A theorem of Hrushovski–Solecki–Vershik applied to uniform and coarse embeddings of the Urysohn metric space

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Abstract

A theorem proved by Hrushovski for graphs and extended by Solecki and Vershik (independently from each other) to metric spaces leads to a stronger version of ultrahomogeneity of the infinite random graph R, the universal Urysohn metric space $\mathbb{U}$, and other related objects. We show how the result can be used to average out uniform and coarse embeddings of $\mathbb{U}$ (and its various counterparts) into normed spaces. Sometimes this leads to new embeddings of the same kind that are metric transforms and besides extend to affine representations of various isometry groups. As an application of this technique, we show that $\mathbb{U}$ admits neither a uniform nor a coarse embedding into a uniformly convex Banach space.
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1. Introduction

A theorem by Hrushovski [18] states that every finite graph Γ can be embedded (as an induced subgraph) into a finite graph Γ' so that each partial isomorphism of Γ is a restriction of a global automorphism of Γ'. Solecki [39] and (independently) Vershik [44] have obtained an analogue of the result for metric spaces: every finite metric space X is contained in a finite metric space Y in such a way that partial isometries of X become restrictions of global self-isometries of Y. Solecki has deduced the result from a powerful general theorem of Herwig and Lascar [16], while Vershik gave a direct construction.

Both theorems are particular cases of a statement where X and Y are drawn from a class of metric spaces whose distance values belong to a given convex subset S of some additive subsemigroup of real numbers. The Hrushovski theorem is recovered for the set of values $S = \{0, 1, 2\}$, while the Solecki–Vershik theorem corresponds to the entire real line. We outline a proof of the result in Section 3; it is also based on the techniques of Herwig and Lascar, but in a different way from Solecki's article.

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Let again $S \subseteq \mathbb{R}$ be a subset as above. Denote by $\mathbb{U}_S$ a version of the universal Urysohn metric space with distance values in S, characterized by the following properties: completeness, ultrahomogeneity (every isometry between finite subspaces is extended to a global self-isometry of the space), and universality for the class of all metric spaces with distance values in S. For $S = \{0, 1, 2\}$ one obtains in this way the infinite random graph R [36,4], while $S = \mathbb{R}$ results in the classical Urysohn metric space $\mathbb{U}$, and $S = \mathbb{Z}$ leads to the integer-valued Urysohn space $\mathbb{U}_{\mathbb{Z}}$. Now the Hrushovski–Solecki–Vershik theorem can be viewed as a stronger version of ultrahomogeneity for the space $\mathbb{U}_S$: every finite subspace X of $\mathbb{U}_S$ is contained in a finite subspace Y of $\mathbb{U}_S$ in such a way that partial isometries of X are restrictions of global isometries of $\mathbb{U}_S$ taking Y to itself. Simple examples (such as the unit sphere in a Hilbert space) show that the property is in general strictly stronger than ultrahomogeneity.

A range of interesting applications of the Hrushovski–Solecki–Vershik property can be found in the original works [18,39,44]. Here we apply the result to analysis of (non)existence of uniform and coarse embeddings of the Urysohn metric space into superreflexive Banach spaces.

Let X and Y be two metric spaces, and let $f : X \to Y$ be an embedding of X into Y as a uniform subspace. This property of f is easily seen to be equivalent to the following: for some non-decreasing functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $0 < \rho_1(x) \leqslant \rho_2(x)$ for $x > 0$ and $\rho_2(x) \to 0$ as $x \to 0$, one has for every $x, y \in X$

$$\rho_1\big(d_X(x, y)\big) \leqslant d_Y\big(f(x), f(y)\big) \leqslant \rho_2\big(d_X(x, y)\big).$$

Here the double inequality only needs to hold for values of the distance $d_X(x, y)$ in a sufficiently small neighbourhood of zero.

If, on the contrary, we are interested in the above property holding for values of the distance $d(x, y)$ in a neighbourhood of *infinity,* we arrive at the relatively recent notion of a *coarse embedding* of X into Y. So is called a function $f : X \to Y$ (not necessarily continuous) such that, for some non-decreasing, unbounded functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho_1 \leqslant \rho_2$ one has for every $x, y \in X$

$$\rho_1\big(d_X(x, y)\big) \leqslant d_Y\big(f(x), f(y)\big) \leqslant \rho_2\big(d_X(x, y)\big).$$

Coarse embeddings are monomorphisms in the *coarse category* of metric spaces, which we will not define here, referring to the book [38] instead. In the same book, the reader can find a detailed motivation for the concept (the *Baum–Connes conjecture*).

The first example of a metric space admitting no *uniform* embedding into the Hilbert space ℓ^2 was constructed by Per Enflo [8]. Gromov asked in [11] if every separable metric space can be *coarsely* embedded into a Hilbert space. The first counter-example was constructed by Dranishnikov, Gong, Lafforgue, and Yu [7], who have used a suitable modification of Enflo's construction. Thus, the two notions are without doubt linked at some fundamental level, though the full extent of this link is not quite obvious. Notice that initially Gromov even used the same term "uniform embedding" to denote what is now known as a coarse embedding.

Of particular interest in relation to the Baum–Connes conjecture are coarse embeddings of metric *spaces of bounded geometry* (for every $R > 0$, the cardinality of each ball $B_R(x)$, $x \in X$, is uniformly bounded in x by a finite number), and especially for finitely generated groups equipped with the Cayley distance. As target spaces Y, one is typically interested in "nice" Banach spaces (the Hilbert space ℓ^2, the spaces ℓ^p, $p > 1$, etc.)

In view of the example of Dranishnikov et al., the standard ultraproduct technique implies immediately that there exists a locally finite metric space (that is, every ball of finite radius contains finitely many points) non-embeddable into ℓ^2 [7]. The same question for spaces of bounded geometry is more difficult. It was resolved by Gromov [13], who has noticed that a disjoint sum of graphs forming an expander family and equipped with the path distance gives such an example of a metric space of bounded geometry. (This construction is treated in detail, e.g., in the book [38].) In the same paper [13], Gromov has outlined a probabilistic construction of a finitely generated infinite group into which a disjoint sum of graphs as above can be coarsely embedded and which therefore admits no coarse embedding into ℓ^2.

Recent results by Kasparov and Yu [21] have brought interest to a more general version of the same coarse embedding problem, where ℓ^2 is replaced with a uniformly convex Banach space. For some remarks on the problem, see [25].

Recall that a Banach space E is *uniformly convex* if the function $\delta : (0, 2] \to \mathbb{R}$ (the *modulus of convexity* for E), defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in E, \; \|x\|, \|y\| \leqslant 1, \; \|x - y\| \geqslant \varepsilon\right\},$$

is strictly positive for all $\varepsilon \in (0, 2]$.

The spaces ℓ^p, $1 < p < \infty$, are uniformly convex. The Hilbert space ℓ^2 coarsely embeds into ℓ_p for $1 \leqslant p \leqslant \infty$, and the spaces ℓ^p, $1 < p \leqslant 2$, admit coarse embeddings into each other (Nowak [32,33]). At the same time, ℓ^p do not coarsely embed into ℓ^2 for $p > 2$ (W.B. Johnson and Randrianarivony [19]). (Notice that all the proofs mentioned in this paragraph are also emulations of analogous results obtained for uniform embeddings much earlier.)

