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On the injective chromatic number of graphs

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Abstract

We define the concepts of an injective colouring and the injective chromatic number of a graph and give some upper and lower bounds in general, plus some exact values. We explore in particular the injective chromatic number of the hypercube and put it in the context of previous work on similar concepts, especially the theory of error-correcting codes. Finally, we give necessary and sufficient conditions for the injective chromatic number to be equal to the degree for a regular graph. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and terminology

Many practical problems can be formulated as graph colouring problems and many of the latter are best seen as problems of graph homomorphisms. There are times, however, when a graph colouring problem does not have an immediate formulation in terms of homomorphisms (the circular chromatic number as originally defined in [13] is an example). Injective colourings defined in this paper have their origin in complexity theory [11] (had the injective chromatic number of the hypercube Q_n been shown to be exponential in n, there would have been consequences to some complexity concerns on Random Access Machines), but related concepts had been studied earlier, see [6]. We begin by giving the basic definitions and background.

Our graphs are finite and simple; we will, however, allow semi-edges in Section 4. We will be colouring elements of finite sets and use the non-negative integers as colours. Thus, for a graph $G=(V(G),E(G))$, a *vertex colouring* (or, simply, a *colouring*) of G is a function $c: V(G)\to\omega$ (ω being the first infinite ordinal). A vertex k-colouring is a function $c: V(G)\to[k]$, with $[k]=\{0,1,\ldots,k-1\}$. We say that a colouring of a graph is *injective* if its restriction to the neighbourhood of any vertex is injective. The *injective chromatic number* $\chi_i(G)$ of a graph G is the least k such that there is an injective k-colouring. Clearly $\Delta(G)\leqslant\chi_i(G)\leqslant|V(G)|$ (as usual, $\Delta(G)$ is the maximum degree of a vertex of G).

An obvious alternate way of looking at the injective chromatic number of a graph G is to consider the *common neighbour graph* $G^{(2)}$ of G defined by $V(G^{(2)})=V(G)$ and $E(G^{(2)})=\{[u,v]$: there is a path of length 2 in G joining u and $v\}$. Then, $\chi_i(G)=\chi(G^{(2)})$.

Other interesting colourings can also be defined. For example, a *strong colouring* requires that the restrictions to closed neighbourhoods of vertices be injective. A natural generalization of the strong colouring is the *distance d colouring*, for which any two vertices of distance at most d have distinct colours. This has been studied in the case of the hypercube. While we do consider the hypercube, we are interested in defining a general concept. There are other possible directions but as it is not our purpose to survey them, we refer to [6] again. It is easy to find relations between the various concepts; we leave this to the interested reader. We shall also leave to the reader the almost trivial results on the injective chromatic number of, for example, the complete graph, the path, the cycle and the star.

For the hypercube, the concept has been studied before by many people as part of a different approach. Lovász [8] introduced *cube-like graphs* as examples of graphs with integral spectrum. A cube-like graph $Q_X(S)$ is defined on the power set 2^X of a finite set X, with two vertices being adjacent if their symmetric difference belongs to a given set $S\subseteq 2^X$ (see also [10]). A particular case is the *distance graph* $Q_n(D)$, which has $\{0,1\}^n$ as vertex set and where two vertices are adjacent if their Hamming distance belongs to $D\subseteq\{1,\ldots,n\}$. Thus, our common-neighbour graph for the hypercube Q_n is $Q_n(\{2\})$ in this terminology. Jaeger [5] proved already that $\chi_i(Q_n(\{2\})\leqslant 2^{\lceil \lg n\rceil}$ and conjectured equality. The conjecture is true for $n=2^m-j$, $0\leqslant j\leqslant 3$ and $m\geqslant 2$, as we prove in Section 3, Theorems 9 and 13. These results were also proved by Linial, Meshulam and Tarsi in [7]. While their methods are essentially the same, the context is not. Our concerns lie in a direction not studied before as far as we know.

2. Basic results

We begin by some general observations. As noted above, $\chi_i(G) \geqslant \Delta(G)$. It is natural to ask whether—or when—the injective chromatic number of a d-regular graph is d. Here we give an easy necessary condition; a complete characterization is given in Section 4.

Lemma 1. *Let G be a d-regular graph G with $\chi_i(G) = d$. Then d divides $n = |V(G)|$.*

Proof. We count the vertices of the same colour. Observe first that since the degree of any vertex of the graph is d, each colour appears exactly once in each neighbourhood and is counted exactly d times in $\sum_{u \in V(G)} 1 = n$. Thus, each colour appears on exactly n/d vertices. It follows that n is a multiple of d. □

This simple result and its consequence that $\chi_i(G) \geqslant d + 1$ when d does not divide $|V(G)|$ will be useful in the next section. For future use we give two more lower bounds.

Lemma 2. *Let G be connected and distinct from K_2. Then $\chi(G) \leqslant \chi_i(G)$.*

Proof. If G is neither a complete graph nor an odd cycle then, by Brooks' theorem, $\chi(G) \leqslant \Delta(G) \leqslant \chi_i(G)$. On the other hand, if G is a complete graph different from K_2 or if G is an odd cycle then $\chi(G) = \chi_i(G)$. □

There are other easy examples of graphs G with $\chi(G) = \chi_i(G)$, namely cycles C_n with $n \not\equiv 2 \pmod 4$. For such n, $\chi(C_n) = \chi_i(C_n) = 3$ when n is odd, and $\chi(C_n) = \chi_i(C_n) = 2$ when $n \equiv 0 \pmod 4$. The graph obtained from C_{6t} by adding all the main diagonals also has $\chi(G) = \chi_i(G) = 3$. We do not have a characterization of such graphs in general.

