JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **192**, 151–179 (1995)

On Bäcklund Transformations for Nonlinear Partial Differential Equations

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Submitted by Augustine O. Esogbue

Received April 29, 1993

In this paper, Bäcklund transformations for nonlinear partial differential equations are obtained without using any structure associated with the equations. We classify all the nonlinear partial differential equations of the form $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ that have Bäcklund transformations whose definition involves only u, u_x, u_{xx}, and a function determined by u, u_x, and u_{xx} via an integrable system of two first-order partial differential equations, and we obtain all such Bäcklund transformations. In particular, a new nonlinear partial differential equation with a Bäcklund transformation (i.e., $u_{xxx} = -\frac{3}{2} \sin 2u\, u_x - \frac{1}{2} u_x^3 + u_t$) is found, and the known Bäcklund transformations for the KdV equation, the MKdV equation, the potential KdV equation, and the potential MKdV equation are recovered without using any knowledge of these equations. Our method is applicable to many other classes of nonlinear partial differential equations. © 1995 Academic Press, Inc.

Bäcklund transformation is a useful tool for generating solutions to certain nonlinear partial differential equations (PDEs), especially, soliton solutions to equations like the Korteweg–de Vries (KdV) equation. Using a Bäcklund transformation for a nonlinear PDE, one obtains a new solution to the equation from a known one.

For example, the system

$$u'_x = u_x - 2\lambda \sin \frac{u + u'}{2},$$

$$u'_t = -u_t + \frac{2}{\lambda} \sin \frac{u - u'}{2},$$

0022-247X/95 $6.00

where $\lambda \neq 0$ is an arbitrary constant, defines a Bäcklund transformation $u \mapsto u'$ of the sine–Gordon equation

$$u_{xt} = \sin u.$$

In fact, if u is a solution to the sine–Gordon equation, then the system above is integrable (i.e., compatible on some domain in the xtu'-space) and yields a new solution u' to the sine–Gordon equation. In particular, from the trivial solution $u = 0$ we obtain the so-called 1-soliton solution

$$u'(x, t) = 4 \arctan \exp\left(\alpha - \lambda x - \frac{1}{\lambda} t\right),$$

where α is also an arbitrary constant and repeated applications of this procedure yield the so-called multiple-soliton solutions.

There is much literature on Bäcklund transformations for nonlinear PDEs. A Bäcklund transformation for the KdV equation and a Bäcklund transformation for the potential KdV equation were derived by Wahlquist and Eastbrook in [12]. Among the numerous Bäcklund transformations found after the work [12] (see, for example, [6, 10] for detailed discussions), a Bäcklund transformation for the modified KdV (MKdV) equation was found by Lamb in [5]. There have been many formulations of Bäcklund transformations. For example, the jet bundle formulation was given by Pirani, Robinson, and Shadwick in [9], Boiti and Tu gave the gauge transformation interpretation of Bäcklund transformations in [1], and Tian worked out several examples for this interpretation in [11]. Despite these advances, several interesting questions remain to be investigated:

1. Which nonlinear PDEs have Bäcklund transformations?
2. How many Bäcklund transformations can a nonlinear PDE have?
3. What is a practical way to find Bäcklund transformations for a given nonlinear PDE?

In this paper, we show that the known Bäcklund transformations for the KdV equation, MKdV equation, potential KdV equation, sin–Gordon equation, and many others allow a unified formulation. Based on this formulation, we give a definition of Bäcklund transformation that is quite general and very practical. Roughly speaking, a Bäcklund transformation is a rule that tells one how to obtain a new solution from an old one in terms of a finite number (called the "*degree*") of functions and all the functions used are determined by the old solution through a finite number (called the "*order*") of times of differentiation and integration. With this definition, we want to address the questions listed above. We will restrict our attention to nonlinear PDEs of the form $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ and Bäcklund transformations for these equations whose definition involves only u, u_x, u_{xx}, and a function determined by u, u_x, and u_{xx}. We require that the equations in question have enough solutions so that at each point in the xt-space, u and all its derivatives except u_{xxx} and the derivatives of u_{xxx} are independent from each other. Therefore, we assume that $\mathcal{F}$ is analytic. It is proved that, up to a rescaling of the xtu-space and a shift in u-direction, the KdV equation and the equations

