The number of part sizes of a given multiplicity in a random Carlitz composition

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Abstract

A number of characteristics of random classical and Carlitz (adjacent parts are different) compositions of integer \( n \) have been studied by Knopfmacher and Prodinger, Hitczenko and Savage, Goh and Hitczenko, and also by Hitczenko, Rousseau and Savage. This paper is an attempt to complement their results by establishing asymptotics of the average multiplicity of a given part size in a random Carlitz composition. An extension of the Problem of Wilf to the Carlitz case is also presented.

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1. Introduction

The aim of this paper is to obtain precise asymptotics, as \( n \to \infty \), for the expected multiplicity of a given part size in a random Carlitz composition of an integer \( n \). Following [6], we will denote a composition of an integer \( n \) by a \( k \)-tuple \((\gamma_1, \ldots, \gamma_k)\) where \( \gamma_j \)'s are positive integers, called parts, such that \( \sum_j \gamma_j = n \). We will call the number \( k \), “the number of parts”, and the values of \( \gamma_j \)'s we will call “part sizes”. There are \( 2^{n-1} \) different compositions of \( n \) [1]. A composition is called Carlitz if the adjacent parts are different, i.e., if \( \gamma_j \neq \gamma_{j+1} \) for \( j = 1, \ldots, k-1 \). We denote the set of all Carlitz compositions of \( n \) by \( \Omega_n \). “Random Carlitz composition” means a composition chosen accordingly to the uniform probability measure on \( \Omega_n \). Many characteristics of
Carlitz compositions, considered in this setting, have been already studied: total number of Carlitz compositions, expected number of parts, expected size of the largest part (Knopfmacher and Prodinger, [7]), expected number of distinct parts sizes (Goh and Hitczenko, [4]). The number of distinct part sizes of a given multiplicity, which characterizes the degree of distinctness of a composition, was studied in several papers for the classical case. The probability that a randomly chosen part size in a random composition has the given multiplicity was studied by Hitczenko and Savage in [6] and by Hitczenko, Savage and Rousseau in [5]. The stochastic properties of the \( m \)-distinctness of random compositions have been studied by Louchard in [9]. In the conclusion of that paper the author emphasizes that an extension of his results to the Carlitz case would constitute an interesting open problem. The present paper is a step in that direction. More precisely, this is an attempt to extend the above mentioned result in [6] to the Carlitz case. Sections 1–4 are devoted to the number of distinct part sizes of a given multiplicity, while an extension of the Problem of Wilf is presented in Section 5.

Let \( U_n^{(m)}(\kappa) \) be the number of part sizes of multiplicity \( m \) in a random Carlitz composition \( \kappa \) of an integer \( n \), and \( \mathcal{I}(m) \) be the set of such part sizes. Then
\begin{align}
E[U_n^{(m)}] &= E\left[ \sum_j I_{\{j \in \mathcal{I}(m)\}} \right] = \sum_j P\{j \in \mathcal{I}(m)\}.
\end{align}
(1.1)

Let \( c(n) \) be the number of Carlitz compositions of an integer \( n \), and \( g(n, j, m) \) be the number of such compositions where part size \( j \) has a multiplicity \( m \). Then:
\begin{align}
P\{j \in \mathcal{I}(m)\} &= \frac{g(n, j, m)}{c(n)}.
\end{align}
(1.2)

It has been shown in [7] that \( c(n) \sim A\rho^{-n} \) where
\[ A = \frac{1}{\rho \sigma'(\rho)} = 0.456387 \ldots, \quad \rho = 0.571349 \ldots. \]

The main result of this paper is the following.

**Theorem 1.**

\[
E[U_n^{(m)}] = \frac{1 + O(1/n)}{\ln(1/\rho)} \left[ \frac{1}{m} + \frac{2}{m!} \Re \sum_{k=1}^{\infty} e^{-2\pi ik[\log_{1/\rho}(n/\sigma'(\rho))]} \Gamma\left(m + i \frac{2\pi k}{\ln(1/\rho)}\right) \right],
\]
\[ n \to \infty \]

where \( \{a\} = a - \lfloor a \rfloor \) is the fractional part of \( a \), and \( \Gamma \) denotes the gamma function.

As it has been discussed in [5], the above series, considered as a function of \( \log_{1/\rho}(n/\sigma'(\rho)) \), has periodic oscillations and the series converges quite rapidly. Even the first term does not exceed \( 10^{-5} \). The corresponding problem for the classical case was
treated in [6], using probabilistic approach, and in [5], where the advantage of generating functions machinery was taken. Since the first one did not seem to be working in the Carlitz case, we followed the second one, which turned out to be very fruitful. In other words, the proof of Theorem 1 will be delivered via singularity analysis of corresponding generating functions.

2. Generating functions

Let \( g(n, j, m) \) be the number of Carlitz compositions, such that part size \( j \) has multiplicity \( m \). We will prove the following statement.

**Proposition 2.** Let \( G_{j,m}(z) \) be the generating function of the sequence \( \{g(n, j, m)\}_{n \geq 0} \). Then

\[
G_{j,m}(z) = 1 + z^m \left( \frac{\sigma(z) - z^j(1 - \sigma(z))}{1 - \sigma(z) + z^j(2 - \sigma(z))} \right)^{m+1}
+ \mathbb{I}_{[m \geq 1]} z^m \frac{(1 + \sigma(z) + z^j \sigma(z))(\sigma(z) - z^j(1 - \sigma(z)))^{m-1}}{(1 - \sigma(z) + z^j(2 - \sigma(z)))^m}
\]  

(2.1)

where

\[
\sigma(z) = \sum_{l \geq 1} (-1)^l (1 - \frac{z^l}{1 - z^l}).
\]

In order to prove of Proposition 2 we need the following lemmas.

