}).
\]
Let \( H_R(\mathfrak{g}) \) denote the homology of the complex
\[
C^\ast_n(\mathfrak{g}) = (ker \pi_2)_n = ker[\mathfrak{g} \otimes \mathfrak{g}^{(n+1)} \rightarrow \mathfrak{g}^{(n+2)}], \quad n \geq 0.
\]
Lemma 5.1. For the affine orthogonal Lie algebra $h_n$, there are natural isomorphisms

\[ HR_{k-3}(so(n); \mathbb{R}) \cong H^k_{\text{Lie}}(so(n); \mathbb{R}) \quad \text{for all } k \geq 3, \]

\[ HR_{k-3}(h_n; \mathbb{R}) \cong H^k_{\text{Lie}}(so(n); \mathbb{R}) \oplus \{ \gamma_n \} \]

where $\gamma_n = \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} X_{ij} \otimes \frac{\delta}{\delta x^i} \wedge \cdots \wedge \frac{\delta}{\delta x^j} \wedge \cdots \wedge \frac{\delta}{\delta x^n}$.


Recall that a Zinbiel algebra is a vector space $V$ equipped with a binary operation $\circ$ which satisfies the relation

\[(a \circ b) \circ c = a \circ (b \circ c) + a \circ (c \circ b) \quad \text{for all } a, b, c \in V.\]

For a Leibniz algebra $g$, let $\gamma \in \text{Hom}(g^{\otimes p}, \mathbb{R})$ and $\beta \in \text{Hom}(g^{\otimes q}, \mathbb{R})$. The co-half shuffle $\gamma \bullet \beta \in \text{Hom}(g^{\otimes (p+q)}, \mathbb{R})$ is defined by

\[(\gamma \bullet \beta)(x_1 \otimes \cdots \otimes x_{p+q}) = \sum_{\sigma \in \text{Sh}_{p-1,q}} (\text{sgn } \sigma) \gamma(x_1, x_{\sigma(2)}, x_{\sigma(3)}, \ldots, x_{\sigma(p)}) \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)});
\]

where $\text{Sh}_{p-1,q}$ is the set of all $(p - 1, q)$-shuffles of $(2, \ldots, p, p + 1, \ldots, p + q)$. Loday showed (see [9]) that the co-half shuffles on cochains induce a Zinbiel algebra structure on $H_{\text{Lie}}^*(g; \mathbb{R})$.

Theorem 5.2. There is an isomorphism of vector spaces

\[ H_{\text{Lie}}^n(h_n) \cong (\mathbb{R} \oplus \langle \tilde{a}_n \rangle) \otimes T^*(\tilde{\gamma}_n), \]

and an algebra isomorphism,

\[ H^*_{\text{Lie}}(h_n) \cong (\mathbb{R} \oplus \langle \tilde{a}_n^d \rangle) \otimes T^*(\tilde{\gamma}_n^d), \]

where

\[ \tilde{a}_n^d = \sum_{\sigma \in S_n} \text{sgn } \sigma d \sigma^{(1)} \otimes d \sigma^{(2)} \otimes \cdots \otimes d \sigma^{(n)}, \]

\[ \tilde{\gamma}_n^d = \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} X_{ij} \otimes d \sigma^{(1)} \otimes \cdots \otimes d \sigma^{(i-1)} \otimes d \sigma^{(i+1)} \otimes \cdots \otimes d \sigma^{(j-1)} \otimes d \sigma^{(j+1)} \otimes \cdots \otimes d \sigma^{(n)} \]

and $H_{\text{Lie}}^*$ is afforded the Zinbiel algebra (dual Leibniz algebra).

Proof. Consider the Pirashvili filtration of the complex

\[ C_n^\text{rel}(g) = \text{ker}(g^{\otimes (n+2)} \to g^{\wedge (n+2)}) \quad n \geq 0, \]

given by

\[ F^k_{m+1}(g) = g^{\otimes k} \otimes \text{ker}(g^{\otimes (m+2)} \to g^{\wedge (m+2)}) \quad m \geq 0, \]

Then $F^k_{m}$ is a subcomplex of $F^k_{m+1}$, and the resulting spectral sequence converges to $H_{\text{Lie}}^*(g)$ with

\[ E_{m,k}^2 \cong H^k_{\text{Lie}}(g) \otimes HR_m(g), \quad m \geq 0, \quad k \geq 0. \]

(See [12] for the cohomological version.)