Lemma 3. *Let G be a diameter 2 graph with independence number α. Then, $\chi_i(G) \geqslant \alpha$.*

Proof. In any independent set of a graph of diameter two, each pair of vertices must have a common neighbour. Therefore any injective colouring of the graph requires at least as many colours as α. □

There is a trivial upper bound on the injective chromatic number, namely, $\chi_i(G) \leqslant |V(G)|$. In what follows, we characterize the graphs for which this upper bound is attained. Clearly, the only such graphs on fewer than four vertices are K_1 and K_3.

Lemma 4. *Let G be an arbitrary graph of order at least four. Then, $\chi_i(G) = |V(G)|$ if and only if either G is a complete graph, or G has diameter 2 and every edge of G is contained in a triangle.*

Proof. Recalling that $V(G)=V(G^{(2)})$ and $\chi_i(G)=\chi(G^{(2)})$, we see, by Brooks' theorem, that the assumption $\chi_i(G)=|V(G)|$ is equivalent to claiming that $G^{(2)}$ is a complete graph. This is, by the definition of the $G^{(2)}$, possible if and only if every pair of distinct vertices of G (adjacent or not) have a common neighbour. □

We now give a slightly less trivial upper bound on $\chi_i(G)$.

Lemma 5. *Let G have maximum degree Δ. Then, $\chi_i(G)\leqslant\Delta(\Delta-1)+1$.*

Proof. Consider the graph $G^{(2)}$ and let $v\in V(G)=V(G^{(2)})$ be an arbitrary vertex. There are at most $\Delta(\Delta-1)$ vertices in G that are at distance 2 from v. This means that the maximum degree of a vertex in $G^{(2)}$ is at most $\Delta(\Delta-1)$. Since $\chi_i(G)=\chi(G^{(2)})$ and the chromatic number is bounded above by the maximum degree plus one, we have $\chi_i(G)\leqslant\Delta(\Delta-1)+1$. □

The bounds given in Lemmas 3 and 5 are sharp for infinitely many graphs G; moreover, both can be attained simultaneously. To see this, let $k-1$ be a prime power and let $\mathscr{P}_k$ be the projective plane of order k (equivalently, a $(k^2-k+1,k,1)$-design). Let $I(\mathscr{P}_k)$ be the incidence graph of $\mathscr{P}_k$, that is, the bipartite graph whose vertices are all points and all lines of $\mathscr{P}_k$, and two vertices u and v are adjacent in $I(\mathscr{P}_k)$ if (without loss of generality) u is a point on the line v in $\mathscr{P}_k$. By the properties of projective planes, $I(\mathscr{P}_k)$ is a diameter 2 graph with $\Delta=k$ and $\alpha=k^2-k+1$. The above observations imply that $\chi_i(I(\mathscr{P}_k))=\Delta(\Delta-1)+1=\alpha$.

The following result which we shall use later seems to be of independent interest from the perspective of degree/diameter problem.

Proposition 6. *Let G be a graph of diameter two and maximum degree Δ, such that every edge of G is contained in a triangle. Then $|V(G)|\leqslant\Delta^2-\Delta$.*

Proof. Assume the contrary and let $|V(G)|\geqslant\Delta^2-\Delta+1$; we may also assume that $\Delta\geqslant 3$. Fix a vertex u of G of degree Δ and let $N(u)=\{u_i\colon 1\leqslant i\leqslant\Delta\}$ be the set of all neighbours of u. Further, let $V=V(G)\backslash(\{u\}\cup N(u))$ and let $N_V(u_i)=N(u_i)\cap V$ for $1\leqslant i\leqslant\Delta$.

Observe first that $|N_V(u_i)|\leqslant\Delta-2$ for $1\leqslant i\leqslant\Delta$, for otherwise there would be no way for the edge u_iu to be contained in a triangle. Thus,

$$|V|\leqslant\sum_{i=1}^{\Delta}|N_V(u_i)|\leqslant\Delta(\Delta-2). \tag{1}$$

But V itself has $\Delta(\Delta-2)$ elements, and hence equality must hold in (1). Consequently, for $1\leqslant i\leqslant\Delta$ we have $|N_V(u_i)|=\Delta-2$ and the sets $N_V(u_i)$ must be pairwise disjoint.

Take now a vertex $v\in N_V(u_1)$ and let $N(v)=\{u_1,v_2,\ldots,v_t\}$ where $t\leqslant\Delta$; note that $v_i\in V$ for each i such that $2\leqslant i\leqslant t$. Since the edge u_1v is in a triangle, we may without loss of generality assume that $v_2\in N_V(u_1)$. Considering the two sets $\{u_i\colon 2\leqslant i\leqslant\Delta\}$ and $\{v_i\colon 3\leqslant i\leqslant t\}$, it quickly follows that there exists a j, $2\leqslant j\leqslant\Delta$, such that $N_V(u_j)\cap N(v)$

$=\emptyset$. But this means that the vertices u_j and v have distance greater than two in G, a contradiction. □

The rest of this section will be devoted to products of graphs. Given graphs $G=(V,E)$ and $H=(U,F)$, recall that

- the Cartesian product of G and H is $G\Box H$ on the vertex set $V\times U$, with $[(v,x),(u,y)] \in E(G\Box H)$ if either $x=y$ and $[v,u]\in E$, or if $v=u$ and $[x,y]\in F$;
- the categorical product of G and H is $G\times H$ on the vertex set $V\times U$, with $E(G\times U)=\{[(v,x),(u,y)]\colon [v,u]\in E,\ [x,y]\in F\}$.

The following fact which relates common neighbour graphs with Cartesian and categorical products is easily proved by inspection.