While working on this article, the author became aware of an announcement by Vincent Lafforgue [23] that a family of expander graphs associated to the groups SL(3) over p-adic fields does not embed into any uniformly convex Banach space.

Another problem that has stimulated the present investigation is a question of the existence of a metric space that admits no uniform embedding into a *reflexive* Banach space. (Cf., e.g., Question 6.5 in [29].) This question was answered in the negative by Kalton [20].

In the present paper, we show that the universal Urysohn metric space $\mathbb{U}$ admits neither uniform nor coarse embedding into a uniformly convex Banach space. Of course, the space $\mathbb{U}$ is far from having a bounded geometry, quite on the contrary. Besides, the results turn out to be outdone not only by Lafforgue's result mentioned above, but by a remarkable theorem of Kalton [20]: the space c_0 admits neither uniform nor coarse embedding into a *reflexive* Banach space. (I have learned about the yet unpublished paper of Kalton from the arXiv preprint [1], which was submitted after the first version of the present article.)

However, our method of proof is rather different from the existing methods for showing noncoarse embeddability of various metric spaces. We are using the Hrushovski–Solecki–Vershik property of the Urysohn space as a basis for an averaging argument, which could be of interest on its own. This is why the present author feels the publication of this work still has some merit.

2. The Urysohn metric space

The universal Urysohn metric space $\mathbb{U}$ is determined uniquely, up to an isometry, by the following description: $\mathbb{U}$ is a complete separable metric space which contains an isometric copy of every separable metric space and is *ultrahomogeneous,* that is, every isometry between two finite subspaces of $\mathbb{U}$ can be extended to a global self-isometry of the space $\mathbb{U}$.

Chapter 5 in the author's book [35] is an introduction to the Urysohn space rather well suited for our needs. Another highly-recommended introductory source is Melleray's article [31] written for the present volume. Other self-contained introductions to the Urysohn space can be found in Gromov's book [12] and articles by Vershik [42,43] and Uspenskij [41].

One obtains numerous variations of the Urysohn space by restricting the set of all possible values that the distances in the metric spaces appearing in the above description of $\mathbb{U}$ can assume. Such variations include the *integer-valued* Urysohn space $\mathbb{U}_{\mathbb{Z}}$ and the *rational Urysohn space,* $\mathbb{U}_{\mathbb{Q}}$. One can also consider the *bounded Urysohn space of diameter one*, $\mathbb{U}_1$, where all distances between two different points take values in the interval [0, 1]. It is an easy exercise, to show that $\mathbb{U}_1$ is isometric to the sphere of radius 1/2 taken in $\mathbb{U}$ around any point. The space $\mathbb{U}_1$ is also known as the *Urysohn sphere.*

Recall that Rado's *infinite random graph, R,* is defined by the following conditions: it is a simple unoriented graph that is universal for all countable graphs (that is, contains an isomorphic copy of every countable graph as an induced subgraph) and ultrahomogeneous in the sense that every isomorphism between two finite induced subgraphs extends to a global automorphism of R [4]. The Urysohn space $\mathbb{U}_{0,1,2}$ whose distances take only values 0, 1 and 2, is easily shown to be isometric to the infinite random graph R equipped with the path-distance (where every edge has length 1).

Now we need to recall another notion, due to Uspenskij [41]. Say that a subspace X of a metric space Y is *g-embedded* into Y if there is a group homomorphism $h : \mathrm{Iso}(X) \to \mathrm{Iso}(Y)$, continuous with regard to the topology of simple convergence on both groups and such that for every isometry $i \in \mathrm{Iso}(X)$ one has

$$h(i)|_X = i.$$

In other words, one can simultaneously extend all isometries from X to Y, preserving the algebraic operations and in a "continuous way".

One of the most useful results of the theory of the Urysohn space, due to Uspenskij [40], says that every separable metric space can be g-embedded into the Urysohn space $\mathbb{U}$. See also [41,35,31]. The same method (Katětov extensions) lead to the following observation, also due to Uspenskij.

Theorem 2.1. *Every compact metric subspace X of $\mathbb{U}$ is g-embedded.*

The following result is obtained along the same lines.

Theorem 2.2. *Every finite metric subspace of the integer Urysohn space $\mathbb{U}_{\mathbb{Z}}$ is g-embedded.*

Proof. The proof is modelled on Uspenskij's original argument, as presented, e.g., in our book [35] on pp. 109–111, 114–115. A function $f : X \to \mathbb{R}$ on a metric space is called a *Katětov function* if

$$\left| f(x) - f(y) \right| \leqslant d_X(x, y) \leqslant f(x) + f(y) \tag{1}$$

for all $x, y \in X$. Such functions are exactly distances from points in metric extensions of X. A 1-Lipschitz function f as above is *controlled by a metric subspace $A \subseteq X$* if f is the largest 1-Lipschitz function on X having a given restriction to A. The set of all Katětov functions controlled by finite subspaces of X, equipped with the supremum metric, is denoted $E(X)$, and contains X in a canonical way under the Kuratowski embedding associating to an $x \in X$ the distance function from x. The completion of the union of a recursively built chain of n-fold extensions $E^n(X) = E(E^{n-1}(X))$ is isometrically isomorphic to the Urysohn space $\mathbb{U}$.

Denote by $E_{\mathbb{Z}}(X)$ the collection of all integer-valued Katětov functions on X controlled by finite subsets, equipped with the uniform distance $d(f, g) = \sup\{|f(x) - g(x)|: \ x \in X\}$. This metric space is nontrivial, because it contains, for instance, all distance functions $d(x, -)$ from points in X. Moreover, the Kuratowski embedding

$$X \ni x \mapsto d(x, -) \in E_{\mathbb{Z}}(X)$$

is easily seen to be an isometric embedding. We will thus identify X with a metric subspace of $E_{\mathbb{Z}}(X)$. The union of an infinite chain of iterated n-fold Katětov extensions of X can be verified to be isometric to $\mathbb{U}_{\mathbb{Z}}$ (cf. [26]). If G is a group acting on X by isometries, then the action lifts to the Katětov extension $E_{\mathbb{Z}}(X)$ through the left regular representation, and the Kuratowski embedding is G-equivariant under this lifting. This means that the action of G goes all the way through to $\mathbb{U}$. $\square$

The space $E_{\mathbb{Z}}(X)$ also appears in [26] and [44]. Here is another application of this construction.

From the viewpoint of coarse geometry, all bounded metric spaces, such as $\mathbb{U}_1$ or R, are alike: they are coarsely equivalent to a single point. Two metric spaces X and Y are *coarsely equivalent* if there is a pair of maps $f : X \to Y$, $g : Y \to X$ such that the compositions fg, gf are uniformly close to the corresponding identity maps, that is, the functions $d_X(gf(x), x)$ and $d_Y(fg(y), y)$ are bounded on X and Y, respectively.