**Lemma 3.** The recurrence

\[
x_{q+1} + \beta x_{q+2} + \gamma x_{q+3} = \alpha_{q+1}, \quad q = 0, 1, \ldots
\]

with the terminal conditions: \( x_{N+1} = x_{N+2} = 0 \), for some \( N \) satisfies the following initial conditions:

\[
x_1 = \sum_{p=1}^{N} \sum_{q=p}^{2p-1} (-1)^{p+1} \frac{(p-1)!}{q!} \beta^{2p-q-1} \gamma^{q-p} \alpha_q.
\]

**Proof.** The above recurrence along with the boundary condition is equivalent to the linear system of algebraic equations \( Mx = c \), where
\[ M = \begin{bmatrix}
1 & \beta & \gamma & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \gamma \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \beta \\
0 & \ddots & \ddots & \ddots & \ddots & 1
\end{bmatrix} \]

\[ x = [x_1, \ldots, x_N]^T, \quad c = [\alpha_1, \ldots, \alpha_N], \]

where \( M \) is \( N \times N \) matrix.

Now we can split \( M \) as the difference: \( M = I - T \), where

\[ T = \begin{bmatrix}
0 & -\beta & -\gamma & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & -\beta \\
0 & \ddots & \ddots & \ddots & \ddots & 0
\end{bmatrix}. \]

Since operator \( T \) is nilpotent (\( T^N = 0 \)), this allows for the following expansion:

\[ M^{-1} = (I - T)^{-1} = I + T + T^2 + \cdots + T^{N-1}. \]

It can be easily seen that the element \((1, q)\) of the matrix \( T^{j-1} \) can be calculated as follows:

\[ t^{(j-1)}_{1,q} = \begin{cases}
0, & \text{if } 1 \leq q < j - 1, \\
(-1)^{j-1} (j-1) \beta^{j-q-1} q^{q-j}, & \text{if } j \leq q \leq 2j - 1, \\
0, & \text{if } q \geq 2j.
\end{cases} \]

This delivers the statement. \( \square \)

**Lemma 4.** Let

\[ G(z, u, v) = \sum_{n,j \geq 1, m \neq 0} g(n, j, m) z^n u^j v^m. \]

Then

\[ G(z, u, v) = \sum_{p \geq 0} \sum_{q=0}^{p-1} (-1)^{p-1} \left( \begin{array}{c} p \\ q \end{array} \right) (\beta(z, v))^{p-q} v^q \alpha(z, z^{p+q} u, v) \]

where
\[ \alpha(z, u, v) = \frac{\psi(z, u, v) + (v + 1)\psi(z, zu, v) + v\psi(z, z^2u, v)}{1 - \sigma(z)}, \]
\[ \beta(z, v) = \frac{2 - (v + 1)\sigma(z)}{1 - \sigma(z)}, \]
\[ \psi(z, u, v) = \sigma(z) \frac{u}{1 - u} + (v - 1)\tau(z, u) - (v - 1) \sum_{s \geq 0} (-1)^s \psi(z, u, v^s), \]
\[ \psi(z, u, v, w) = \sum_{l \geq 1} (-1)^{l+1} \frac{z^{l+1} uv^l w}{1 - z^{l+1} uv^l w}. \]
\[ \sigma(z) = \sum_{l \geq 1} (-1)^l \frac{z^l}{1 - z^l}, \quad \tau(z, u) = \sum_{l \geq 1} (-1)^l \frac{z^l u}{1 - z^l}. \]

**Proof.** Let us denote the number of Carlitz compositions of integer \( n \) with \( k \) parts, such that part size \( j \) has multiplicity \( m \) by \( a_k(n, j, m) \), and the number of Carlitz compositions satisfying all of the above plus having number \( p \) as the last part by \( a_k(n, j, m, p) \). We also define \( b_k(n, j, m) = a_k(n, j, m, j) \). Using classical technique of “adding a slice” (see [7] for example), one arrives at the recurrence:
\[
a_{k+1}(n, j, m, p) = \begin{cases} 
    a_k(n - p, j, m) - a_k(n - p, j, m, p), & \text{if } p \neq j, \\
    a_k(n - j, j, m - 1) - b_k(n - j, j, m - 1), & \text{if } p = j.
\end{cases}
\]

For \( k \geq 1 \) we define generating functions:
\[
f_k(z, u, v, w) = \sum_{n, j, p \geq 1, m \geq 0} a_k(n, j, m, p) z^n u^j v^m w^p,
\]
\[
h_k(z, u, v) = \sum_{n, j \geq 1, m \geq 0} b_k(n, j, m) z^n u^j v^m,
\]
which allows:
\[
f_k(z, u, v, 1) = \sum_{n, j \geq 1, m \geq 0} a_k(n, j, m) z^n u^j v^m.
\]

Applying standard techniques (see for example, [4, Section 2]) we come up with the following functional equation:
\[
f_{k+1}(z, u, v, w) = \frac{zw}{1 - zw} f_k(z, u, v, 1) - f_k(z, u, v, zw)
+ (v - 1) f_k(z, zuv, v, 1) - (v - 1) h_k(z, zuv, v). \quad (2.2)
\]
One may notice that the above equation contains two unknown sequences: $f_k$ and $h_k$. They are however related in the following way. We have:
\[
b_{k+1}(n, j, m) = a_k(n - j, j, m - 1) - b_k(n - j, j, m - 1).
\]
Iterating this recurrence yields the following formula:
\[
b_{k+1}(n, j, m) = \sum_{q \geq 1} (-1)^{q+1} a_{k-q+1}(n - qj, j, m - q).
\]
Hence:
\[
h_k(z, u, v) = \sum_{n, j \geq 1} \sum_{m \geq 0} \sum_{q \geq 1} (-1)^{q+1} a_{k-q}(n - qj, j, m - q)z^j u^j v^m
\]
\[
= \sum_{q \geq 1} (-1)^{q+1} v^q f_{k-q}(z, z^q u, v, 1).
\]