Lemma 7. *For any G, H we have $(G\Box H)^{(2)}=(G^{(2)}\Box H^{(2)})\cup(G\times H)$.*

Now we prove an upper bound for the injective chromatic number of the Cartesian product of two graphs in terms of χ_i of the constituents.

Lemma 8. *If G and H are connected graphs both distinct from K_2, then $\chi_i(G\Box H)\leqslant \chi_i(G)\chi_i(H)$.*

Proof. It is well known that the chromatic number of a union of two graphs (on the same set of vertices) is bounded above by the product of their chromatic numbers. Since $\chi(G^{(2)})=\chi_i(G)$, the known inequalities for the chromatic number of the Cartesian and categorical product together with Lemma 7 imply that

$$\chi_i(G\Box H)\leqslant \max\{\chi_i(G),\chi_i(H)\}\cdot\min\{\chi(G),\chi(H)\}. \tag{2}$$

We may without loss of generality assume that $\chi_i(G)\geqslant\chi_i(H)$ and $|V(H)|\geqslant 3$; then we also have $\chi_i(H)\geqslant 2$. But then, using (2) and Lemma 2, we obtain $\chi_i(G\Box H)\leqslant \chi_i(G)\chi(H)\leqslant\chi_i(G)\chi_i(H)$. □

We note that this upper bound is sharp, for example, if $G=K_m$ and $H=K_n$ where $m,n\geqslant 3$. For less trivial examples of pairs of graphs G,H for which $\chi_i(G\Box H)= \chi_i(G)\chi_i(H)$ one can take $G=K_m\Box K_2$ and $H=K_n\Box K_2$ for any $m,n\geqslant 3$.

3. The cube

In coding theory there is an interest in the problem of determining the injective chromatic number of the n-dimensional hypercube in general and in the question of when it is equal to the degree n of the n-cube Q_n, in particular (recall that Q_n is the graph defined on the vertex set $\{0,1\}^n$ by $[a,b]\in E(Q_n)$ if and only if a and b differ in exactly one coordinate). As already mentioned in the introduction, the answer to the particular question given in Theorems 9 and 13 is also contained in [7]. The context and the terminology of [7], however, are different. To save the reader the effort of translating the various concepts, and in order to make the present paper self-contained, we include simple proofs. Note also that the sufficiency of the condition of Theorem 9 is only implicit in [7] and is observed explicitly from coding theory in [3].

Theorem 9. *Let Q_n be the n-dimensional cube. Then $\chi_i(Q_n)=n$ if and only if $n=2^r$ for some $r\geqslant 0$.*

Proof. Necessity follows from Lemma 1. For sufficiency, let $n=2^r$, $r\geqslant 0$. We exhibit a neighbourhood injective colouring of Q_n with n colours. As usual, let Q_n consists of vertices $a=(a_i)$ with $a_i\in\{0,1\}$ for each i, $0\leqslant i\leqslant n-1$. For any such i let $i^r=(i_0,\ldots,i_{r-1})$, $i_j\in\{0,1\}$ for $0\leqslant j\leqslant r-1$, be the binary representation of length r of i. Working over $GF(2^r)$, we define a colouring c of $V(Q_n)$ by

$$c(a)=\sum_{i=0}^{n-1} a_i i^r,$$

so that each colour is a vector in $\mathbf{Z}_2^r$ representing the binary form of the colour. Clearly, the number of colours is n. It remains to show neighbourhood injectivity.

Suppose a and b are neighbours of the same vertex $d=(d_i)$. Then for some p,q, $p\neq q$, $a_p=d_p+1$, $b_q=d_q+1$, $a_q=d_q$, $b_p=d_p$ and $a_i=b_i=d_i$ for $0\leqslant i\leqslant n-1$, $i\neq p,q$. Suppose a and b have the same colour. Then

$$c(a)=\sum_{i=0}^{n-1} a_i i^r=\sum_{i=0}^{n-1} b_i i^r=c(b),$$

means that

$$(d_p+1)p^r+d_q q^r=d_p p^r+(d_q+1)q^r$$

and so $p^r=q^r$, that is, $p=q$. $\square$

The unstated corollary that $\chi_i(Q_{2^r-1})=2^r$ for $r\geqslant 0$ will be improved in Theorem 13. There is one more that we do state.

Corollary 10. *For any n, $\chi_i(Q_n)\leqslant 2^{\lceil \lg n\rceil}$; thus, $\chi_i(Q_n)\leqslant 2n-2$.*

The rest of this section contains a few more facts on injective colourings of cubes. Let H_n be the graph whose vertices are all the binary strings of length n that contain an even number of 1's, and where two vertices are adjacent if they (as strings) differ in exactly two places (that is, if their Hamming distance is 2). It is easy to see that the common neighbour graph $Q_n^{(2)}$ corresponding to the n-cube Q_n has exactly two components, both isomorphic to H_n.

Let Q_n^2 be the graph obtained from the n-cube Q_n by joining any pair of vertices at distance 2 in the n-cube by a new edge; formally, $Q_n^2=Q_n\cup Q_n^{(2)}$. Clearly, two vertices (binary strings) of Q_n^2 are adjacent if and only if they differ in at most two places. This graph has been studied quite frequently; for a brief survey, including other cube-like graphs, and the folklore fact that Q_n^2 is isomorphic to H_{n+1} see [6].

In accordance with standard terminology, a (binary) *code* of length n is an arbitrary subset S of vertices of the n-cube Q_n. The code S is *single-error-correcting* if the Hamming distance of any two distinct vertices of S is at least 3. The *code covering number* $\gamma(Q_n)$ of the n-cube is the minimum number t of single-error-correcting codes $S_1,\dots,S_t$ such that $V(Q_n)=S_1\cup\cdots\cup S_t$.