From the proof of Lemma 5.1, it is clear that for $k \leq n - 2$, $\partial : H^k_{\text{Lie}}(h_n; \mathbb{R}) \to HR_k(h_n; \mathbb{R})$ is an isomorphism, so from the long exact sequence relating Lie and Leibniz homologies, we have in particular

\[ H^1_{\text{Lie}}(h_n; \mathbb{R}) = H^1_{\text{Lie}}(h_n; \mathbb{R}) = 0 \quad \forall n \geq 3; \quad H^2_{\text{Lie}}(h_n; \mathbb{R}) \cong \frac{H^0_{\text{rel}}(h_n)}{im \partial} \cong 0 \quad \forall n > 3. \]

In fact we prove inductively that $H^k_{\text{Lie}}(h_n; \mathbb{R}) \cong 0$ for $1 \leq k \leq n - 2$. Indeed, since $so(n)$ is a semi-simple Lie algebra, $H^k_{\text{Lie}}(so(n); \mathbb{R}) = 0$, $k \geq 1$. So from the long exact sequence

\[ \cdots \to H^k_{\text{Lie}}(so(n)) \to H^k_{\text{Lie}}(h_n) \to H^k_{\text{rel}}(so(n)) \to H_{k-1}^k(so(n)) \to \cdots \]

induced by the definition of $H^k_{\text{Lie}}(so(n))$, we have that $\partial : H^k_{\text{Lie}}(so(n)) \to H^k_{\text{rel}}(so(n))$ is an isomorphism for $k \geq 3$. The inclusion $so(n) \hookrightarrow h_n$ induces a map between exact sequences

\[ H^k_{\text{Lie}}(h_n; \mathbb{R}) \to H^k_{\text{Lie}}(so(n); \mathbb{R}) \to H^k_{\text{rel}}(h_n; \mathbb{R}) \to H^k_{\text{rel}}(so(n); \mathbb{R}) \to H^k_{\text{Lie}}(so(n); \mathbb{R}) \]

\[ H^k_{\text{Lie}}(so(n); \mathbb{R}) \to H^k_{\text{Lie}}(so(n); \mathbb{R}) \to H^k_{\text{rel}}(so(n); \mathbb{R}) \to H^k_{\text{rel}}(so(n); \mathbb{R}) \to H^k_{\text{Lie}}(so(n); \mathbb{R}) \]
and the inclusion $F^r_m(\Delta(n)) \hookrightarrow F^r_m(h_n)$ induces a map of spectral sequences, thus a map

$$H_0^\rel(\Delta(n)) \otimes H_0^\rel(n) \longrightarrow H_0^\rel(h_n) \otimes H_0^\rel(n). \quad (4.2.0)$$

Since $H_0^\rel(\Delta(n)) \cong H_0^\rel(n)$, all classes in $H_0^\rel(h_n) \otimes H_0^\rel(n)$ are mapped by the boundary maps $d^2_{p,q}$, $d^3_{p,q}$, ... of the spectral sequence to the zero class. So by the map (4.2.0), all classes in $H_0^\rel(h_n) \otimes H_0^\rel(n)$ are absolute cycles (i.e. cycles for all boundary maps of the spectral sequence) in $H_0^\rel(h_n)$. So if by inductive hypothesis $H_0^\rel(h_n) \cong 0$ for $1 \leq r < k \leq n - 2$, we clearly have

$$H_0^\rel(h_n) \cong E^\infty_{r,0} \cong H_{r+3}^\lie(\so(n)); \quad \mathbf{R}.$$ 

Therefore we have from the long exact sequence induced by $H_0^\rel(h_n)$ that $H_0^\rel(h_n, \mathbf{R}) \cong 0$. 