Applying this to (2.2), and noticing that $f_0(z, u, v, w) = u/(1 - u)$, we can sum up over $k \geq 0$. We get:
\[
\sum_{k \geq 0} f_{k+1}(z, u, v, w) = \frac{zw}{1 - zw} \sum_{k \geq 0} f_k(z, u, v, 1) - \sum_{k \geq 0} f_k(z, u, v, zw)
\]
\[
+ (v - 1) \sum_{k \geq 0} f_k(z, zw u, v, 1)
\]
\[
- (v - 1) \sum_{k \geq 0} \sum_{q \geq 1} (-1)^{q+1} v^q f_{k-q}(z, z^q u, v, 1)
\]
\[
+ \frac{u}{1 - u}.
\]

Denoting $F(z, u, v, w) = \sum_{k \geq 1} f_k(z, u, v, w)$ and noticing that
\[
\sum_{k \geq 0} \sum_{q \geq 1} (-1)^{q+1} v^q f_{k-q}(z, z^q u w, v, 1)
\]
\[
= \varphi(z, u, v, w) + \sum_{q \geq 1} (-1)^q v^q F(z, z^q u w, v, 1),
\]
we arrive at the following functional equation:
\[
F(z, u, v, w) = \frac{zw}{1 - zw} \left( \frac{u}{1 - u} + F(z, u, v, 1) \right) + (v - 1) \frac{zuw}{1 - zw}
\]
\[
+ (v - 1) F(z, zw u, v, 1) - (v - 1) \varphi(z, u, v, w)
\]
\[
- (v - 1) \sum_{q \geq 1} (-1)^q v^q F(z, z^q u w, v, 1) - F(z, u, v, zw).
\]
Substituting \( w \) for \( zw \) in the above equation, and iterating it as in [7, Section 1], or [4, Section 2], after plugging in \( w = 1 \) and denoting \( G(x, u, v) = F(z, u, v, 1) \), we get:

\[
G(z, u, v) = \sigma(z) \left( \frac{u}{1-u} + G(z, u, v) \right) + (v-1)\tau(z, u)
- (v-1) \sum_{l \geq 0} (-1)^l \psi(z, u, v, z') + (v-1) \sum_{l \geq 1} (-1)^{l+1} G(z, z' u, v)
- (v-1) \sum_{l, q \geq 1} (-1)^{l+q} v^q G(z, z'^{l+q} u, v),
\]

or, in a more compact form:

\[
\left[ 1 - \sigma(z) \right] G(z, u, v) = \psi(z, u, v) - (v-1) \sum_{l \geq 1, q \geq 0} (-1)^{l+q} v^q G(z, z'^{l+q} u, v).
\]

Since

\[
\sum_{l \geq 1, q \geq 0} (-1)^{l+q} v^q G(z, z'^{l+q} u, v)
= \frac{1}{v} \left[ \sum_{l \geq 1} (-1)^l v^l G(z, z' u, v) - \sum_{l \geq 1} (-1)^l G(z, z' u, v)
+ \sum_{l \geq 1, q \geq 0} (-1)^{l+q} v^q G(z, z'^{l+q} u, v) \right],
\]

one can write

\[
\left[ 1 - \sigma(z) \right] G(z, u, v) = \psi(z, u, v) - \sum_{l \geq 1} (-1)^l v^l G(z, z'^{l+1} u, v)
+ \sum_{l \geq 1} (-1)^l G(z, z' u, v).
\]

Substituting \( u \) for \( zu \) and iterating as before we finally arrive at the following functional equation:

\[
G(z, u, v) = \alpha(z, u, v) - \beta(z, v) G(z, zu, v) - v G(z, z^2 u, v).
\]

We denote \( x_{q+1} = G(z, z^q u, v) \), \( \alpha_{q+1} = \alpha(z, z^q u, v) \). After applying Lemma 3, and letting \( N \to \infty \), the claim of the lemma follows. □

Now we can return to
Proof of Proposition 2. Using the binomial formula, it can be verified that

\[
(\beta(z, v))^{p-q} = \frac{1}{(1 - \sigma(z))^{p-q}} \sum_{r=0}^{p-q} (-1)^{p-q-r} \binom{p-q}{r} (2 - \sigma(z))^r (\sigma(z))^{p-q-r} v^{p-q-r}.
\]

Combining this with the result of Lemma 4 allows the following extraction from the generating function \(G(z, u, v)\):

\[
G_j(z, v) = \left[u^j\right] G(z, u, v) = \sum_{p \geq 0} \sum_{q=0}^{p-q} (-1)^q \binom{p}{q} (2 - \sigma(z))^q (\sigma(z))^{p-q} \frac{(1 - \sigma(z))^{p-q}}{(1 - \sigma(z))^{p-q}}
\times v^{p-r} z^j (p+q) v^m.
\]

After collecting alike terms w.r.t. power of \(v\) in \(\alpha(z, u, v)\) and denoting \(p - r = m\) we obtain the following:

\[
G_j(z, v) = \gamma_j(z, v) \sum_{m \geq 0} z^m \sum_{p \geq 0} (-1)^p \binom{p}{m} \left(\frac{\sigma(z)}{1 - \sigma(z)} - z^j\right)^m \binom{2 - \sigma(z)}{1 - \sigma(z)} v^m.
\]

where

\[
\gamma_j(z, v) = \frac{\sigma(z)(1 + z^j) - z^j}{1 - \sigma(z)} + \frac{\sigma(z)(1 + z^j)z^j + z^j(1 - z^j) + z^{2j}}{1 - \sigma(z)} v,
\]

or, after changing the order of summation and using the binomial summation formula we arrive at:

\[
G_j(z, v) = \gamma_j(z, v) \sum_{m \geq 0} z^m \left(\frac{\sigma(z)}{1 - \sigma(z)} - z^j\right)^m \binom{2 - \sigma(z)}{1 - \sigma(z)} v^m.
\]