Lemma 11. $\chi_i(Q_{n+1})=\gamma(Q_n)$.

Proof. Observe first that $\chi_i(Q_{n+1})=\chi(Q_{n+1}^{(2)})=\chi(H_{n+1})=\chi(Q_n^2)$. Now, the colour classes of any proper colouring of Q_n^2 obviously are one-error-correcting codes, and vice versa, any decomposition of the vertex set of the n-cube into single-error-correcting codes yields a proper vertex-colouring of Q_n^2. It follows that $\chi_i(Q_{n+1})=\chi(Q_n^2)=\gamma(Q_n)$. □

Corollary 12. $\chi_i(Q_3)=\chi_i(Q_4)=4$, *and* $\chi_i(Q_k)=8$ *for* $5\leqslant k\leqslant 8$.

Proof. We already know that $4=\chi_i(Q_4)\geqslant\chi_i(Q_3)$. On the other hand, $\chi_i(Q_3)=\gamma(Q_2)$, and the latter number is trivially equal to 4. To prove the rest, it is sufficient to show that $\chi_i(Q_5)=8$. Again, we know that $8=\chi_i(Q_8)\geqslant\chi_i(Q_5)=\gamma(Q_4)$. Since no three vertices of Q_4 can have Hamming distance $\geqslant 3$ from each other, we need 8 single-error-correcting codes to cover the 4-cube, which gives $\gamma(Q_4)=8$. □

Let $\sigma(n)$ denote the maximum number of codewords in a single-error-correcting binary code of length n. The definition of the code covering number implies immediately that

$$\sigma(n)\gamma(Q_n)\geqslant 2^n. \tag{3}$$

The function $\sigma(n)$ has been studied extensively in coding theory, but the exact values are largely unknown. In our next observation we are making use of the fact that

$$\sigma(2^m-j)=2^{2^m-m-j},\quad 1\leqslant j\leqslant 4. \tag{4}$$

Here, the lower bound for all $j\leqslant 4$ is obtained by considering Hamming codes of appropriate length. The upper bound for $j=1,2$ comes from the well-known sphere-packing bound and its improvement for codes of even length. Upper bounds for $j=3,4$ have been obtained using linear programming and can be found in [9] and in [2]; see also [1] or Chapter 17 of [12].

Theorem 13. $\chi_i(Q_{2^m-j})=2^m$ *for* $0\leqslant j\leqslant 3$.

Proof. We know from Corollary 10 that 2^m is an upper bound on χ_i. On the other hand, (3) and (4) show that, for $0\leqslant j\leqslant 3$,

$$\chi_i(Q_{2^m-j})=\gamma(Q_{2^m-j-1})\geqslant 2^{2^m-j-1}/\sigma(2^m-j-1)=2^m. \qquad \square$$

The first unknown value of χ_i for cubes is $\chi_i(Q_9)$. We have that $13 \leqslant \chi_i(Q_9) \leqslant 14$; the lower bound comes from (3) using the known value of $\sigma(8)=20$. The upper bound was found by G.F. Royle by computer search (personal communication quoted in [6]).

We conclude this section with the following observation.

Lemma 14. $\chi_i(Q_{2n+1}) \leqslant 2\chi_i(Q_{n+1})$.

Proof. It is sufficient to show that $\gamma(Q_{2n}) \leqslant 2\gamma(Q_n)$. Let $S_1, \ldots, S_t$ be a covering of $V(Q_n)$ by $t=\gamma(Q_n)$ single-error-correcting codes of length n. We construct a covering of $V(Q_{2n})$ by $2t$ codes as follows. For two binary strings $\bar{x}=(x_1,\ldots,x_n)$, $\bar{y}=(y_1,\ldots,y_n)$ of length n let $\bar{x}|\bar{y}$ denote the binary string $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ of length $2n$. Let V_e and V_o denote the subsets of all binary strings of length n with even and odd number of 1's, respectively. Now, for $b \in \{e,o\}$ and for $1 \leqslant j \leqslant t$ we define $S_{b,j} = \{\bar{x} \mid (\bar{x}+\bar{y});\ \bar{x} \in V_b,\ \bar{y} \in S_j\}$. It is a matter of routine to show that each $S_{b,j}$ is a single-error-correcting code of length $2n$ (this also follows from a standard direct-sum-like construction in coding theory). The fact that these $2t$ codes cover all vertices of Q_{2n} is obvious from the construction. □

4. Extremal graphs

As we have seen, for each graph G with maximum degree Δ we have $\Delta \leqslant \chi_i(G) \leqslant \Delta^2 - \Delta + 1$. In this section, we characterize the extremal d-regular graphs for the lower bound, that is, the d-regular graphs for which the lower bound is actually attained. We also characterize all extremal graphs for the upper bound. We start with the description of all d-regular graphs whose injective colouring number is equal to d; for that we need to recall a few facts from the theory of covering graphs. This theory often needs to allow graphs with loops and semi-edges, which we describe first.

A *semi-edge* is an edge with just one end incident to a vertex; its other end is "dangling". For the purpose of this paper, by a *semi-graph* we understand a simple graph $H=(V,E)$ with a subset $S \subset V$ of vertices at which semi-edges are attached (one semi-edge at each vertex of S). It is convenient to denote the semi-edge attached at a vertex $v \in S$ simply by v^*. Such a semi-graph is then denoted by $H^*=(V,E,S)$. In what follows, we will regard a vertex with a semi-edge as *self-adjacent*. It follows that for the neighbourhood $N_{H^*}(v)$ of any vertex $v \in V$ we have $v \in N_{H^*}(v)$ if and only if $v \in S$. This has to be taken into account when injective colourings of semi-graphs are considered: If $v \in S$ then no edge at v can have both end vertices coloured the same.