The real line $\mathbb{R}$ is coarsely equivalent to the subspace $\mathbb{Z}$. As a function $f : \mathbb{Z} \to \mathbb{R}$, one can take the canonical embedding, and as a function $g : \mathbb{R} \to \mathbb{Z}$, the integer part function. This observation generalizes as follows: every metric space is coarsely equivalent to its subspace forming an ε-net for some $\varepsilon > 0$. As one of the authors of [26] (Lionel Nguyen Van Thé) has pointed out to the present author, the proof of Proposition 1 (Section 2.1 in [26]) can be modified so as to establish the following result.

Theorem 2.3 *(Jordi López-Abad and Lionel Nguyen Van Thé).* *The Urysohn space $\mathbb{U}$ contains as a 1-net an isometric copy of the integer Urysohn space $\mathbb{U}_{\mathbb{Z}}$.*

Corollary 2.4. *The Urysohn space $\mathbb{U}$ is coarsely equivalent to the integer-valued Urysohn space $\mathbb{U}_{\mathbb{Z}}$.*

Since the composition of two coarse embeddings is a coarse embedding, and every coarse equivalence is a coarse embedding, it follows from Corollary 2.4 that for the purpose of considering coarse embeddings, there is no difference between $\mathbb{U}$ and $\mathbb{U}_{\mathbb{Z}}$.

3. The Hrushovski–Solecki–Vershik homogeneity property

Definition 3.1. Let us say that a metric space X has the *Hrushovski–Solecki–Vershik property* if for every finite subspace Y of X there exists a finite Z with $Y \subseteq Z \subseteq X$ such that every partial isometry of Y extends to a self-isometry of Z.

Proposition 3.2. *For a complete separable metric space X the Hrushovski–Solecki–Vershik property implies ultrahomogeneity.*

Proof. Required global isometries of X are built up recursively. Namely, if A and B are two finite metric subspaces of X, and $i : A \to B$ is an isometry, then there exists a finite $Y \subseteq X$ and a self-isometry j_0 of Y whose restriction to A coincides with i. Now enumerate an everywhere dense subset $X' = \{x_i : i \in \mathbb{N}_+\}$ of X, and choose an increasing chain

$$Y = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \subseteq \cdots$$

of subspaces of X and their self-isometries j_n in such a way that $Y_{n+1} \supseteq Y_n \cup \{x_n\}$ and $j_{n+1}|_{Y_n} = j_n$. The mapping j defined by the rule $j(x) = j_n(x)$ whenever $x \in Y_n$ is a self-isometry of X', and it extends by continuity over all of X. $\quad\square$

Corollary 3.3. *A metric space X has the Hrushovski–Solecki–Vershik property if and only if for every finite subspace Y of X there exists a finite Z with $Y \subseteq Z \subseteq X$ such that every partial isometry of Y extends to a global self-isometry of X taking Z to itself.*

The converse to Proposition 3.2 is not true: the Hrushovski–Solecki–Vershik property does not in general follow from ultrahomogeneity.

The topology of pointwise convergence on the isometry group $\mathrm{Iso}(X)$ of a metric space X coincides with the compact-open topology and is a group topology. If X is in addition separable and complete, then $\mathrm{Iso}(X)$ is a Polish group. The following belongs to Vershik.

Proposition 3.4. *If a separable metric space X has the Hrushovski–Solecki–Vershik property, then the isometry group $\mathrm{Iso}(X)$ contains an everywhere dense locally finite subgroup.*

Proof. A dense locally finite subgroup is built recursively, much like in the proof of Proposition 3.2. $\quad\square$

Example 3.5. The Hilbert space ℓ^2, as well as all Euclidean spaces $\ell^2(n)$ (including the real line $\mathbb{R}$), are ultrahomogeneous [3, Chapter IV, §38, Property 4]. However, each of them fails the Hrushovski–Solecki–Vershik property.

Indeed, let $x \neq 0$ be an element of a Euclidean space E (finite or infinite-dimensional), and let T_x denote the translation $y \mapsto y + x$. Let $Y = \{0, x, 2x, 3x\}$, $A = \{0, x, 2x\}$, and let $i = T_x|_A$. Let j be an isometry of some larger subspace $Z \supseteq Y$ of E extending the partial isometry i. The points $j(0) = x$, $j(x) = 2x$ and $j(2x)$ form a metric space isometric to $(0, x, 2x)$ and therefore lie on a straight line, and consequently one must have $j(2x) = 3x$. An inductive argument shows that Z contains all elements of the form nx and thus is infinite.

A similar argument (using a rotation by an irrational angle along a grand circle instead of a translation along a straight line) gives the following.

Example 3.6. The unit sphere $\mathbb{S}$ of every Hilbert space of positive dimension (including the circle $\mathbb{S}^1$ and the infinite-dimensional unit sphere $\mathbb{S}^\infty$ of ℓ^2) does not have the Hrushovski–Solecki–Vershik property.

The sphere $\mathbb{S}^\infty$ has a somewhat weaker property: every finite collection of isometries can be simultaneously approximated in the strong operator topology with elements of a finite subgroup of isometries (Kechris [22], a remark on p. 186). Notice that the sphere $\mathbb{S}^\infty$ is ultrahomogeneous.

The following two results explain the origin of the name that we gave to the property above. The first one is an equivalent reformulation of a result by Hrushovski [18].

Theorem 3.7. *The infinite random graph (or, equivalently, the Urysohn space $\mathbb{U}_{0,1,2}$) has the Hrushovski–Solecki–Vershik property.*

The second result has been established by Solecki [39] and, independently, by Vershik [44]. The following is an equivalent reformulation.

Theorem 3.8. *The Urysohn metric space $\mathbb{U}$, the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$ and the integer Urysohn space $\mathbb{U}_{\mathbb{Z}}$ all have the Hrushovski–Solecki–Vershik property.*

Solecki's proof is a corollary of deep results of the paper [16] of Herwig and Lascar, while Vershik's proof is direct. In Appendix A we present a deduction of the theorem from results of [16], although in a somewhat different way from Solecki's argument.

Hrushovski's theorem admits a very simple direct combinatorial proof, cf. [16, Section 4.1]. This author has been unable to find an analogue for finite metric spaces.

4. Averaging distances

Let $X_\alpha, \alpha \in A$, be a family of metric spaces. Choose a point x_α in each X_α. Define the set

$$\ell^\infty(X_\alpha, x_\alpha, A) = \left\{ y \in \prod_{\alpha \in A} X_\alpha \;\middle|\; \sup_{\alpha \in A} d(x_\alpha, y_\alpha) < \infty \right\}.$$

Now let ξ be an ultrafilter on the index set A. Equip $\ell^\infty(X_\alpha, x_\alpha, A)$ with the pseudometric

$$d(y, z) = \lim_{\alpha \to \xi} d(y_\alpha, z_\alpha).$$

The *metric space ultraproduct* along ξ of the family (X_α) centred at (x_α) is the metric quotient of the pseudometric space $\ell^\infty(X_\alpha, x_\alpha, A)$. If ξ is a free ultrafilter, then the ultraproduct is complete.

In a particular case where every X_α is a normed space and $x_\alpha = 0$, we obtain the familiar concept of the ultraproduct of a family of normed spaces. In nonstandard analysis, Banach space ultraproducts are known as *nonstandard hulls*. For more on them, see [15] and references therein.