Introducing the notation: \(G_{j,m}(z) = [v^m]G_j(z, v)\) and \(C_{j,m}(z) = 1 + G_{j,m}(z)\) delivers the claim of Proposition 2. \(\square\)
3. Singularities of the generating function

It is a known fact (see [7]) that the generating function of total number of Carlitz compositions has the unique singularity in the disc \( |z| \leq 0.75 \). This singularity is the unique real root, \( \rho \), of the equation \( \sigma(x) = 1 \) on \([0, 1]\). The numerical approximation of that root is \( \rho = 0.571349 \ldots \). It is easy to see that the singularities of the function \( G_{j,m}(z) \) are zeros of the following function:

\[
Q_j(z) = 1 + 2z^j - (1 + z^j)\sigma(z).
\]

We need the following claims:

Claim 1. \( Q_j(z) \) has a unique zero in the disc \( |z| \leq 0.75 \) for all \( j \geq 2 \). We denote it as \( \rho_j \).

Claim 2. \( \rho_j \) are strictly decreasing for \( j \geq 2 \) and \( \rho_j \to \rho \) as \( j \to \infty \).

Proof of Claims. In order to prove the first claim we represent \( Q_j(z) \) as follows:

\[
Q_j(z) = (1 + z^j)\varphi_j(z),
\]

where

\[
\varphi_j(z) = f(z) + h_j(z),
\]

\[
f(z) = 1.8 - \sum_{m=1}^{10} \frac{z^m}{1+z^m}, \quad h_j(z) = 0.2 - \frac{1}{1+z^j} - \sum_{m \geq 11} \frac{z^m}{1+z^m}.
\]

Consider the equation \( f(z) = 0 \). It is equivalent to the polynomial equation of order 45. It can be verified using Maple that it has a unique (and hence real) root in the disc \( |z| \leq 0.73 \). The numerical approximation of that root is \( z^* = 0.7238862780 \ldots \). The next step is to compute a lower estimate for \( \min_{|z|=0.73} |f(z)| \). It can be noticed that for any two \( z_0, z_1 \in \{z: |z| = r\} \) the following inequalities take place:

\[
\left| f(z_1) - f(z_0) \right| \leq \sum_{m=1}^{10} \frac{|z_1^m - z_0^m|}{|1+z_1||1+z_0|} \leq |z_1 - z_0| \sum_{m \geq 1} \frac{m z_0^{m-1}}{(1-r^m)^2} \leq \frac{|z_1 - z_0|}{(1-r)^4}.
\]

Hence \( f(z) \) is a Lipschitz function with the constant \( L = (1-r)^{-4} \) on the circle \( \{z: |z| = r\} \). By taking five points on the circle and using Maple one can obtain the following estimate:

\[
\min_{|z|=0.73} |f(z)| \geq 2.54.
\]
On the other hand for \( r = 0.73 \):

\[
|h_2(z)| \leq 0.2 + \frac{1}{1 - r^2} + \sum_{m \geq 11} \frac{r^m}{1 - r^m} \leq \frac{1}{1 - r^2} + \frac{r^{11}}{(1 - r^{11})(1 - r)} \leq 2.41.
\]

Since \( r^{j_2} < r^{j_1} \) for any \( j_2 > j_1 \) it follows that \( |h_j(z)| \leq 2.41 \) \( \forall j \geq 2, |z| \leq r \). Let us define a disk \( \bar{\gamma} = \{ z : |z| \leq 0.73 \} \) and let the circle \( \gamma = \{ z : |z| = 0.73 \} \) be a boundary of that disk. Clearly, both functions \( f(z) \) and \( h_j(z) \) are analytic on \( \bar{\gamma} \). We just have shown that \( |f(z)| > |h_j(z)| \) on \( \gamma \). According to the Rouche theorem, the number of zeros of the function \( f(z) + h_j(z) \) on the open disk \( \bar{\gamma} \setminus \gamma \) is equal to the number of zeros of the function \( f(z) \) in that region. \( f(z) \) has a unique (and, hence, real) root inside the disk \( \bar{\gamma} \), and so does \( \phi_j(z) = f(z) + h_j(z) \). Claim 1 follows.

In order to handle the second claim let us consider the function of a real variable \( p = p(t), t \in [t_0, \infty) \) for some \( t_0 \geq 1 \), implicitly defined by the equation \( S(t, p) = 0 \), where:

\[
S(t, p) = 1 + 2p' - (1 + p')\sigma(p).
\]

Clearly

\[
-p'_t = \frac{2p' \ln p - p' \ln p \sigma(p)}{2tp^{t-1} - tp^{t-1}\sigma(p) - (1 + p')\sigma'(p)} = \frac{p' \ln(1/p)(2 - \sigma(p))}{(1 + p')\sigma'(p) - tp^{t-1}(2 - \sigma(p))}.
\]

We denote:

\[
A = \min_{|p| \leq 0.73} \{2 - \sigma(p)\}, \quad B = \max_{|p| \leq 0.73} \{\sigma'(p)\} > 0, \quad C = \frac{A}{B}.
\]

Since \( \sigma(x) \) is monotonically increasing as a function of real variable and \( \sigma(0.73) < 1.96 \), we make a conclusion that \( A > 0 \) and, hence, \( C > 0 \). Hence:

\[
-p'_t \geq \frac{p' \ln(1/p)}{(1 + p')C - tp^{t-1}}.
\]