$$u_{xxx} = (q_0 \pm \tfrac{3}{2} u^2)\, u_x + u_t,$$

$$u_{xxx} = p_0 + q_0 u_x + 3u_x^2 + u_t,$$

$$u_{xxx} = p_0 + q_0 u_x \pm \tfrac{1}{2} u_x^3 + u_t,$$

$$u_{xxx} = p_0 + q_0 u_x + \tfrac{3}{2} u_x^2 - \tfrac{1}{2} u_x^3 + u_t \qquad \text{with } q_0 > -1,$$

$$u_{xxx} = p_0 + q_0 u_x + \tfrac{3}{2} u_x^2 + \tfrac{1}{2} u_x^3 + u_t \qquad \text{with } q_0 < 1,$$

$$u_{xxx} = (q_0 - \tfrac{3}{2} \sin 2u) u_x - \tfrac{1}{2} u_x^3 + u_t$$

are the only nonlinear PDEs of the form $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ that have Bäcklund transformations whose definition involves only u, u_x, u_{xx}, and a function ϕ determined by u, u_x, and u_{xx} via an integrable system of the form

$$\phi_x = \Omega(\phi, u, u_x, u_{xx}),$$

$$\phi_t = \Theta(\phi, u, u_x, u_{xx}).$$

We also determine all such Bäcklund transformations for these nonlinear PDEs. In the case of the KdV equation, the MKdV equation, the potential KdV equation, and the potential MKdV equation, the known Bäcklund transformations are the only Bäcklund transformations of the above form (up to an equivalence).

Our method is very general and practical. It can be used to discuss various types of Bäcklund transformations for many other classes of nonlinear PDEs. It is also interesting to find out if one can obtain all the Bäcklund transformations for a fixed nonlinear PDE like the KdV equation or the MKdV equation using our method.

We review the known Bäcklund transformations for the sine–Gordon equation, the KdV equation, the MKdV equation, and the potential KdV equation and their unified formulation in Section 1, and we give the definition of Bäcklund transformation that will be used in this paper. In Section 2, we determine the nonlinear PDEs of the form $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ that have Bäcklund transformations whose definition involves only u, u_x, u_{xx}, and a function determined by u, u_x, and u_{xx}, and we find all such Bäcklund transformations for these equations.

We are greatly indebted to Josef Dorfmeister, Franz Pedit, and Rudolf Schmid for their interest in this work. This work was done during the author's visit at Emory University in 1991–1993 and the hospitality of the Department of Mathematics and Computer Science there is very appreciated. We also thank the referees for their suggestions, which led to many improvements in our presentation. The author was partially supported by NSF Grant DMS-9205293.

1. Examples and a Definition of Bäcklund Transformations

A Bäcklund transformation $w \mapsto w'$ for the potential KdV equation

$$w_t = 3w_x^2 + w_{xxx},$$

defined by

$$\begin{aligned} w'_x &= \lambda - w_x - \tfrac{1}{2}(w - w')^2, \\ w'_t &= -w_t + (w - w')(w_{xx} - w'_{xx}) - 2(w_x^2 + w_x w'_x + w'^2_x), \end{aligned} \tag{1.1}$$

where λ is an arbitrary constant, was found by Wahlquist and Estabrook in [12]. From this Bäcklund transformation $w \mapsto w'$ for the potential KdV equation they obtained a Bäcklund transformation $u \mapsto u'$ for the KdV equation

$$u_t = 6uu_x + u_{xxx},$$

defined by

$$u \mapsto w = \int u\, dx \mapsto w' \mapsto u' = w'_x.$$

In [11], it is shown that the above Bäcklund transformation for the KdV equation can also be defined as follows. Let ϕ be a solution to the integrable system