However, even in the case where $x_\alpha \neq 0$, the construction is of interest.

Proposition 4.1. *The ultraproduct of a family of normed spaces X_α centred at an arbitrary family of points (x_α) becomes an affine normed space in a natural way, and for different choices of (x_α) all affine normed spaces arising in this way are pairwise affinely isomorphic and isometric.*

Proof. Every linear translation in the product $\prod_{\alpha \in A} X_\alpha$ preserves the values of the distance $\lim_{\alpha \to \xi} d(y_\alpha, z_\alpha)$ and consequently defines an isometry between ultraproducts centered at different points. Since every self-isometry of a normed space is an affine map, it follows that every isometry between two such ultraproducts transports the affine structure in a unique way. An alternative way to furnish each ultraproduct with an affine structure is through noticing that the space $\ell^\infty(X_\alpha, x_\alpha, A)$ is an affine subspace of the linear space $\prod_{\alpha \in A} X_\alpha$, of which the ultraproduct is a quotient affine space. $\quad\square$

For a normed space E and a set Z, denote by $\ell^2(E, Z)$ the ℓ^2-type sum of $|Z|$ copies of E.

Theorem 4.2. *Let G be a locally finite group acting by isometries on a metric space X and having a dense orbit. Suppose that X admits a mapping ϕ into a normed space E such that for some functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ one has*

$$\rho_1\big(d_X(x, y)\big) \leqslant \|\phi(x) - \phi(y)\| \leqslant \rho_2\big(d_X(x, y)\big). \tag{2}$$

Then there is a map ψ of X into a Banach space ultrapower of $\ell^2(E, \xi)$, satisfying the same inequalities (2) and such that the action of G on $\psi(X)$ extends to an action of G by affine isometries on the affine span of $\psi(X)$.

Proof. Denote by Ξ the set Ξ of all finite subgroups of G. For every $F \in \Xi$ define a map $\psi_F : X \to \ell^2(E, F)$ as follows:

$$\psi_F(x)(g) = \frac{1}{|F|^{1/2}} \psi\left(g^{-1}x\right).$$

Since the distance $d = d_X$ is G-invariant, the map ψ_F satisfies the inequalities (2), e.g., for the right-hand side inequality:

$$\left\|\psi_F(x) - \psi_F(y)\right\| = \left(\sum_{g \in F} \frac{1}{|F|} \left\|\psi\left(g^{-1}x\right) - \psi\left(g^{-1}y\right)\right\|_E^2\right)^{1/2}$$

$$\leqslant \frac{1}{|F|^{1/2}} \left(|F| \cdot \rho_2^2\left(d(x, y)\right)\right)^{1/2}$$

$$= \rho_2\left(d(x, y)\right).$$

Make $\ell^2(E, F)$ into an F-module via the left regular representation:

$$^h f(g) = f\left(h^{-1}g\right), \quad g, h \in F.$$

The map ψ_F is F-equivariant: for every $g \in F$,

$$^h \psi_F(x)(g) = \psi_F(x)\left(h^{-1}g\right) = \frac{1}{|F|^{1/2}} \psi\left(g^{-1}hx\right) = \psi_F(hx)(g).$$

Choose an ultrafilter ξ on Ξ with the property that for each $F \in \Xi$ the set of all $\Phi \in \Xi$ containing F is in ξ. Select a point $x^* \in X$ whose G-orbit is dense in X. Denote by V the ultraproduct of the spaces $\ell^2(E, F)$, $F \in \Xi$, along the ultrafilter ξ, centred at the family of points $(\psi_F(x^*))_{F \in \Xi}$.

For every $h \in G$, the family $(\psi_F(hx^*))_{F \in \Xi}$ is at a finite distance from $(\psi_F(x^*))_{F \in \Xi}$:

$$\sup_{F \in \Xi} \left\|\psi_F(hx^*) - \psi_F(x^*)\right\| = \sup_{F \in \Xi} \frac{1}{|F|^{1/2}} \left(\sum_{g \in F} \left\|\psi\left(g^{-1}hx^*\right) - \psi\left(g^{-1}x^*\right)\right\|_E^2\right)^{1/2}$$

$$\leqslant \sup_{F \in \Xi} \frac{1}{|F|^{1/2}} \left(|F| \cdot \rho_2\left(d\left(g^{-1}hx^*, g^{-1}x^*\right)^2\right)\right)^{1/2}$$

$$= \rho_2\left(d(hx^*, x^*)\right).$$

This has two consequences. First, since the G-orbit of x^* is dense in X, the family of mappings (ψ_F) determines a well-defined mapping $\psi : X \to V$ (that is, for every $x \in X$ the image $\psi(x)$ is a well-defined element of V). Second, for every h the translation by h determines an isometry of V, and in this way the group G acts on V by isometries. making the mapping ψ G-equivariant.

It remains to notice that the space $\ell^2(E, \xi)$ contains all spaces $\ell^2(E, F)$ as normed subspaces, so a suitable metric space ultrapower of $\ell^2(E, \xi)$ contains our metric space ultraproduct of $\ell^2(E, F)$. Finally, each metric ultrapower of $\ell^2(E, \xi)$ is isometrically affinely isomorphic to the Banach space ultrapower. $\quad\Box$

Let us introduce a natural concept: say that a group G of isometries of a metric space X is *almost n-transitive* if for every $\varepsilon > 0$ and every isometry i between two subspaces $A, B \subseteq X$ of cardinality n each there is a $g \in G$ with the property that for all $a \in A$ one has $d(i(a), ga) < \varepsilon$.

Remark 4.3. Every dense locally finite subgroup of the group of isometries $\mathrm{Iso}(X)$ of a separable metric space having the Hrushovski–Solecki–Vershik property (Proposition 3.4) is almost n-transitive for every n.

Corollary 4.4. *Let X be a metric space admitting a locally finite almost 2-transitive group of isometries. Let ψ be a mapping of X into a normed space E satisfying inequalities (2). Then there is a map ψ of X into a Banach space ultrapower of $\ell^2(E, \xi)$, satisfying the same inequalities (2) and such that the image $\psi(X)$ is a metric transform of X: the distance $\|\psi(x) - \psi(y)\|$ is a function of $d(x, y)$ alone. Furthermore, the action of the group of isometries of X extends to a representation by affine isometries on the affine span of $\psi(X)$, making ψ an equivariant map.*

5. Non-existence of uniform embeddings

Uniform convexity is a *metric* property of a Banach space, which can be lost if a norm is replaced by an equivalent one. The corresponding property of Banach spaces invariant under isomorphisms is *superreflexivity*. A Banach space is superreflexive if every Banach space ultrapower of it is a reflexive Banach space. It can be shown that a Banach space is superreflexive if and only if it admits an equivalent uniformly convex norm (cf. [10]). In our context, speaking of superreflexive Banach spaces is more appropriate, because both coarse and uniform structures are invariant under Banach space isomorphisms.