Since \( tp^{t-1} < t^{0.73t-1} \rightarrow 0 \) as \( t \rightarrow \infty \), we conclude that

\[
\exists t_0 \geq 1 \ \forall t > t_0 \quad tp^{t-1} < C.
\]

Hence:

\[
-p'_t \geq \frac{p' \ln(1/p)}{(1 + p')C - C} > 0.
\]

This delivers the second claim. \( \Box \)
4. Asymptotics

We now use the generating function $G_{j,m}(z)$ of the sequence \{\(g(n,j,m)\)\}, \(n \geq 0\) in order to analyze the asymptotic behavior of the quantity \(g(n,j,m)\) as \(n \to \infty\). According to (2.1), poles of \(G_{j,m}(z)\) are zeros of the following function of a complex variable:

\[
Q_j(z) = 1 - \sigma(z) + z^j [2 - \sigma(z)].
\]  
(4.1)

As it has been demonstrated above, it has a zero \(\rho_j\), such that \(0.57 < \rho_j < 0.66\) and no other zeros in the disc \(|z| \leq 0.73\). The function \(G_{j,m}(z)\) can be represented as:

\[
G_{j,m}(z) = \frac{P_{j,m}(z)}{[Q_j(z)]^{m+1}}.
\]

\(\rho_j\) is a simple root of \(Q_j(z)\) and \(P_{j,m}(\rho_j) \neq 0\). Thus \(G_{j,m}(z)\) has the following Laurent expansion:

\[
\frac{P_{j,m}(z)}{[Q_j(z)]^{m+1}} = \sum_{k=1}^{m+1} \frac{c_{-k}}{(z - \rho_j)^k} + \sum_{s \geq 0} c_s (z - \rho_j)^s.
\]

Following [5] we obtain an asymptotic estimate of \(g(n,j,m) = G_{j,m}(z)\). The theoretical background of such an estimate could be found in [10, Theorem 5.2.1], or elsewhere.

\[
g(n,j,m) = \left(\frac{n + m}{m}\right) \frac{P_{j,m}(\rho_j)}{[-\rho_j Q_j'(\rho_j)]^{m+1}(\rho_j/n)^n}\left(1 + O\left(\frac{1}{n}\right)\right).
\]  
(4.2)

Combining now (1.2) and (4.2) we get:

\[
P\{j \in \mathcal{N}(m)\} = \frac{1}{A} \left(\frac{n + m}{m}\right) \frac{P_{j,m}(\rho_j)}{[-\rho_j Q_j'(\rho_j)]^{m+1}(\rho_j/n)^n}\left(1 + O\left(\frac{1}{n}\right)\right).
\]  
(4.3)

In order to estimate components in this expression we want to apply the “bootstrapping method” (see, for example, [4]) to the equation \(Q_j(\rho_j) = 0\). First we re-write it in a form:

\[
\sigma(\rho_j) = 2 - \frac{1}{1 + \rho_j^j}.
\]  
(4.4)

Recalling that \(\rho\) is the real root of the equation \(\sigma(z) = 1\) in the unit disc, we denote \(\rho_j = \rho + \varepsilon_j\), where \(\varepsilon_j > 0\). We get:

\[
\sigma(\rho + \varepsilon_j) = 2 - \frac{1}{1 + (\rho + \varepsilon_j)^j}.
\]
Applying now the Taylor expansion on both sides we obtain:

\[ \varepsilon_j = \frac{\rho_j}{\sigma'(\rho)} \pm O(\rho^{2j}). \]

Hence

\[ \rho_j = \rho + \frac{\rho_j}{\sigma'(\rho)} \pm O(\rho^{2j}). \quad (4.5) \]

**Lemma 5.** \( Q_j(x) \) as a function of a real variable is monotonically decreasing on \([\rho, 1)\).

**Proof.** According to (4.1):

\[ -Q'_j(x) = jx^{j-1} [\sigma(x) - 2] + \sigma'(x) [1 + x^j]. \]

Assuming that \( \sigma(x) \) and \( \sigma'(x) \) are both monotonically increasing on \([\rho, 1)\), one can write:

\[ -Q'_j(x) > -jx^{j-1} + \sigma'(\rho)(1 + \rho^j) > 3.8 + 3.8\rho^j - j\rho^{j-1}. \]

Since function \( f(x) = x^j - 1, x > 1 \), assumes its maximum at \( x^* = 1/\ln(1/\rho) = 1.786495 \ldots \) so that \( f(x^*) = 1.15028 \ldots \), we can write:

\[ -Q'_j(x) > 3.8 + 3.8\rho^j - 1.16 > 0. \]

Since monotonicity of \( \sigma(x) \) is trivial, it remains to show that \( \sigma''(x) > 0 \) for \( x \in [\rho, 1) \).