Recall that for given graphs $X=(V(X),E(X))$ and $Y=(V(Y),E(Y))$, a mapping $\phi: V(X) \to V(Y)$ is a graph homomorphism of X into Y if it is edge-preserving, that is, if $[\phi(u),\phi(v)] \in E(Y)$ whenever $[u,v] \in E(X)$. We write $X \to Y$ if there is a homomorphism from X to Y and we abuse notation to write simply $\phi: X \to Y$. Each graph homomorphism $\phi: X \to Y$ naturally induces a mapping $\phi_E: E(X) \to E(Y)$ defined by $\phi([u,v]) = [\phi(u),\phi(v)]$. This extends naturally to graphs with loops or semi-edges.

With the semi-graph $H^*=(V,E,S)$ we may associate a graph $H^o=(V,E,L)$ obtained from $H^*=(V,E,S)$ by replacing each semi-edge with a loop at the corresponding vertex. The natural mapping $h: H^o \to H^*$ which is an identity when restricted to the original graph $H=(V,E)$ and which maps a loop to a semi-edge at the same vertex will be called a *loop folding*. More precisely, the identity map $h: V \to V$ induces the edge identity map $h_E: E \to E$ and a bijection $h_L: L \to S$ which sends the loop $[u,u]$ to the semi-edge u^*. This term comes from the obvious visualization of g as a mapping that "folds" loops at vertices in S onto semi-edges. The inverse mapping h^{-1} will be the *loop unfolding*.

Now let $G^*=(V',E',S')$ be another semi-graph. We say that a mapping $f: V' \to V$ is a *semi-homomorphism* of G^* into H^* (and use the notation $f: G^* \to H^*$) if there exists a graph homomorphism $f': G^o \to H^o$ such that $f_{E' \cup S'} = h_L \circ f'_{E' \cup S'} \circ g_S^{-1}$, where $g: G^o \to G^*$ and $h: H^o \to H^*$ are loop foldings. We note that f' may map some non-loop edges of G' onto loops of G^o, and this way the semi-homomorphism f may "fold" some edges of G' onto semi-edges of G^*. Of course, f will always map semi-edges of G^* onto semi-edges of H^*.

The reason for introducing semi-edges and preferring them to loops lies in the topological coverings we intend to use. Roughly speaking, an edge can be "folded" and thus it can cover a semi-edge. In the reverse direction, a semi-edge can "lift" to a single edge, but a loop (topologically) lifts to a pair of parallel edges and never to a single edge. In the next part, we describe the basics of the theory of covering spaces needed later. From now on we will no longer be distinguishing semi-graphs in the notation by a star, as no confusion will be likely.

Let $H=(V,E,S)$ be a semi-graph. Each edge of H can be assigned one of the two possible directions; an edge with a direction is an *arc*. For each such arc x, the symbol x^- will denote the *reverse* of the arc x, that is, the arc obtained from the same underlying edge by choosing the orientation opposite to x; note that $(x^-)^- = x$. Semi-edges are not assigned any orientation (as there is no natural orientation inherited from loop folding); nevertheless, they will be referred to as arcs as well. Thus, if $D(H)$ denotes the set of all arcs of H, then $|D(H)| = 2|E| + |S|$. If x is an arc arising from a semi-edge, we set $x^- = x$ by definition. It follows that the mapping $x \mapsto x^-$ induces an involutory permutation of the set $D(H)$ whose fixed points are exactly the semi-edges in S.

Let m be a positive integer and let Σ_m be the full symmetric group of permutations of the set $\{1,2,\ldots,m\}$. A *permutation voltage assignment* on H in the group Σ_m is any mapping $\alpha: D(H) \to \Sigma_m$ such that $\alpha(x^-) = (\alpha(x))^{-1}$ for each arc x of H. The elements $\alpha(x)$ are called *permutation voltages*; the above condition can be now rephrased by saying that reverse arcs receive inverse permutation voltages. Observe that if x is a semi-edge then $\alpha(x)$ is necessarily an involutory permutation. In order to simplify the notation we shall write α_x instead of $\alpha(x)$; the image of an element i under α_x will be denoted $i\alpha_x$.

Given a graph H and a permutation voltage assignment α on H in the group Σ_m, we may define a new graph H^α, called a *lift* of H with respect to α, as follows. The vertex and the arc set of the lift are $V(H^\alpha) = V(H) \times \{1,2,\ldots,m\}$ and $D(H^\alpha) = D(H) \times \{1,2,\ldots,m\}$. As to the incidences in the lift, consider an arbitrary arc x of H, emanating from a vertex u and terminating at a vertex v (x could be a semi-edge, in which case $u=v$). For each $i \in \{1,2,\ldots,m\}$, there is an arc (x,i) in the lift H^α emanating from the vertex (u,i) and terminating at the vertex $(v,i\alpha_x)$. Moreover, the arc (x,i) represents a semi-edge in H^α if and only if $u=v$ and $i\alpha_x=i$. Note that the lift H^α is well-defined and *undirected*; if $i\alpha_x\neq i$ then the arcs (x,i) and $(x,i)^-=(x^-,i\alpha_x)$ are a pair of mutually reverse arcs constituting an undirected edge of H^α. The lift need not be connected; for a necessary and sufficient condition on connectivity of lifts (which is not of our concern here) we refer to [4].