Again from Lemma 5.1, we have the isomorphism $H_{n-3}(h_n; \mathbf{R}) \cong \text{im } \partial \oplus \langle \gamma_n \rangle$ where $\partial : H_{n-3}^\lie(h_n; \mathbf{R}) \longrightarrow H_{n-3}^\rel(h_n; \mathbf{R})$. We have from an articulation of the boundary map $\partial$ in the long exact sequence induced by $H_0^\rel(h_n)$ that

$$H_{n-1}(h_n; \mathbf{R}) \cong \frac{H_{n-3}^\rel(h_n)}{\text{im } \partial} \cong \langle \gamma_n \rangle;$$

where $\gamma_n$ is symmetrized and lifted to $\tilde{\gamma}_n \in T^*(h_n)$ by antisymmetrization.

All classes in $H_{n-1}(h_n) \otimes H_{n}^\rel(\so(n))$ are not absolute cycles: Indeed if $\partial = \gamma_n \otimes z$ where $z = x_1 \wedge \cdots \wedge x_k \in \wedge^k(\so(n))$ is a generator of $H_k^\rel(\so(n); \mathbf{R})$, we lift $z$ to $\tilde{z}$ in $T^*(\so(n))$ and $\gamma_n$ to $\tilde{\gamma}_n$ to have as $d(\tilde{\gamma}_n) = 0$ and using invariance $[\tilde{\gamma}_n, x_i] = 0$ for $i = 1, 2, \ldots, k$;

$$d(\gamma_n) = d(\tilde{\gamma}_n) \otimes d(\tilde{z}) \quad (4.2.1)$$

which is a representative of a non-zero class in $H_{n-1}(h_n) \otimes H_{n}^\rel(\so(n))$. Hence again the terms $E^\infty_{r,0} \cong H_{r+3}^\lie(\so(n)); \mathbf{R}$. It follows from (4.2.1) that all classes in $H_{n-1}(h_n) \otimes H_{n}^\rel(\so(n))$ with representatives lifted to $\tilde{\gamma}_n \otimes d(\tilde{z})$, are not absolute cycles. Also, let $[\theta] \in H_{n}^\rel(\so(n))$ be represented by a sum

$$\theta = \sum_{j=1}^{n+1} \gamma_j \otimes x_{1,j} \wedge x_{2,j} \wedge x_{3,j} \wedge \cdots \wedge x_{k-2,j} \quad (4.2.2)$$

with $x_{i,j} \in \so(n)$. Let $y_{k-2}$ be the antisymmetrization of $\sum_{j=1}^{n+1} x_{1,j} \wedge x_{2,j} \wedge x_{3,j} \wedge \cdots \wedge x_{k-2,j}$ on $T^*(\so(n))$. Then we use invariance and the fact that $[\tilde{\gamma}_n, \tilde{\gamma}_n] = 0$ to show that

$$d(\tilde{\gamma}_n) \otimes \theta = \tilde{\gamma}_n \otimes d(y_{k-2})$$

which corresponds to a non-zero class in $H_{2n-2}(h_n; \mathbf{R}) \otimes H_{k-3}(h_n)$. Similarly, since $\tilde{\gamma}_n \otimes 2 \notin \text{im } \partial$ with $\partial : H_{2n-4}(h_n; \mathbf{R}) \longrightarrow H_{2n-4}(h_n; \mathbf{R})$, it corresponds to a non-zero class in $H_{2n-2}(h_n)$ and all classes in $H_{2n-2}(h_n) \otimes H_{n}^\rel(h_n)$ except $\tilde{\gamma}_n \otimes 2$ are not absolute cycles in $H_{n}^\rel(h_n)$, and $\tilde{\gamma}_n \otimes 3 \notin \text{im } \partial$. By induction on $k$, $\tilde{\gamma}_n \otimes k$ corresponds to a non-zero class in $H_{k+1}(h_n)$ and all classes in $H_{k+1}(h_n) \otimes H_{n}^\rel(h_n)$ are not absolute cycles, except $\tilde{\gamma}_n \otimes k+1 \notin \text{im } \partial$. At this point, $H_{k}^\rel(h_n)$ is completely determined for $k < n$; to determine $H_{k}^\rel(h_n)$ for $k \geq n$, we first notice that

$$\alpha_n \in \ker \partial, \quad \partial : H_{n}^\lie(h_n; \mathbf{R}) \longrightarrow H_{n}^\lie(h_n; \mathbf{R}).$$