Indeed:

\[ \sigma''(x) = \frac{d^2}{dx^2} \sum_{p \geq 1} \frac{x^p}{1+x^p} = \frac{d}{dx} \left[ \frac{1}{(1+x)^2} + \sum_{p \geq 1} \frac{(p+1)x^p}{(1+x^{p+1})^2} \right] \]

\[ = \frac{2}{(1+x)^3} + \sum_{p \geq 1} \left( (p+1)(p+2) \frac{x^p}{(1+x^{p+2})^3} - (p+1)^2 \frac{x^{p+1}}{(1+x^{p+1})^3} \right) > 0. \]

This completes the proof of Lemma 5. \( \square \)

**Lemma 6.**

(i) For all \( j \geq 1 \):

\[ \rho_j < \rho + \rho^j. \]
(ii) There exists \( j_0 \geq 1 \) such that for all \( j \geq j_0 \):

\[
\rho_j > \frac{1}{1/\rho - \rho^{j+2}}.
\]

**Proof.** Let us define:

\[
a_j = \frac{1}{1/\rho - \rho^{j+2}}, \quad b_j = \rho + \rho^j.
\]

(i) In view of Lemma 5, it is enough to show that \( Q_j (b_j) < 0 \). Since \( \sigma(x) \) is convex on \([0, 1]\), we have:

\[
-Q_j (b_j) = \left[1 + (\rho + \rho^j)^j\right] \sigma(\rho + \rho^j) - 1 - 2(\rho + \rho^j) > \left[1 + (\rho + \rho^j)^j\right] \sigma(\rho + \rho^j) - 1 - 2(\rho + \rho^j)
\]

\[
= \rho^j \left[ \sigma'(\rho) + \sigma'(\rho) \rho^j (1 + \rho^{j-1})^j - (1 + \rho^{j-1})^j \right] > \rho^j \left[ 3.8 + \sigma'(\rho) \rho^j (1 + \rho^{j-1})^j - e^{1.1503} \right] > 0.
\]

This takes care of part (i).

(ii) We can rewrite:

\[
a_j = \rho + \frac{\rho^{j+4}}{1 - \rho^{j+3}}.
\]

Using (4.5) we obtain:

\[
\rho_j - a_j > \frac{\rho^j}{\sigma'(\rho)} - \frac{\rho^{j+4}}{1 - \rho^{j+3}} - O(\rho^{2j}) \geq \rho^j \left[ \frac{1}{\sigma'(\rho)} - \frac{\rho^4}{1 - \rho^4} \right] - O(\rho^{2j}) \geq \rho^j (0.03 - O(\rho^j)).
\]

This completes the proof of Lemma 6. \( \square \)

Now we are ready to start proving the Theorem 1. First, we combine (1.1) and (4.3), and observe that:

\[
E[U_n^{(m)}] = \frac{n^m}{A^m!} \left(1 + O\left(\frac{1}{n}\right)\right) \sum_{j \geq 1} \frac{P_{j,m}(\rho_j)}{[-\rho_j Q_j(\rho_j)]^{m+1}} \frac{1}{(\rho_j/\rho)^n}.
\]  

(4.6)

Second, by using Lemma 6 along with the fact that \((1 + x)^n < e^{nx}\) for \(x > -1\), we get:

\[
\frac{1}{(\rho_j/\rho)^n} > e^{-n\rho^{j-1}}, \quad \text{for } j \geq 1.
\]  

(4.7)
and
\[ \frac{1}{(\rho_j/\rho)^n} < e^{-np^{j+3}}, \quad \text{for } j \geq j_0. \] (4.8)

Now we want to find \( q(n) \) such that all the terms in the sum (4.6), corresponding to \( j \leq q(n) \) are negligible. Let:
\[ q(n) = \ln n - \ln \ln n - \ln (m + 1) - 3; \]
then for \( j_0 \leq j \leq q(n) \) we can estimate \( nm/(\rho_j/\rho)^n \) using the following inequality:
\[ \frac{nm}{(\rho_j/\rho)^n} < n me^{-np^{j+3}} \leq \frac{1}{n}, \]
which yields the following estimate for the sum of corresponding terms in (4.6):
\[ \sum_{j=1}^{q(n)} P_{j,m}(\rho_j) \left[ -\rho_j Q'_j(\rho_j) \right]^{m+1} (\rho_j/\rho)^n = O\left( \frac{\ln n}{n} \right). \] (4.9)

Hence it is enough to only take into consideration the terms with \( j > q(n) \). We will need the following lemma.

**Lemma 7.** For \( j > q(n) \) the following asymptotic estimate takes place:
\[ \frac{1}{(\rho_j/\rho)^n} = e^{-np^{j-1}/\pi(\rho)} \left( 1 + O\left( \frac{\ln^2 n}{n} \right) \right). \]

**Proof.** Using (4.5), we obtain:
\[ e^{-np^{j-1}/\pi(\rho)} = \frac{1}{(\rho_j/\rho)^n} = e^{-np^{j-1}/\pi(\rho)} = \left[ 1 + \frac{\rho^{j-1}}{\sigma'(\rho)} \pm O(\rho^{2j}) \right]^n \]
\[ \leq e^{-np^{j-1}/\pi(\rho)} = e^{-n(1+O(\rho^{j-1}))\rho^{j-1}/\sigma'(\rho)}. \]

Then
\[ e^{-np^{j-1}/\pi(\rho)} = \frac{1}{(\rho_j/\rho)^n} < e^{-np^{j-1}/\pi(\rho)} \frac{n\rho^{j-1}O(\rho^{j+1})}{\sigma'(\rho)} = e^{-np^{j-1}/\pi(\rho)} O\left( \frac{\ln^2 n}{n} \right). \]

On the other hand
\[
\frac{1}{(\rho_j/\rho)^n} e^{-\frac{\rho_j}{\sigma'(\rho)}} = \left[ 1 + \left( 1 \pm O(\rho^{j+1}) \right) \frac{\rho_j}{\sigma'(\rho)} \right]^{-n} e^{-\frac{\rho_j}{\sigma'(\rho)}} = e^{-\frac{\rho_j}{\sigma'(\rho)}} \left[ e^{\frac{\rho_j}{\sigma'(\rho)}} \ln \left( 1 + \left( 1 + O(\rho^{j+1}) \right) \frac{\rho_j}{\sigma'(\rho)} \right) \right] - 1 = e^{-\frac{\rho_j}{\sigma'(\rho)}} \left[ e^n O(\rho^j) - 1 \right] = e^{-\frac{\rho_j}{\sigma'(\rho)}} O\left( \ln^2 n \right).
\]