Theorem 5.1. *The universal Urysohn metric space $\mathbb{U}$ cannot be embedded, as a uniform subspace, into a superreflexive Banach space.*

Proof. Assume, towards a contradiction, that an embedding, $\phi : \mathbb{U} \to E$, does exist. Choose a dense locally finite subgroup of $\mathrm{Iso}(\mathbb{U})$ (Proposition 3.4). This group is almost n-transitive for each n. By Corollary 4.4, there exist a mapping, ψ, of $\mathbb{U}$ into the Banach space ultrapower V of $\ell^2(E)$, as well as a representation π of G by affine isometries of the affine span F of $\psi(\mathbb{U})$ in V, making ψ equivariant.

Since the mapping ψ is a uniform isomorphism on its image and in particular a homeomorphism, the topology of pointwise convergence on X on the group G coincides with the topology of pointwise convergence on $\psi(X)$ and consequently on the affine span of $\psi(X)$. The action of G on F extends by continuity to an action of all of $\mathrm{Iso}(\mathbb{U})$ on F, and ψ still remains equivariant. Consequently, the affine representation of $\mathrm{Iso}(\mathbb{U})$ on F is faithful.

This representation is a continuous homomorphism from G to the group of affine isometries of F, that is, the semidirect product $\mathrm{Iso}(F) \ltimes F_+$. Here $\mathrm{Iso}(F)$ denotes the group of linear isometries, while F_+ is the additive group of F upon which $\mathrm{Iso}(F)$ acts in a canonical way. Let $\pi : \mathrm{Iso}(F) \ltimes F_+ \to \mathrm{Iso}(F)$ be the standard projection (a quotient homomorphism).

The Polish group $\mathrm{Iso}(\mathbb{U})$ is universal [41]. In particular, it contains, as a topological subgroup, the group $\mathrm{Homeo}_+[0, 1]$ of all homeomorphisms of the unit interval preserving endpoints, equipped with the standard compact-open topology. We have, therefore, a faithful continuous affine representation of this group in a superreflexive Banach space.

According to Megrelishvili [28], the only continuous representation of $\mathrm{Homeo}_+[0, 1]$ by linear isometries in a reflexive Banach space is the trivial (identity) representation. Therefore, the composition of three homomorphisms

$$\mathrm{Homeo}_+[0, 1] \hookrightarrow \mathrm{Iso}(\mathbb{U}) \to \mathrm{Iso}(F) \ltimes F_+ \xrightarrow{\pi} \mathrm{Iso}(F)$$

is a trivial map. This means the image of $\mathrm{Homeo}_+[0, 1]$ is contained, as a topological subgroup, in the kernel of π, that is, the Abelian Polish group F_+, which is absurd. $\quad\square$

6. Non-existence of coarse embeddings

We will begin by recalling a useful test for a space not to be (super)reflexive. Let $\varepsilon > 0$ and let n be either a natural number or the symbol ∞. An (n, ε)-*tree* in a normed space E is a binary tree T of depth n whose nodes are elements of E such that for every node x its children nodes y and z have the properties: $x = (y + z)/2$ and $\|y - z\| \geqslant \varepsilon$.

Theorem 6.1. *(See, e.g., [10, p. 295].) If a normed space E contains a uniformly bounded family of (n, ε)-trees for some $\varepsilon > 0$ and all natural n, then E is not superreflexive. If E contains a bounded (∞, ε)-tree for some $\varepsilon > 0$, then E is non-reflexive.*

Here is a consequence that we will be using.

Corollary 6.2. *(Cf. a similar statement in [10, Exercise 9.22, p. 308].) Let a normed space E have the following property: for some $\varepsilon > 0$ and $M > 0$ and for every n, the M-ball around zero contains a sequence of closed convex subsets K_i, $i = 1, 2, \ldots, n$, such that $K_{2i} \cup K_{2i+1} \subseteq K_i$ and K_{2i} and K_{2i+1} are at a distance at least ε from each other for all i. Then E is not superreflexive.*

Proof. For every n one can easily construct a (n, ε)-tree contained in the M-ball of E by recursion, starting with the leaves and using the fact that the midpoint of two nodes belonging to K_{2i} and K_{2i+1}, respectively, is contained in K_i. The assumption on the distance between K_{2i} and K_{2i+1} assures that the two children nodes are always at least ε-apart from each other. $\quad\square$

We need to recall a classical result by Day.

Theorem 6.3. *(See Day [5].) The ℓ_p-type direct sum of normed spaces, $1 < p < \infty$, is uniformly convex if and only if they have a common modulus of convexity.*

Recall that a normed space E is *uniformly smooth* [2, Appendix A], [10, Chapter 9] if $\|x + y\| + \|x - y\| = o(\|y\|)$ as $\|y\| \to 0$, uniformly for all x in the unit sphere of E. More precisely, the *modulus of uniform smoothness*, δ, of E is defined for $\tau > 0$ by

$$\rho(\tau) = \sup\left\{ \frac{\|x + \tau h\| + \|x - \tau h\|}{2} : \|x\| = \|h\| = 1 \right\}.$$

Now E is uniformly smooth if and only if

$$\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0.$$

An ultraproduct of a family of uniformly smooth normed spaces of the same modulus of smoothness is again uniformly smooth, which is straightforward. The same is true of the ℓ^p-direct sum, $1 < p < \infty$. (The result can be deduced from Day's theorem by using duality, as there is a correspondence between the moduli of uniform convexity and of uniform smoothness of E and of E^*.)

Every superreflexive space admits an equivalent norm that is both uniformly convex and uniformly smooth. A space E is uniformly convex iff E^* is uniformly smooth, and vice versa. (See [10, Chapter 9].)

Every point x of the unit sphere of a uniformly smooth normed space E is a smooth point, that is, there exists a unique linear functional $j(x) = \varphi \in E^*$ of norm one such that $\varphi(x) = 1$. (See, e.g., [2, p. 70].)

Theorem 6.4. *The universal Urysohn metric space $\mathbb{U}$, as well as the integer-valued Urysohn space, $\mathbb{U}_{\mathbb{Z}}$, do not admit a coarse embedding into a superreflexive Banach space.*

Proof. Assume, towards a contradiction, that a coarse embedding $\phi : \mathbb{U}_{\mathbb{Z}} \to E$ exists. Since the space E is super-reflexive, it admits an equivalent norm that is at the same time uniformly convex and uniformly smooth.

By Corollary 4.4, there exist a coarse embedding ψ of $\mathbb{U}_{\mathbb{Z}}$ into an ultrapower of $\ell^2(E)$ and a strongly continuous representation of $\mathrm{Iso}(\mathbb{U}_{\mathbb{Z}})$ by affine isometries in a closed subspace F spanned by $\psi(\mathbb{U}_{\mathbb{Z}})$, such that ψ is an $\mathrm{Iso}(\mathbb{U}_{\mathbb{Z}})$-equivariant mapping. Since ℓ^2-sums and ultraproducts preserve uniform convexity and uniform smoothness, and these properties are inherited by normed subspaces, the norm on the space F is both uniformly convex and uniformly smooth.

It follows from the same Corollary 4.4 that the coarse embedding ψ is a metric transform:

$$\forall x, y \in \mathbb{U}_{\mathbb{Z}}, \quad \|\psi(x) - \psi(y)\| = \rho\big(d(x, y)\big),$$

where $\rho(r) \to \infty$ as $r \to \infty$. By rescaling the norm of F if necessary, one can assume without loss in generality that $\rho(1) = 1$.