Observe that the mapping $\pi:H^\alpha\to H$ which erases the second coordinate is a semi-homomorphism. In topological graph theory where graphs are treated as 1-dimensional complexes, the mapping π is known as *covering projection*, and the lift H^α is a *covering space* of H. If $u\in V(H)$ and $x\in D(H)$, the pre-images $\pi^{-1}(u)$ and $\pi^{-1}(x)$ are the *fibres* above the vertex u and the arc x, respectively. It follows from the above description that each fibre above a vertex contains m vertices and, similarly, each fibre above an arc contains exactly m arcs. These statements extend to *edges* as follows. If e is any edge of H which is *not* a semi-edge, then $|\pi^{-1}(e)|=m$. But care should be taken when counting *edges* in the pre-image of a semi-edge. If x is a semi-edge of H then the pre-image $\pi^{-1}(x)$ certainly consists of m *arcs* (namely, those of the form (x,i), $1\leqslant i\leqslant m$), but the ones for which $i\alpha_x\neq i$ will combine into edges while the others (with $i\alpha_x=i$) will be semi-edges. Thus, setting $r=|\{i;\ i\alpha_x=i\}|$, we see that the fibre above x consists of r semi-edges and $(m-r)/2$ *edges* (arising from pairs of mutually reverse arcs).

We are now ready to prove the characterization result for d-regular semi-graphs with injective chromatic number equal to d. Let K_d^* be the semi-graph obtained from a complete graph K_d by attaching a semi-edge at each vertex.

Theorem 15. *Let $G=(V,E,S)$ be a d-regular semi-graph, $d\geqslant 1$. Then, $\chi_i(G)=d$ if and only if there exists a positive integer m and a permutation voltage assignment α on the semi-graph $H=K_d^*$ in the group Σ_m such that the lift H^α is isomorphic to G.*

Proof. *Necessity*. Let G be a d-regular semi-graph such that $\chi_i(G)=d$ and let ϕ : $V(G)\to\{1,2,\ldots,d\}$ be a corresponding injective coloring. It is clear that ϕ is, at the same time, a semi-homomorphism $G\to K_d^*$ with vertex set $\{1,2,\ldots,d\}$. Since ϕ is an injective d-coloring of a d-regular semi-graph, for each $i\in\{1,2,\ldots,d\}$ the set $V_i=\phi^{-1}(i)$ induces a semi-graph $G'\subset G$ whose each component is either an edge (with both ends colored i) or a vertex with an attached semi-edge.

We first show that the number $|V_i|$ is independent of i. This is trivially true if $d=1$, so we may assume $d\geqslant 2$. Take any $j\neq i$ and consider the arc x of K_d^* which emanates from the vertex i and terminates at the vertex j. We claim that the semi-homomorphism pre-image $\phi_{E\cup S}^{-1}(x)$ consists of $|V_i|$ arcs no two of which share a common vertex. Indeed, combining d-regularity with injective d-colorability again, we see that for each $v\in V_i$ there exists precisely one arc x_v in the semi-graph G, emanating from v, whose terminal vertex is colored j. Moreover, if $v'\in V_i$ is a vertex different from v then the arcs x_v and $x_{v'}$ cannot share a common terminal vertex (its existence would violate the definition of an injective coloring). It follows that the set $\phi_{E\cup S}^{-1}(x)$ constitutes a matching between the sets V_i and V_j. Hence, $|V_1|=|V_2|=\cdots=|V_d|=m$, for some m.

For each $i \in \{1, 2, \ldots, d\}$ let us fix a numbering of vertices in the set V_i; without loss of generality, we may assume that $V_i = \{(i,1), (i,2), \ldots, (i,m)\}$. As we have seen in the preceding paragraph, for each arc $x = ij$ $(i \neq j)$ of the semi-graph K_d^*, the pre-image $\phi_{E\cup S}^{-1}(x)$ is a perfect matching of arcs from V_i to V_j. Thus, for each t, $1 \leqslant t \leqslant m$, there is precisely one arc in $\phi_{E\cup S}^{-1}(x)$ (which will be denoted (x,t) from now on) emanating from the vertex (i,t) and terminating in V_j; let $(j, t\alpha_x)$ denote the terminal vertex of the arc (x,t). This defines, for each $x = ij$, $i \neq j$ and $1 \leqslant i$, $j \leqslant d$, a permutation $\alpha_x \in \Sigma_m$. Observe that for the reverse arc $y = x^- = ji$, the arc $(y, t\alpha_x) \in \phi_{E\cup S}^{-1}(y)$ which emanates from the vertex $(j, t\alpha_x)$ must terminate at the vertex (i,t). According to the way we have defined the permutations α, we have $(i,t) = (i, t\alpha_x\alpha_y)$. Hence, if y is the reverse of the arc x then α_y is the inverse permutation to α_x.

It remains to take care of semi-edges. Let $i \in \{1, 2, \ldots, d\}$ and let x be the semi-edge at the vertex i in K_d^*. As already noted, each component of $\phi_{E\cup S}^{-1}(x)$ is either an edge or a semi-edge. Let $e_1, e_2, \ldots, e_\ell$ be the collection of all *edges* in $\phi_{E\cup S}^{-1}(x)$; of course, $\ell \leqslant m/2$. Let e_r be incident to vertices (i, s_r) and (i, t_r), where $1 \leqslant r \leqslant \ell$ and all the s_r and t_r are pairwise distinct. Then we define α_x to be the product of the ℓ two-cycles $(s_1t_1)(s_2t_2)\ldots(s_\ell t_\ell)$.

It is now straightforward to verify that our semi-graph G is indeed isomorphic to the lift H^α of the semi-graph $H = K_d^*$ with respect to the permutation voltage assignment $\alpha : D(H) \to \Sigma_m$ which assigns to every arc x of H the permutation α_x described above.