So, $\tilde{\alpha}_n$ generates a non-zero class in $H_{n}(h_n; \mathbf{R})$ mapping to the class $\alpha_n \in H_{n}^\lie(h_n; \mathbf{R})$. So we have in addition to the steps above in the determination of $H_{k+1}(h_n)$ to examine the boundary maps on $\alpha_n \otimes \theta$ for $\theta \in H_{n}^\rel(h_n)$. Indeed, assume on one hand that $[\theta] \in H_{n}^\rel(h_n)$ is represented by a sum

$$\theta = \sum_{j=1}^{n} x_{1,j} \otimes x_{2,j} \wedge x_{3,j} \wedge \cdots \wedge x_{k+1,j}$$

where $x_{i,j} \in \so(n)$ and $d(\theta) = 0$. Let $\tilde{\theta} \in T^*(\so(n))$ be the antisymmetrization of $\theta$. By invariance, $[\tilde{\alpha}_n, x_{i,j}] = 0$ for each $x_{i,j}$. This yields the conclusion that $\tilde{\alpha}_n \otimes \tilde{\theta}$ represents an absolute cycle in $H^\rel(h_n)$. However, assuming on the other hand $\theta$ as in (4.2.2), using invariance and the fact that $[\tilde{\alpha}_n, \tilde{\gamma}_n] = 0$, we obtain $d(\tilde{\alpha}_n \otimes \tilde{\theta}) = \tilde{\alpha}_n \otimes \tilde{\gamma}_n \otimes d(y_{k-2})$ which is a representative of a non-zero class in $H_{2n-1}(h_n; \mathbf{R}) \otimes H_{k-3}(h_n)$. To compute

$$\partial : H_{n}^\lie(h_n; \mathbf{R}) \longrightarrow H_{n-1}^\lie(h_n; \mathbf{R}) \otimes H_{k-3}(h_n),$$

on classes of the form $\tilde{\alpha}_n \otimes H^\lie_*(\so(n))$, let $[\theta'] \in H^\lie_*(\so(n))$ with $\partial(\theta') = \theta$. By lifting $\alpha_n \otimes \theta'$ to $\tilde{\alpha}_n \otimes \theta'$ in $T^*(h_n)$ and using invariance, we have

$$\partial(\alpha_n \otimes \theta') = \tilde{\alpha}_n \otimes \theta'.$$

Hence $\tilde{\alpha}_n \otimes \tilde{\gamma}_n$ also corresponds to a non-zero class in $H_{2n-1}(h_n; \mathbf{R})$ since
\[ \tilde{\alpha}_n \otimes \tilde{\gamma}_n \notin \text{im } \partial; \quad \partial : H_{n+4}^{\text{Lie}}(\mathfrak{h}_n; \mathbb{R}) \longrightarrow H_n^{\text{rel}}(\mathfrak{h}_n; \mathbb{R}). \]

By induction on \(k\), \(\tilde{\alpha}_n \otimes \tilde{\gamma}_n^{\otimes k}\) corresponds to a non-zero class in \(H_{n+k(n-1)}(\mathfrak{h}_n; \mathbb{R})\).