Choose an arbitrary point $z_0 \in \mathbb{U}_{\mathbb{Z}}$ and denote $G = \mathrm{Iso}(\mathbb{U}_{\mathbb{Z}})_{z_0}$ the isotropy subgroup of z_0 in $\mathrm{Iso}(\mathbb{U}_{\mathbb{Z}})$. Make the affine space F into a linear space by setting $0_F = \psi(z_0)$. The action of G on F becomes a linear continuous isometric representation.

Fix a natural number m with $\rho(m) \geqslant 6$, and let $x, y \in S_m(z_0)$ be such that $d(x, y) = m$. Then $\|\psi(x) - \psi(y)\| = \rho(m)$ and by the Hahn–Banach theorem, there is a linear functional $\varphi \in F^*$ of norm 1 such that

$$\varphi\big(\psi(x) - \psi(y)\big) = \rho(m).$$

Since F is uniformly smooth, such a φ is unique (the support functional of $\psi(x) - \psi(y)$).

Let $x = x_0, x_1, \ldots, x_{m-1}, x_m = y$ be a sequence of elements in $S_m(z_0)$ satisfying $d(x_i, x_j) = |i - j|$ for all $i, j = 0, 1, \ldots, m$. Any two subsequent values $\varphi(\psi(x_i))$, $i = 0, 1, \ldots, m$, differ by at most 1.

There is an isometry $f \in G$ interchanging x_j and x_{n-j} for every j. In particular, f flips x and y. The corresponding linear isometry of E^* will take φ to the support functional of $\psi(y) - \psi(x)$, that is, to $-\varphi$. This means that $\varphi(\psi(x_j)) = -\varphi(\psi(x_{m-j}))$ for every $j = 0, 1, 2, \ldots, m$, in particular, $\varphi(\psi(x)) = -\varphi(\psi(y))$. Without loss in generality one can assume that $\varphi(\psi(x))$ is negative.

Denote $k = \max\{j = 0, 1, 2, \ldots, m: \varphi(\psi(x_j)) < 0\}$. Then $\varphi(\psi(x_{k+1})) \geqslant 0$. Let z and w be arbitrary points of $S_m(z_0)$ such that $d(x, z) = k$, $d(x, w) = k + 1$, $d(y, z) = m - k$, and $d(y, w) = m - k - 1$. There exists a $g \in G$ stabilizing x and y and taking $z \mapsto x_k$ and $w \mapsto x_{k+1}$. The extension of g to F^* will leave φ fixed, because of its uniqueness as a support functional. We conclude that $\varphi(\psi(z)) = \varphi(\psi(x_k))$ and $\varphi(\psi(w)) = \varphi(\psi(x_{k+1}))$. This can be summarized as follows: the functional φ assumes the constant value $\varphi(\psi(x_k)) < 0$ at all points of $\psi(S_k(x) \cap S_{m-k}(y))$, and the constant value $\varphi(\psi(x_{k+1})) \geqslant 0$ at all points of $\psi(S_{k+1}(x) \cap S_{m-k-1}(y))$. Denoting $\gamma = |\varphi(\psi(x_k))| > 0$, one concludes: the closed convex hulls of $\psi(S_k(x) \cap S_{m-k}(y))$ and of $\psi(S_{k+1}(x) \cap S_{m-k-1}(y))$ are at least $\gamma > 0$ apart.

Our choice of m assures that $k \geqslant 2$, and in particular the intersections $S_k(x) \cap S_{m-k}(y)$ and $S_{k+1}(x) \cap S_{m-k-1}(y)$ are infinite. In fact, they are both isometrically isomorphic to the Urysohn metric space $\mathbb{U}_{0,1,\ldots,k}$ of diameter $2k \geqslant 4$.

Let N be given. Choose $2N + 2$ points a_i, b_i, $i = 0, 1, 2, \ldots, N$, on the sphere $S_m(z_0)$ so that $a_0 = x$, $b_0 = y$ and the distances between any two distinct points from among them is given by:

$$d(z, w) = \begin{cases} 1, & \text{if } z = a_i, \ w = a_j, \ i \neq j, \\ 1, & \text{if } z = b_i, \ w = b_j, \ i \neq j, \\ m, & \text{if } z = a_i, \ w = b_i, \\ m - 1, & \text{if } z = a_i, \ w = b_j, \ i \neq j. \end{cases}$$

For an arbitrary sequence $\varepsilon = (\varepsilon_i)_{i \leqslant N} \in \{0, 1\}^N$, define the function f_ε on $\{z_0\} \cup \{a_i\}_{i \leqslant N} \cup \{b_i\}_{i \leqslant N}$ by the conditions:

$$f_\varepsilon(z_0) = m, \qquad f_\varepsilon(a_i) = k + \varepsilon_i, \qquad f_\varepsilon(b_j) = m - k - \varepsilon_j.$$

Now one can verify, by considering 17 separate cases, that f_ε is an (integer-valued) Katětov function and so a distance function from some point $x^* \in S_m(z_0)$. It means that the intersection

$$T_\varepsilon = \bigcap_{i=0}^{N} S_{k+\varepsilon_i}(a_i) \cap \bigcap_{i=1}^{N} S_{m-k-\varepsilon_i}(b_i) \cap S_m(z_0)$$

is non-empty.

Associate to every $\varepsilon \in \{0, 1\}^N$ the closed convex hull C_ε of $\psi(T_\varepsilon)$, which is a weakly compact subset of F. If $\varepsilon \leqslant \delta$, $\varepsilon \neq \delta$, then C_ε and of C_δ are at a distance of at least $\gamma > 0$ from each other, where the constant γ was defined previously in this proof. Indeed, suppose $0 \leqslant i \leqslant N$ be such that $\varepsilon_i \neq \delta_i$, and let h be an isometry of $\mathbb{U}_\mathbb{Z}$, preserving z_0 and taking $x \mapsto a_i$ and $y \mapsto b_i$. The linear functional $\varphi \circ h$ has norm one and assumes constant values on C_ε and on C_δ, differing between themselves by at least γ.

Denote by $\mathcal{T}$ the prefix binary tree associated to the Hamming cube $\{0, 1\}^N$, i.e., $\mathcal{T}$ consists of all prefix strings of elements $\varepsilon \in \{0, 1\}^N$, with $\sigma \leqslant \tau$ if and only if σ is a prefix of τ. The system C_σ, $\sigma \in \mathcal{T}$, forms a binary tree under inclusion, and for every two nodes σ, τ at the same level the distance between C_σ and C_τ is at least γ. By Corollary 6.2, the space F is not uniformly convex. $\quad\square$

7. Open questions

(1) The most interesting open question at the moment of writing this article seems to be whether or not every finitely generated group admits a coarse embedding into a superreflexive Banach space.

(2) It remains unknown whether a direct sum of graphs forming an expander family can ever admit a coarse embedding into a superreflexive Banach space (cf. open problem ♯ 9, submitted by Piotr Nowak on the list [34]).

(3) Is there a proof of the Solecki–Vershik property along the lines of a simple combinatorial proof of the Hrushovski theorem given in [16, Section 4.1]?