This completes the proof of Lemma 7. □

Considering (4.9), one can write:

\[
E[U(m)] = \frac{n^m}{A m!} \left( 1 + O\left( \frac{1}{n} \right) \right) \sum_{j=q(n)} P_{j,m}(\rho_j) \left[ -\rho_j Q_j'(\rho_j) \right]^{m+1} e^{-\frac{\rho_j}{\sigma'(\rho)}}. (4.10)
\]

Now estimating coefficients:

\[
P_{j,m}(\rho_j) = \rho_j^{m+1} \left( \sigma(\rho_j) - \rho_j \left( 1 - \sigma(\rho_j) \right) \right)^{m+1} = \rho_j^{m+1} \left( 1 + O(\rho^j) \right),
\]

\[
[-\rho_j Q_j'(\rho_j)]^{m+1} = \rho_j^{m+1} \left[ \sigma'(\rho) \right]^{m+1} \left( 1 + O(\rho^j) \right).
\]

Combining this with (4.10) we obtain:

\[
E[U(m)] = \frac{1 + O(1/n)}{A \sigma'(\rho)^{m+1}} \frac{n^m}{m!} \sum_{j \geq 1} \rho_j^{m+1} e^{-\frac{\rho_j}{\sigma'(\rho)}}. (4.11)
\]

Now we need to compute the following sum:

\[
\frac{n^m}{m!} \sum_{j \geq 1} \rho_j^{m+1} e^{-\frac{\rho_j}{\sigma'(\rho)}}.
\]

In order to calculate this sum, following [5], we use Mellin’s formula:

\[
e^{-\lambda} = \frac{1}{2\pi i} \int_{s-i \infty}^{s+i \infty} \lambda^{-z} \Gamma(z) dz,
\]

assuming \( \lambda = np^j/\sigma'(\rho) \). This technique is explained in great detail in [8, pp. 131–134]. The method of Mellin’s transform is also treated in [3, Chapter 7]. After some calculations this yields:

\[
\frac{n^m}{m!} \sum_{j \geq 1} \rho_j^{m+1} e^{-\frac{\rho_j}{\sigma'(\rho)}} = \frac{n^m}{2\pi im!} \int_{s-i \infty}^{s+i \infty} \left( \frac{n}{\sigma'(\rho)} \right)^{-z} \Gamma(z) \left( \frac{1}{\rho} \right)^{m-z-1} dz.
\]
Letting now $s = m - 1/2$ one arrives at the expression:

$$\frac{n^m}{m!} \sum_{j \geq 1} \rho^j e^{-\frac{\sigma'(\rho)}{\ln(1/\rho)}} \left[ \frac{1}{m} + \frac{2}{m} \text{Re} \left[ \sum_{k=1}^{\infty} e^{-2\pi i k \frac{\log(1/\rho)}{\sigma'(\rho)}} \Gamma \left( m + i \frac{2\pi k}{\ln(1/\rho)} \right) \right] \right],$$

which proves Theorem 1.

5. The Problem of Wilf for Carlitz compositions

The Problem of Wilf for classical compositions was solved in [6], see also [5]. This is how the problem is defined: determine asymptotically as $n \to \infty$ the probability that a randomly chosen part size of a random composition of $n$ in has multiplicity $m$. For partitions this problem was solved in [2]. Based upon Theorem 1, it is easy now to formulate the solution to the problem extended to the case of Carlitz compositions. Recall that a composition $\kappa$ is a $k$-tuple $(\gamma_1, \ldots, \gamma_k)$, and the number of distinct part sizes of $\kappa$ can be defined as following:

$$D_n(\kappa) = 1 + \sum_{v=2}^{k} 1_{\{\gamma_v \neq \gamma_j, j=1,\ldots,v-1\}}.$$

We consider the following experiment. In a random Carlitz composition defined in Section 1 we chose uniformly at random one of all distinct part sizes. We want to calculate the unconditional probability that the multiplicity of a chosen part size is $m$. We denote this event as $A_n^{(m)}$. Clearly:

$$P\{A_n^{(m)}\} = \frac{U_n^{(m)}}{D_n}.$$

(5.1)

It has been shown in [6] and [5] that in the classical case

$$(\ln n)P\{A_n^{(m)}\} \to \frac{1}{m} \frac{1}{\ln n} \to \frac{1}{m}.$$  

We will show, that in the Carlitz case

$$(\ln n)P\{A_n^{(m)}\} \to \frac{1}{B^2 m} \frac{1}{\ln n} \to \frac{1}{m},$$

where $B = 1/\ln(1/\rho)$. 
Following the technique developed in [5], we will show that $D_n$ is heavily concentrated around its mean and hence:

$$P\left[ A_n^{(m)} \right] \sim \frac{E[U_n^{(m)}]}{E[D_n]}.$$ 

The following theorem takes place.

**Theorem 8.**

$$(\log_{1/\rho} n) P\left[ A_n^{(m)} \right] = \frac{1}{B m} + h_m(c \ln n) + o(1), \quad \text{as } n \to \infty,$$

where $B = 1/\ln(1/\rho)$, and $h_m$ is a mean zero functions of period 1 whose Fourier coefficients are given by

$$\phi_i^{(m)} = \frac{1}{B m!} \Gamma \left( m - \frac{2 \pi i l}{\ln(1/\rho)} \right), \quad l \neq 0,$$

and $c$ is some well-defined constant.