Summing up,
\[
H_{\ell}(\mathfrak{h}_n; \mathbb{R}) \cong \begin{cases} \{\tilde{\gamma}^{\otimes r}\}, & \text{for } r = k(n-1) \\ \{\tilde{\alpha} \otimes \tilde{\gamma}^{\otimes r}\}, & \text{for } r = n + k(n-1) \\ 0, & \text{else} \end{cases}
\]

Hence, the graded vector space isomorphism
\[ H_{\ell}(\mathfrak{h}_n) \cong (\mathbb{R} \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n). \]

For the cohomology, we use the vector space isomorphism
\[ H^*_L(\mathfrak{h}_n; \mathbb{R}) \cong \text{Hom}(H_{\ell}(\mathfrak{h}_n; \mathbb{R}), \mathbb{R}), \]

to conclude that
\[ H^*_L(\mathfrak{h}_n) \cong (\mathbb{R} \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n), \]

where \(\tilde{\alpha}_n^d = \sum_{\sigma \in S_n} \text{sgn}(\sigma) dx^{\sigma(1)} \otimes dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)},\)
\[
\tilde{\gamma}_n = \sum_{1 \leq i < j \leq n, \sigma \in S_{n-2}} (-1)^{i+j+1} X_{ij} \otimes dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(j)} \otimes \cdots \otimes dx^{\sigma(n)}
\]
\[ + \sum_{1 \leq i < j \leq n, \sigma \in S_{n-2}} (-1)^{i+j+1} dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(i)} \otimes \cdots \otimes dx^{\sigma(j)} \otimes \cdots \otimes dx^{\sigma(n)} \otimes \tau_{ij}, \]

\(X^*_n := -x_i dx^j + x_j dx^i\) and \(dx_i\) is the dual of \(\frac{\partial}{\partial x_i}\) with respect to the basis of \(\mathfrak{h}_n\) given in Section 2. The Zinbiel algebra structure on \(H^*_L(\mathfrak{h}_n)\) is given by the Zinbiel products (see Lemma A.1):
\[
\tilde{\gamma}_n^d \ast \tilde{\gamma}_n^d = (n!) (n! + (-1)^{n+1}) \tilde{\gamma}_n^{\otimes 2}, \quad \tilde{\alpha}_n^d \ast \tilde{\alpha}_n^d = 0.
\]
\[
\tilde{\gamma}_n^d \ast \tilde{\alpha}_n^d = \frac{(n!)^2}{2} \tilde{\alpha}_n \otimes \tilde{\gamma}_n, \quad \text{and} \quad \tilde{\alpha}_n^d \ast \tilde{\gamma}_n = k_n \tilde{\alpha}_n \otimes \tilde{\gamma}_n
\]
for some non-zero real \(k_n\). One checks that \(k_3 = 36\) and \(k_4 = 144\). \(\Box\)

**Corollary 5.3.** As graded vector spaces,
\[ H_{\ell}(\mathfrak{h}_n; \mathfrak{h}_n) \cong (\mathbb{R} \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^{n+1}(\tilde{\gamma}_n). \]

**Proof.** We apply the isomorphism \(H_{\ell+b}(\mathfrak{h}_n; \mathfrak{h}_n) \cong H_{\ell+b+1}(\mathfrak{h}_n; \mathbb{R})\) [10]. \(\Box\)

**Appendix**

In this appendix, we sketch the calculus of the Zinbiel products of the proof of Theorem 5.2.

**Lemma A.1.**
\[
\tilde{\gamma}_n^* \ast \tilde{\gamma}_n^* = (n!) (n! + (-1)^{n+1}) \tilde{\gamma}_n^{\otimes 2}, \quad \tilde{\alpha}_n^* \ast \tilde{\alpha}_n^* = 0.
\]
\[
\tilde{\gamma}_n^* \ast \tilde{\alpha}_n^* = \frac{(n!)^2}{2} \tilde{\alpha}_n \otimes \tilde{\gamma}_n, \quad \text{and} \quad \tilde{\alpha}_n^* \ast \tilde{\gamma}_n = k_n \tilde{\alpha}_n \otimes \tilde{\gamma}_n
\]
for some non-zero real \(k_n\).

**Proof.** We show that \(\tilde{\alpha}_n^d \ast \tilde{\alpha}_n^d = 0\). Indeed, for \(n = 3\), none of the \(X_{ij}^*\)'s appears in the expression of \(\tilde{\alpha}_3^d \ast \tilde{\alpha}_3^d\), so \((\tilde{\alpha}_3^d \ast \tilde{\alpha}_3^d)(\tilde{\gamma}_3^{\otimes 3}) = 0\).