(4) Is the following "coarse analogue" of Holmes' theorem [17] true? Suppose the Urysohn space $\mathbb{U}$ is coarsely embedded into a normed space E in such a way that the image of $\mathbb{U}$ spans E. Then E is coarsely equivalent to the Lipschitz–free Banach space over $\mathbb{U}$. (Cf. also [35, p. 112], as well as [31].)

(5) Does there exist an analogue of the universal Urysohn metric space in the coarse category among spaces of bounded geometry and exponential growth? Cf. related constructions in [7].

(6) A *regular embedding* of a (simple, non-oriented) graph Γ into a metric space X is a map from the set of vertices of Γ to X such that the distance between images of two adjacent vertices is always α, and between non-adjacent ones is always β, where $\alpha < \beta$. It is well known that many finite graphs do not admit regular embeddings into the Hilbert space. (Cf. [27].) Does the infinite random graph R admit a regular embedding into a reflexive Banach space?

Acknowledgements

Appendix A. Alternative proof of Solecki–Vershik Theorem 3.8

As brought to this author's attention by S. Solecki, the first version of his article [39] contained a proof of Theorem 3.8 along the same lines as outlined below. We include the proof just to make the paper reasonably self-contained.

Let X be a finite metric space. Denote by P the set of all partial isometries p of X whose domain dom p is non-empty. Let $F = F(P)$ be the free group on P. Every word $w \in F$ defines in a unique way a partial isometry of X (possibly one with empty domain), under the convention that the empty word e corresponds to the identity map of X. In this way, one obtains a *partial action* of $F(P)$ on X, that is, a map from $F(P)$ to the set of partial isometries of X satisfying the properties that for all $x \in X$:

(1) $e \cdot x = x$,
(2) if $u \cdot x$ is defined, then $u^{-1} \cdot u \cdot x = x$,
(3) if $u \cdot v \cdot x$ is defined, then $(uv) \cdot x = u \cdot v \cdot x$.

Cf. [9,30].

A *globalization* of a pair consisting of a metric space X and a partial action by a group G on X is a metric space Y containing X as a metric subspace and equipped with a global action of G in such a way that for every $g \in G$ the partial isometry of X defined by g is a restriction of the corresponding global isometry of Y. A *universal globalization* of X is a globalization Z with the property that the embedding of X into any other globalization uniquely factors through the embedding $X \hookrightarrow Z$.

Given a finite metric space X, we will consider it as equipped with a canonical partial action of the free group $F = F(P)$. Since every finite subspace is g-embedded into the Urysohn metric space, every isometric embedding $X \hookrightarrow \mathbb{U}$ determines (in a more than one way) a globalization of X.

The universal globalization of a pair $(X, F(P))$ as above was constructed by Megrelishvili and Schröder in [30]. We will denote this globalization by $U(X)$. The construction of a globalization of a partial action appears in many previous works and, as stressed by the anonymous referee, goes back at least to Mackey. A contribution of [30] is, in particular, establishing that the canonical mapping from X to $U(X)$ is an isometry under a rather weak set of assumptions.

At the set level, this globalization is the quotient set of $F \times X$ modulo the equivalence relation

$$(uv, x) \sim (u, v \cdot x) \quad \text{whenever } v \cdot x \text{ is defined.} \tag{3}$$

As in [30], we will denote the equivalence class of a pair (u, x) by $[u, x]$. The action of F on $U(X)$ is defined by $g \cdot [u, x] = [gu, x]$.

The universal globalization $U(X)$ admits an alternative description as a homogeneous factor-space of the group $F(P)$. Here is a repetition, *mutatis mutandis*, of a construction presented in [16, pp. 1987–1988]. Choose a point $a_0 \in X$ and denote by H_0 a subgroup of F generated by the set

$$X_0 = \left\{ p^{-1} \cdot p' \colon p, p' \in P, \ p(a_0) = p'(a_0) \right\} \cup \left\{ p_3^{-1} \cdot p_1 \cdot p_2 \colon p_1, p_2, p_3 \in P, \ p_1 \circ p_2(a_0) = p_3(a_0) \right\}.$$

Let now H be any subgroup of F satisfying $H_0 < H$.

For every $a \in X$ there is a partial isomorphism $p \in P$ taking a_0 to a (for instance, one with $\operatorname{dom} p = \{a_0\}$, $\operatorname{im} p = \{a\}$). Furthermore, if p' has the property $p'(a_0) = a$, then the left cosets pH and $p'H$ coincide. Therefore, the map ϕ from X to the homogeneous space F/H given by the formula

$$\phi(a) = pH, \quad \text{where } p \in P \quad \text{and} \quad p(a_0) = a,$$

is well-defined. If moreover $H \cap X_1 = \emptyset$, where

$$X_1 = \left\{ p^{-1} \cdot p' \colon p, p' \in P, \ p(a_0) \neq p'(a_0) \right\},$$

then ϕ is injective. We will assume this condition to be satisfied, and will identity X with its image under ϕ in F/H.

Every $g \in F$ determines a left translation of the factor-space F/H, which we will denote $\tilde{g}$. It is easy to see that for every $p \in P$ and every $a \in \operatorname{dom} p$ one has $\tilde{p}(a) = p(a)$. Indeed, this condition means, in full, $p\phi(a) = \phi(p(a))$, or $pp_1 H = p_2 H$, where $p_1(a_0) = a$ and $p_2(a_0) = p(a)$. Since $p_2^{-1} pp_1(a_0) = a_0$, one has $p_2^{-1} pp_1 \in H_0 \subseteq H$, and the condition holds.

In order to make F/H into a metric space, we first turn it into an edge-coloured graph. Namely, we add an edge labeled with a real number r between two elements α and α' if and only if there are $a, a' \in X$ and a $g \in F$ with

$$d(a, a') = r, \qquad ga = \alpha, \quad \text{and} \quad ga' = \alpha'. \tag{4}$$

(Again, we identify a with the corresponding coset $\phi(a)$, etc.)

Since for every word $w \in F(P)$ and each $p \in P$ the cosets wH and wpH are adjacent, with the corresponding edge carrying the weight $d(a_0, p(a_0))$, an inductive argument shows that the graph built on F/H is connected. Now we equip F/H with the corresponding path distance, which is a left-invariant pseudometric.

There are a few potential problems that may arise here. Firstly, is the edge-labeling as above uniquely defined? Secondly, is the path distance a genuine metric (that is, the distance between two distinct points is non-zero)? Thirdly, will the restriction of this distance to X coincide with the original metric on X? (A priori, it is only bounded by d_X from above.)

In the case where $H = H_0$, the answer to all three questions is positive, and it follows from the construction of Megrelishvili and Schröder mentioned above.

Indeed, consider the following formula for an arbitrary $g \in F$:

$$\psi(gH_0) = [g, a_0].$$

Since every element of H_0 stabilizes a_0, one has $[gh, a_0] = [g, h(a_0)] = [g, a_0]$, and the map ψ from F/H_0 to $U(X)$ is well-defined. Clearly, ψ is surjective and F-equivariant. Further, it is not difficult to verify that H_0 is precisely the stabilizer of the class $[e, a_0] \in U(X)$, and so the map ψ is a bijection. Lifting the metric from $U(X)$ to F, one concludes that the path distance constructed above is a metric extending the distance d_X.