$$\begin{aligned} \phi_x &= -\lambda + \phi^2 + u, \\ \phi_t &= 4\lambda(-\lambda + \phi^2) + 2(\lambda + \phi^2)u + 2u^2 + 2\phi u_x + u_{xx}, \end{aligned} \tag{1.2}$$

where λ is an arbitrary constant; then

$$u' = 2\lambda - 2\phi^2 - u. \tag{1.3}$$

Moreover, it is proved in [11] that the Bäcklund transformation for the MKdV equation

$$v_t = \tfrac{3}{2} v^2 v_x + v_{xxx}$$

found by Lamb in [5] can also be defined by

$$v \mapsto v' = \frac{4\lambda\psi}{1 + \psi^2} + v, \tag{1.4}$$

where λ is an arbitrary constant and ψ is a solution to the integrable system

$$\begin{aligned} \psi_x &= \lambda\psi + \tfrac{1}{2}(1 + \psi^2)v, \\ \psi_t &= \lambda^3\psi + \tfrac{1}{2}\lambda^2(1 + \psi^2)u + \tfrac{1}{2}\lambda\psi u^2 + \tfrac{1}{4}(1 + \psi^2)u^3 \\ &\quad + \tfrac{1}{2}\lambda(1 - \psi^2)u_x + \tfrac{1}{2}(1 + \psi^2)u_{xx}. \end{aligned} \tag{1.5}$$

Note that the original Bäcklund transformation found by Bäcklund (see [2]), i.e., the Bäcklund transformation for the sine–Gordon equation mentioned in the introduction, can also be put into this form: $u' = \sigma$ with σ being a solution to the integrable system

$$\begin{aligned} \sigma_x &= u_x - 2\lambda \sin\frac{u + \sigma}{2}, \\ \sigma_t &= -u_t + \frac{2}{\lambda}\sin\frac{u - \sigma}{2}, \end{aligned} \tag{1.6}$$

where λ is an arbitrary constant. Motivated by these examples and many others, the following definitions seem appropriate.

Let $n > 0$ and $k \geq 0$ be integers. Consider a system

$$\begin{aligned} \Phi_x &= \Omega(\phi_1, \phi_2, \ldots, \phi_n, u, u_x, u_t, \ldots, \partial_t^k u), \\ \Phi_t &= \Theta(\phi_1, \phi_2, \ldots, \phi_n, u, u_x, u_t, \ldots, \partial_t^k u) \end{aligned} \tag{1.7}$$

on the (row) vector Φ of n functions $\phi_1, \phi_2, \ldots, \phi_n$, where Ω and Θ are two vectors of non-zero smooth functions such that one of them involves a kth order derivative of u.

Definition 1.8. We call the system (1.7) an *integrable system associated with a nonlinear partial differential equation* $\mathcal{F}(u, u_x, u_t, u_{x\cdots xt\cdots t}) = 0$ if it is integrable on a non-empty open subset of the Φ-space when and only when u is a solution to a fixed set of partial differential equations implied by $\mathcal{F}(u, u_x, u_t, u_{x\cdots xt\cdots t}) = 0$. The number n will be called the *degree* of (1.7), while the number k will be called the *order* of (1.7) if one cannot reduce it using only the differential equation $\mathcal{F}(u, u_x, u_t, u_{x\cdots xt\cdots t}) = 0$ in question.

For example, the systems (1.1), (1.2), and (1.5) are integrable systems of degree 1 and order 2 associated with the potential KdV equation, the KdV equation, and the MKdV equation, respectively, and the system (1.6) is an integrable system of degree 1 and order 1 associated with the sine–Gordon equation. The functions defined by an integrable system associated with a nonlinear partial differential equation were called *pseudopotentials* of the equation by Wahlquist and Estabrook in [13] and studied extensively by several authors (see, for example, [7, 8]). From [11], when $n = 1$ and Ω and Θ are quadratic in the pseudopotential, this type of integrable systems comes from $sl(2, \mathbb{R})$-linear systems associated with the nonlinear PDEs, which can be regarded as degree-2 integrable systems by our definition.