**Proof.** According to (2.1) the g.f. of the number of Carlitz compositions not using part size $j$ is:

$$G_{j,0}(z) = \frac{1 + z^j}{1 - \sigma(z) + z^j[2 - \sigma(z)]} = \frac{1 + z^j}{Q_j(z)}.$$

If $\vartheta(\kappa) = \vartheta_n(\kappa)$ is the set of distinct part sizes of a random composition $\kappa$, and $D_n = |\vartheta(\kappa)|$, then

$$E[D_n] = E \left[ \sum_j I_{j \in \vartheta(\kappa)} \right] = \sum_j P\{ j \in \vartheta(\kappa) \}.$$

We can write:

$$P\{ j \in \vartheta(\kappa) \} = 1 - \frac{1}{c(n)} [z^n] G_{j,0}(z).$$

Hence, according to the Residue theorem:

$$[z^n] G_{j,0}(z) = - \frac{1 + \rho_j}{\rho_j^{n+1} Q'_j(\rho_j)} + O(1).$$

Thus

$$\frac{1}{c(n)} [z^n] G_{j,0}(z) = A \rho^{n-\nu} \rho_j^{\nu} [\rho_j Q'_j(\rho_j)] + O(\rho^n).$$
It directly follows from (4.4) and (4.5) that

\[
\frac{1 + \rho_j^i}{-\rho_j Q_j'(\rho)} = \frac{\rho \sigma'(\rho) [1 + O(\rho^{j-1})]}{1 - \rho^j - O(\rho^j)} = \rho \sigma'(\rho) [1 + O(\rho^{j-1})].
\]

Hence

\[
P\{ j \in \vartheta(\kappa) \} = 1 - \frac{\rho \sigma'(\rho)}{A(\rho_j / \rho)^n} [1 + O(\rho^{j-1})].
\]

Using (4.7), we can write:

\[
P\{ j \in \vartheta(\kappa) \} = 1 - \frac{\rho \sigma'(\rho)}{A e^{-n \rho_j^{j-1}}} \leq n \rho_j^{j-1} - \ln \frac{\rho \sigma'(\rho)}{A} < n \rho_j^{j-1}.
\]

Choosing

\[b = \frac{\ln n + \ln \ln^2 n}{\ln(1/\rho)},\]

we obtain

\[
\sum_{j>b} P\{ j \in \vartheta(\kappa) \} \leq n \sum_{j>b} \rho_j^{j-1} = O\left(\frac{1}{\ln^2 n}\right).
\]

For the lower bound we have

\[
P\{ j \in \vartheta(\kappa) \} \geq 1 - \frac{\rho \sigma'(\rho)}{A e^{-n \rho_j^{j+3}}} [1 + O(\rho^{j-1})].
\]

Hence

\[
1 - P\{ j \in \vartheta(\kappa) \} \leq \frac{\rho \sigma'(\rho)}{A e^{-n \rho_j^{j+3}}} [1 + O(\rho^{j-1})],
\]

and

\[
\sum_{1 \leq j \leq a} [1 - P\{ j \in \vartheta(\kappa) \}] \leq C \sum_{1 \leq j \leq a} e^{-n \rho_j^{j+3}} = O\left(\frac{1}{n}\right),
\]

provided that

\[a = \left\lfloor \frac{\ln n - \ln \ln n}{\ln(1/\rho)} \right\rfloor - 3.
\]

Now we can write
\[
P\{A_n^{(m)}\} = E\left[\frac{U_n^{(m)}}{D_n}I_{\{a \leq D_n \leq b\}}\right] + E\left[\frac{U_n^{(m)}}{D_n}I_{\{a \leq D_n < b\}}\right] + E\left[\frac{U_n^{(m)}}{D_n}I_{\{D_n < a\}}\right] + E\left[\frac{U_n^{(m)}}{D_n}I_{\{D_n > b\}}\right].
\]

All we need to show is that the last two terms of this sum can be neglected. Since \(0 \leq U_n^{(m)}/D_n \leq 1\), we have for the first term

\[
E\left[\frac{U_n^{(m)}}{D_n}I_{\{D_n < a\}}\right] \leq P\{D_n < a\} \leq \sum_{1 \leq j \leq a} P\{j \in \mathcal{I}_n^{(0)}\}.
\]

Using (4.3) and (4.10) for \(m = 0\) and (4.7) we obtain:

\[
P\{j \in \mathcal{I}_n^{(0)}\} \leq \left(1 + O\left(\frac{\ln n}{n}\right)\right)e^{-\ln n} = O\left(\frac{1}{n}\right).
\]

which yields for \(j \leq a\):

\[
P\{j \in \mathcal{I}_n^{(0)}\} \leq \left(1 + O\left(\frac{\ln n}{n}\right)\right)e^{-\ln n} = O\left(\frac{1}{n}\right).
\]

Thus

\[
\sum_{1 \leq j \leq a} P\{j \in \mathcal{I}_n^{(0)}\} \leq \sum_{j=1}^{a} C \frac{a}{n} C a = O\left(\frac{\ln n}{n}\right) = o\left(\frac{1}{\ln n}\right).
\]

For the second term we can write

\[
E\left[\frac{U_n^{(m)}}{D_n}I_{\{D_n > b\}}\right] \leq P\{D_n > b\} \leq \sum_{j>b} \left(1 - P\{j \in \mathcal{I}_n^{(0)}\}\right) = \sum_{j>b} \left(1 - \frac{1}{(\rho j/\rho)^n}\right)
\]

\[
< \sum_{j>b} \left(1 - e^{-npj^{-1}}\right) < \sum_{j>b} npj^{-1} = Cnp^b = \frac{C}{\ln^2 n} = o\left(\frac{1}{\ln n}\right).
\]

Hence

\[
P\{A_n^{(m)}\} \sim \frac{E[U_n^{(m)}]}{\log_{1/\rho} n},
\]

which in view Theorem 1 delivers the claim of Theorem 8. □
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