Sufficiency. Let G be isomorphic to a lift H^α of the semi-graph $H = K_d^*$. Since G is d-regular, we have $\chi_i(G) \geqslant d$. On the other hand, the covering projection $\pi : G \to H$ is a semi-homomorphism; hence $\chi_i(G) \leqslant \chi_i(H) = d$. □

We now turn to describing the connected graphs with maximum degree Δ whose injective coloring number is largest possible, that is, $\Delta^2 - \Delta + 1$. The case $\Delta = 1$ is trivial and the corresponding graphs are K_2 and K_1^* (a single vertex with a semi-edge). If $\Delta = 2$ then the only connected graphs G with $\Delta(G) = 2$ and $\chi_i = 3$ are the cycles C_ℓ of length ℓ not congruent to $0 \pmod 4$, or odd non-trivial paths (odd number of vertices) with semi-edges on the endpoints. In what follows, we therefore assume that $\Delta \geqslant 3$. We also recall the incidence graph $I(\mathscr{P}_k)$ of a finite projective plane $\mathscr{P}_k$ of order $k-1$, introduced in Section 2.

Theorem 16. *Let G be a connected graph of maximum degree $\Delta \geqslant 3$. Then $\chi_i(G) = \Delta^2 - \Delta + 1$ if and only if there exists a projective plane of order $\Delta - 1$ and G is isomorphic to $I(\mathscr{P})$.*

Proof. Sufficiency follows from the discussion after Lemma 5; we therefore focus on necessity. Let G be a connected regular graph of maximum degree Δ such that $\chi_i(G) = \Delta^2 - \Delta + 1$. We know that the maximum degree of the common neighbour graph $G^{(2)}$ is at most $\Delta^2 - \Delta$, and hence $\chi_i(G) = \chi(G^{(2)}) \leqslant \Delta^2 - \Delta + 1$. Since we assume that $\Delta \geqslant 3$, by Brooks' theorem we have $\chi_i(G) = \Delta^2 - \Delta + 1$ if and only if $G^{(2)}$ has a connected component K isomorphic to a complete graph of order $\Delta^2 - \Delta + 1$. Let W be the vertex set of K; we recall that $W \subset V(G) = V(G^{(2)})$.

Let $U \subset V(G)$ be the set of all vertices of G which have a neighbour in W; at the moment we do not make any assumptions about the intersection $U \cap W$. For each vertex $u \in U$ let $d_W(u)$ be the number of all neighbours of u which are contained in W; note that $1 \leqslant d_W(u) \leqslant \Delta$. Take an arbitrary vertex $w \in W$. We know that $N(w)$ is a subset of U. Since W induces a complete subgraph of $G^{(2)}$, for *each* vertex $w' \in W$ different from w there must exist in $N(w)$ a common neighbour u of both w and w' in G.

It follows that

$$|W \backslash \{w\}| \leqslant \sum_{u \in N(w)} (d_W(u) - 1). \tag{5}$$

Taking into account that $|W| = \Delta^2 - \Delta + 1$ and $|N(w)| \leqslant \Delta$ and $d_W(u) \leqslant \Delta$, we obtain from (5):

$$\Delta^2 - \Delta \leqslant \sum_{u \in N(w)} (d_W(u) - 1) \leqslant \Delta(\Delta - 1). \tag{6}$$

However, the leftmost and the rightmost part of (6) are equal; this is possible if and only if $|N(w)| = d_W(u) = \Delta$ for each $u \in N(w)$. Since the above argument is valid for an arbitrary $w \in W$, we conclude that $d_W(u) = \Delta$ for *each* vertex $u \in U$ and $|N(w)| = \Delta$ for *each* $w \in W$.

Assume first that $U \cap W = \emptyset$. Then W is necessarily an independent set in the graph G. Also, it follows from the above that the subgraph B of G induced by the set $U \cup W$ is connected and Δ-regular; the connectivity assumption on G now implies that, in fact, $G = B$. Moreover, we see that G is bipartite, with the obvious bipartition (U, W), and regularity shows that $|U| = |W|$. Analyzing the preceding counting arguments one can easily see that (5) and (6) imply, for any pair w, w' of distinct vertices in W, the existence of precisely one vertex $u \in U$ adjacent to both w and w' in G. It follows that interpreting vertices in W as points and sets $N(u)$, $u \in U$ as blocks of a design, we have obtained a BIBD with parameters $(\Delta^2 - \Delta + 1, \Delta, 1)$, that is, a projective plane $\mathscr{P}$ of order $\Delta - 1$; at the same time we see that $G = I(\mathscr{P})$.

It remains to consider the case when $U \cap W \neq \emptyset$; we show that this assumption leads to a contradiction. Thus, let $u \in U \cap W$. As we saw earlier, we have $d_W(u) = \Delta$, and therefore $N(u) \subset W$. Take any $w \in W$ such that $w \notin \{u\} \cup N(u)$. Since uw is an edge of $G^{(2)}$, there must exist a vertex $u' \in N(u) \cap N(w)$; hence $W \subset N(N(u)) = \bigcup \{N(u'); u' \in N(u)\}$. However, by the definition of U we have $N(u) \subset U$ and hence also $N(N(u)) \subset U$. This shows that $W \subset U$. On the other hand, recalling again that $N(u') \subset W$ for all $u' \in U$ and combining this fact with $W \subset U$ we quickly obtain $U \subset W$. Summing up, we have shown that $U \cap W \neq \emptyset$ implies that $U = W$. It follows that $V(G) = W$, and so G is a Δ-regular graph with $\Delta^2 - \Delta + 1 = \chi_i(G) = |V(G)|$. By Lemma 4, either G is a complete graph, or G has diameter 2 and every edge of G lies in a triangle. If G is a complete graph then $\Delta = |V(G)| = \Delta^2 - \Delta$, which is impossible for $\Delta \geqslant 3$. Finally, if G has diameter 2 and every edge of G is contained in a triangle, Proposition 6 shows that $|V(G)| \leqslant \Delta^2 - \Delta$, a contradiction again. This completes the proof. □

5. Complexity

One should ask about the complexity of the problem *Injective Chromatic Number* (*ICN*):

Instance: A graph $G=(V,E)$ and an natural number k.
Question: Is there an injective k-colouring of G?
It is easy to see that the *ICN* is in *NP*. But, as suspected, more is true.