DEFINITION 1.9. Assume that (1.7) is an integrable system associated with a nonlinear partial differential equation

$$\mathcal{F}(u, u_x, u_t, u_{x\cdots xt\cdots t}) = 0. \tag{1.10}$$

A map $u \mapsto u' = F(\phi_1, \phi_2, \ldots, \phi_n, u, u_x, u_t, \ldots, \partial_t^k u)$ is called a *Bäcklund transformation* for (1.10) if u' is always a solution to (1.10) for any solution u to (1.10) and any solution $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$ to the corresponding system (1.7). We will say that this map is of *degree n* if (1.7) is of degree n and that this map is of *order k* if either (1.7) is of order k or (1.7) is of order $<k$ but F involves a kth order derivative of u that cannot be replaced by lower order derivatives of u using only the differential equation (1.10) in question.

For example, the Bäcklund transformations for the KdV equation, the MKdV equation, and the potential KdV equation listed above are of degree 1 and order 2, while the one for the sine–Gordon equation is of degree 1 and order 1.

If $u \mapsto u' = F(\phi, u, u_x, u_t, \ldots, \partial_t^k u)$ is a Bäcklund transformation defined via an integrable system

$$\phi_x = \Omega(\phi, u, u_x, u_t, \ldots, \partial_t^k u),$$
$$\phi_t = \Theta(\phi, u, u_x, u_t, \ldots, \partial_t^k u),$$

and f is an invertible smooth function, then this Bäcklund transformation can also be written as $u \mapsto u' = F(f(\psi), u, u_x, u_t, \ldots, \partial_t^k u)$ with ψ defined by

$$\psi_x = \frac{1}{f'(\psi)}\,\Omega(f(\psi), u, u_x, \ldots, \partial_l^k u),$$

$$\psi_t = \frac{1}{f'(\psi)}\,\Theta(f(\psi), u, u_x, \ldots, \partial_l^k u).$$

This freedom in choosing the integrable system defining ϕ sometimes gives us a way to normalize Ω or Θ. For example, if we normalize the system (1.5) into the system

$$\phi_x = \lambda \sin \phi + u,$$

$$\phi_t = \lambda^2[\lambda \sin \phi + u] + \frac{\lambda}{2} \sin \phi \, u^2 + \frac{1}{2} u^3 + \lambda \cos \phi \, u_x + u_{xx}$$

via the substitution $\psi = \tan(\phi/2)$, then the Bäcklund transformation (1.4) for the MKdV equation can be written as

$$v \mapsto v' = 2\lambda \sin \phi + u.$$

We will frequently use this fact in the next section. More general substitutions can also be used to normalize an integrable system and, hence, the corresponding Bäcklund transformation. For example, using the relation

$$\phi = -\frac{w + w'}{2}$$

we can rewrite the system (1.1) as

$$\phi_x = -\frac{\lambda}{2} + \phi^2 + 2\phi w + w^2,$$

$$\phi_t = 2\lambda\left(-\frac{\lambda}{2} + \phi^2 + 2\phi w + w^2\right) + (\lambda + 2\phi^2 + 4\phi w + 2w^2)w_x$$
$$- \frac{1}{2} w_x^2 + 2(\phi + w)w_{xx},$$

and the corresponding Bäcklund transformation for the potential KdV equation becomes

$$w' = -2\phi - w.$$

Our method originated from [3] and relies on the requirement that the nonlinear PDEs $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ in question have enough solutions so that at each point in the xt-space, u and all its derivatives except u_{xxx} and the derivatives of u_{xxx} are independent from each other; namely, we can treat them as independent variables in the functions $\mathcal{F}$, F, Ω, and Θ when we want to determine $\mathcal{F}$, F, Ω, and Θ. Moreover, by our definition of associated integrable systems and Bäcklund transformations, we can deal with $\phi_1, \phi_2, \ldots, \phi_n$ in $\mathcal{F}$, F, Ω, and Θ in the same way. We also note that the way we determine associated integrable systems is the direct approach for finding pseudopotentials [4].