All that remains to be done, is to show that the same three conclusions hold for at least one subgroup $H < F$ of finite index containing H_0. Notice that the condition that the path distance, d, be a metric on F/H is not essential: as long as the restriction of d to X coincides with the distance on X, one can replace $(F/H, d)$ with the associated metric space, that is, the quotient space under the relation $x \sim y \iff d(x, y) = 0$.

The two conditions to be verified are that the edge labeling be uniquely defined, and that the path distance on F/H be an extension of the distance d_X on X. Both can be reformulated in a unified way as follows. Let $a, b \in X$. Say that two finite sequences of pairs (c_i, d_i) of elements of X and of elements x_i of F, $i = 1, 2, \ldots, n$, form a *bad configuration* for (a, b), if the following conditions are met:

(1) $x_1 \cdot c_1 = a$,
(2) $x_{i+1} c_i = x_i d_i$ for $i = 1, 2, \ldots, n$,
(3) $x_n d_n = b$, and
(4) $\sum_{i=1}^{n} d_X(c_i, d_i) < d_X(a, b)$.

In particular, non-existence of bad configurations implies the uniqueness of labeling.

We want to stress again that here we identify X with its image in F/H under ϕ. A more accurate rendering of the existence of a bad configuration in F/H is therefore given by the following *ad hoc* concept.

Definition A.1. Let X be a finite metric space and H a subgroup of $F(P)$ containing H_0. A *bad configuration* for X modulo H is a collection of elements $p, q, p_i, q_i \in P$ and $x_i \in F(P)$, $i = 1, 2, \ldots, n$, such that

(1) $x_1 p_1 \equiv p \mod H$,
(2) $x_{i+1} p_i \equiv x_i q_i \mod H$ for $i = 1, 2, \ldots, n$,
(3) $x_n q_n \equiv q \mod H$, and
(4) $\sum_{i=1}^{n} d_X(p_i(a_0), q_i(a_o)) < d_X(p(a_0), q(a_0))$.

What we want, is to avoid bad configurations for X modulo H by carefully choosing a finite index subgroup H that in addition contains H_0. Remark that there are only finitely many theoretically possible bad configurations for any given finite metric space X, so if we can learn to choose a subgroup H of finite index containing H_0 and avoiding a given bad configuration, we are done, as the intersection of finitely many subgroups of finite index has finite index by Poincaré's theorem.

Recall that a group G is *residually finite* if homomorphisms from G to finite groups separate points. For example, free groups are residually finite. Equivalently, residual finiteness means that for every finite subset $A \subseteq F \setminus \{e\}$, where F denotes a non-Abelian free group, there is a normal subgroup $N \lhd F$ of finite index disjoint from A. In fact, the collection of subgroups of free groups of finite index is much richer than that, as shows the following surprising result.

Theorem A.2. *(See Hall [14].) Every finitely generated subgroup H of a non-Abelian free group F is the intersection of subgroups $N < F$ of finite index.*

This result can be further strengthened.

Theorem A.3. *(See Ribes and Zalesskiĭ [37].) Let $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of a non-Abelian free group F. Then for every $g \in F \setminus H_1 H_2 \ldots H_n$ there are finite index subgroups $K_1, K_2, \ldots, K_n < F$ such that $H_i < K_i$, $i = 1, 2, \ldots, n$, and $g \notin K_1 K_2 \ldots K_n$.*

A further refinement of the above theorems forms the core result of the above mentioned paper by Herwig and Lascar. To state it, we need to recall their terminology from [16]. A *left-system* is a finite set of equations of the form

$$x \equiv_i y \cdot g \quad \text{or} \quad x \equiv_i g,$$

where $i = 1, 2, \ldots, n$, x, y belong to a finite set of unknowns, X, and g are elements of a free non-Abelian group F. If $\mathcal{H} = (H_1, H_2, \ldots, H_n)$ is a sequence of subgroups of F, a solution of a left-system as above modulo $\mathcal{H}$ is a family $g_x, x \in X$ of elements of F such that for every equation of the form $x \equiv_i y \cdot g$ one has

$$g_x \equiv g_y \cdot g \mod H_i,$$

and for every equation of the form $x \equiv_i g$ one has

$$g_x \equiv g \mod H_i.$$

Theorem A.4. *(See Herwig and Lascar [16].) Let $n \in \mathbb{N}$, let $\mathcal{H} = (H_1, H_2, \ldots, H_n)$ be a sequence of subgroups of a free non-Abelian group F, and let (E) be a left-system of equations in F. Assume that (E) has no solutions in F modulo $\mathcal{H}$. Then there exist finite index subgroups $K_1, K_2, \ldots, K_n$ of F such that $H_i < K_i$, $i = 1, 2, \ldots, n$, and the left-system (E) has no solutions in F modulo $\mathcal{K} = (K_1, K_2, \ldots, K_n)$.*

Notice that the theorem of Ribes and Zalesskiĭ is obtained from the above result if one considers the left-system of the form

$$\begin{cases} x_n \equiv_n g, \\ x_{n-1} \equiv_{n-1} x_n, \\ \quad \vdots \\ x_2 \equiv_2 x_3, \\ x_1 \equiv_1 e. \end{cases}$$

This system has no solutions in F modulo $\mathcal{H}$ if and only if $g \notin H_1 H_2 \ldots H_n$.

Now it remains to notice that the existence of a bad configuration for a finite metric space X modulo a subgroup H (Definition A.1) is equivalent to the existence of a solution to a left-system of equations:

$$\begin{cases} x_1 \equiv p p_1^{-1}, \\ x_2 \equiv x_1 \cdot q_1 p_1^{-1}, \\ x_3 \equiv x_2 \cdot q_2 p_2^{-1}, \\ \quad \vdots \\ x_n \equiv q q_n^{-1}. \end{cases}$$

Since there is no solution of this left-system modulo H_0, there is a subgroup H of finite index containing H_0 and such that there are no solution modulo H either. This finishes the proof of Theorem 3.8 for the case of the Urysohn space.

Note that the above argument in effect leads to the following slight technical generalization. As usual, we say that a subset S of a totally ordered set X is *convex* if for every $x, y, z \in X$ the conditions $x \leqslant y \leqslant z$ and $x, z \in S$ imply $y \in S$.

Theorem A.5 *(General form of Hrushovski–Solecki–Vershik theorem). Let S be a convex subset of an additive sub-semigroup T of $\mathbb{R}$, containing zero. Then the universal Urysohn metric space $\mathbb{U}_S$ whose distance takes values in S has the Hrushovski–Solecki–Vershik property.*

Remark that under our assumptions on S, the space $\mathbb{U}_S$ always exists and is unique, according to Theorem 1.4 and Example 1.5.3 in [6]. Now in the case $S = \{0, 1, 2\}$ (a convex subset of the semigroup $\mathbb{N}$ of natural numbers) one recovers Hrushovski's theorem 3.7, while the case $S = \mathbb{R}$ gives the Solecki–Vershik theorem 3.8.

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