Theorem 17. *The problem* ICN *is* NP-*complete*.

Proof. We use a reduction from Chromatic Number. Let $G=(V,E)$ be a graph and let k be a natural number. Without loss of generality, we may assume that G is connected and has at least 3 edges. We will construct a graph $\tilde{G}$ with the property that $\chi(G)\leqslant k$ if and only if $\chi_i(\tilde{G})\leqslant k+3|E|$. The reduction is clearly polynomial.

Let $V(\tilde{G})=V\cup V_E\cup V_A$ with $V_E=\{x_{uv}\colon\ uv\in E\}$ and $V_A=\{y^u_{uv}\colon\ uv\in E\}$. Let $E(\tilde{G})=E_I\cup E_E\cup E_A$ with $E_A=\{y^s_{st}y^u_{uv}\colon\ st,uv\in E\}$, $E_E=\{x_{st}x_{uv}\colon\ st,uv\in E\}$, and $E_I=\{ux_{uv},uy^u_{uv},x_{uv}y^u_{uv}\colon\ uv\in E\}$. In human terms, $\tilde{G}$ is obtained from G by adding, for each edge uv of G, three new vertices x_{uv},y^u_{uv},y^v_{uv} whose role is to ensure that if $\tilde{G}$ is injectively coloured, then u and v, as well as u and x_{uv}, u and y^u_{uv} get different colours. The complete graphs (of order $\geqslant 3$) on each of the sets V_E and V_A guarantee that in an injective colouring the colours on the respective vertices are also distinct.

More precisely, let first $c: V\to[k]$ be a proper k-colouring of G. Define an injective colouring $\tilde{c}: V(\tilde{G})\to[k]\cup E\cup\{(uv,u)\colon\ uv\in E\}$ of $\tilde{G}$, using $k+3|E|$ colours as follows. Put $\tilde{c}(u)=c(u)$ for each $u\in V$ and $\tilde{c}(x_{uv})=uv$ for each $x_{uv}\in V_E$. Set $\tilde{c}(y^u_{uv})=(uv,u)$. To see that $\tilde{c}$ is injective, note that any two vertices in $V_A\cup V_E$ have distinct colours, so we only need to worry about vertices with neighbours in both V and $V_E\cup V_A$. Consider first a vertex $x\in V_E$. All its neighbours in $V_E\cup V_A$ have colours distinct from each other and from those of vertices in V. Further, $x=x_{uv}$ has precisely two neighbours in V, u and v, and these have different colours since in G they form an edge. Now, a vertex $y\in V_A$ has only one neighbour in V and its colour is distinct from all the other distinct colours of the neighbours of y. Thus $\tilde{c}$ is injective.

Conversely, let an injective ℓ-colouring $\tilde{c}$ of $\tilde{G}$ be given. We claim that its restriction to V is a proper colouring of G using exactly $\ell-3|E|$ colours. Observe first that since the graphs induced by V_A and V_E are complete (of order $\geqslant 3$), no two vertices in either set have the same colour. Further for any $st,uv\in E$, the vertices x_{st} and y^u_{uv} have a common neighbour, namely x_{uv} if $\{s,t\}\neq\{u,v\}$ and u otherwise. Thus, the $3|E|$ vertices in $V_A\cup V_E$ all have distinct colours in $\tilde{c}$. Moreover, each vertex of V has in $\tilde{G}$ a common neighbour with *each* vertex of V_A, and a similar statement holds for V_E. Therefore, the colours of $\tilde{c}$ that appear on vertices in V are distinct from the $3|E|$ colours in $V_A\cup V_E$. Now, since $\tilde{c}$ is injective and any two vertices in V which are adjacent in G have a common neighbour in $\tilde{G}$, the restriction of $\tilde{c}$ to V properly colours G with the remaining $\ell-3|E|$ colours. □

Note added in proof

Since the acceptance of this paper more related work appeared, see [14] and [15]. Further we can prove slightly more than Theorem 17.

Theorem 18. *The problem INC is NP-complete for every fixed* $k \geqslant 3$.

Proof. Consider a k-regular graph G. It is known to be NP-complete to decide if G can be edge-colored properly by k colors (cf. Holyer [16] for $k=3$ and [17] for $k > 3$). Without loss of generality we may assume that G is not the complete graph K_{k+1}. Denote by G^* the graph obtained from G by subdividing each edge e by a new extra vertex x_e. We claim that $\chi_i(G^*) = k$ if and only if $\chi_e(G) = k$ (here and later χ_e denotes the chromatic index).

Let f be an independent k-coloring of G^*. Then $g: E(G) \to \{1, 2, \ldots, k\}$ defined by $g(e) = f(x_e)$ is a proper edge coloring of G.

On the other hand, if g is a k-edge coloring of G, define $f: V(G^*) \to \{1, 2, \ldots, k\}$ by $f(x_e) = g(e)$ for edges e of G, and use any proper k-vertex coloring of G for coloring the vertices of $V(G) \subset V(G^*)$ (such a coloring exists by Brooks's theorem. Then f is an independent coloring of G^* and it uses k colors.

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