For \(n > 3\), there is no chain on degree \(2n\).

To prove that \(\tilde{\gamma}_n^d \ast \tilde{\alpha}_n^d = \frac{(n!)^2}{2} \tilde{\gamma}_n \otimes \tilde{\alpha}_n\), notice that the only \((n - 2, n)-\)shuffle that fixes 2 is the identity, so we have \((\tilde{\gamma}_n^d \ast \tilde{\alpha}_n^d)(\tilde{\gamma}_n \otimes \tilde{\alpha}_n) = (n - 2)! (n!)^2 n! = \frac{(n!)^2}{2}\).

We show that \(\tilde{\gamma}_n^d \ast \tilde{\gamma}_n^d = (n!) (n! + (-1)^{n+1}) \tilde{\gamma}_n^{\otimes 2}\). Indeed, only two \((n - 2, n - 1)-\)shuffles yield possible non-zero terms. These shuffles are the identity and
\[
\begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & \cdots & 2n-3 \\ n & n+1 & \cdots & 2n-3 & 1 & \cdots & n-1 \end{pmatrix}
\]
both with positive signature. Note that the first is the only shuffle fixing \( n - 1 \) and the second is the only shuffle which takes \( 2n - 3 \) to \( n - 1 \). We then have

\[
(\tilde{\gamma}_n^d \bullet \tilde{\gamma}_n^d)(\tilde{\gamma}_n^{\otimes 2}) = \left(2(2 + (-1)^{n+1}) \binom{n}{n-2}(n-2)! + 2 \left(\frac{n(n-1)(n-2)!}{2}\right)\right)
\]

which simplifies to \((\tilde{\gamma}_n^d \bullet \tilde{\gamma}_n^d)(\tilde{\gamma}_n^{\otimes 2}) = (n!)(n! + (-1)^{n+1})\).

To show that \( \tilde{\alpha}_n^d \bullet \tilde{\gamma}_n^d = k_n \tilde{\alpha}_n \otimes \tilde{\gamma}_n \) for some non-zero real \( k_n \). Notice that \( \tilde{\alpha}_n^d \bullet \tilde{\gamma}_n^d \) is the summation of \( \binom{2n-2}{n-1} \) cochains of the types

1. \((dx^1 \otimes \cdots \otimes dx^n) \bullet (X^n_0 \otimes dx^1 \cdots \hat{dx}^i \cdots dx^n)\)
2. \((dx^1 \otimes \cdots \otimes dx^n) \bullet (dx^1 \cdots \hat{dx}^i \cdots \otimes dx^n \otimes X^n_0)\)

The only non-zero terms are obtained by evaluating:

(a) cochains of the first type on the chains

\[
\frac{\partial}{\partial x^{ij}} \otimes \cdots \otimes \frac{\partial}{\partial x^{ij}} \otimes X^n_0 \otimes \frac{\partial}{\partial x^{ij}} \otimes \cdots \otimes \frac{\partial}{\partial x^{ij}}
\]

using the \((n - 1, n - 1)\)-shuffle identity, and on the chains

\[
\frac{\partial}{\partial x^{ij}} \otimes \cdots \otimes \frac{\partial}{\partial x^{ij}} \otimes \frac{\partial}{\partial x^{ij}} \otimes \cdots \otimes \frac{\partial}{\partial x^{ij}} \otimes X^n_0
\]

using the \((n - 1, n - 1)\)-shuffle satisfying \( \alpha(2n - 2) = n, \alpha(k) = k \) for \( k = 1, \ldots, n - 1 \).

(b) cochains of the second type on the chain \((c1)\) using the \((n - 1, n - 1)\)-shuffle satisfying \( \alpha(n) = 2n - 2, \alpha(k) = k \) for \( k = 1, \ldots, n - 1 \), and on the chains \((c2)\) using the \((n - 1, n - 1)\)-shuffle identity. Therefore \((\tilde{\alpha}_n^d \bullet \tilde{\gamma}_n^d)(\tilde{\alpha}_n \otimes \tilde{\gamma}_n) \neq 0. \Box

References