2. Bäcklund Transformations for $u_{xxx} = \mathcal{F}(u, u_x, u_t)$

In this section, we want to demonstrate how to find Bäcklund transformations for nonlinear PDEs of the form $u_{xxx} = \mathcal{F}(u, u_x, u_t)$, where $\mathcal{F}$ is analytic and $(\partial/\partial u_t)\mathcal{F} \neq 0$. We start from simple ones and then look for more general ones. In the latter case, the formats of associated integrable systems, Bäcklund transformations, and nonlinear PDEs are given first (see Lemmas 2.11 and 2.23 and Corollary 2.22); the actual determination of the Bäcklund transformations is carried out second (see Theorem 2.43); and finally some remarks follow.

THEOREM 2.1. *There is no Bäcklund transformation*

$$u \mapsto v = F(\phi, u) \tag{2.2}$$

for a nonlinear partial differential equation

$$u_{xxx} = \mathcal{F}(u, u_t) \tag{2.3}$$

defined via an associated integrable system of the form

$$\begin{aligned} \phi_x &= \Omega(\phi, u), \\ \phi_t &= \Theta(\phi, u, u_{xx}), \end{aligned} \tag{2.4}$$

where $(\partial/\partial\phi)F \neq 0$ *and* $(\partial/\partial u_{xx})\Theta \neq 0$.

Proof. From (2.3) and (2.4) we have

$$\phi_{xt} = \Theta \frac{\partial \Omega}{\partial \phi} + u_t \frac{\partial \Omega}{\partial u}, \qquad \phi_{tx} = -\Omega \frac{\partial \Theta}{\partial \phi} + u_x \frac{\partial \Theta}{\partial u} + \mathcal{F} \frac{\partial \Theta}{\partial u_{xx}}.$$

Therefore, the compatibility condition $\phi_{xt} = \phi_{tx}$ of (2.4) implies that

$$\frac{\partial \Theta}{\partial u} = 0 \qquad \text{and} \qquad \frac{\partial \Omega}{\partial u} = \frac{\partial \mathcal{F}}{\partial u_t} \frac{\partial \Theta}{\partial u_{xx}},$$

which imply that

$$\mathcal{F}(u, u_t) = f(u) + g(u)u_t, \qquad \Theta(\phi, u, u_{xx}) = \theta(\phi) + \kappa(\phi)u_{xx}$$

for some smooth functions f, g, θ, and κ. Hence, without loss of generality, (2.4) can be written as

$$\begin{aligned} \phi_x &= \Omega(\phi, u), \\ \phi_t &= \theta(\phi) + u_{xx}. \end{aligned}$$

So, $\phi_{xt} = \phi_{tx}$ if and only if

$$\frac{\partial \Omega}{\partial \phi} = 0, \qquad \frac{\partial \Omega}{\partial u} = g, \qquad 0 = \Omega\theta' + f.$$

Thus,

$$\Omega(\phi, u) = G(u), \qquad \theta(\phi) = a - b\phi, \qquad f(u) = bG(u)$$

for some constants a, b, and some smooth function G satisfying $G' = g$. Therefore, the nonlinear partial differential equation has the form

$$u_{xxx} = bG(u) + g(u)u_t, \tag{2.5}$$

where g is not constant, and the associated integrable system has the form

$$\begin{aligned} \phi_x &= G(u), \\ \phi_t &= a - b\phi + u_{xx}. \end{aligned} \tag{2.6}$$

By (2.2) and (2.6),

$$v_{xxx} = \mathcal{G}(\phi, u, u_x, u_{xx}) + g(u)\frac{\partial F(\phi, u)}{\partial u} u_t,$$

$$bG(v) + g(v)v_t = \mathcal{H}(\phi, u, u_x, u_{xx}) + g(v)\frac{\partial F(\phi, u)}{\partial u} u_t$$

for some smooth functions $\mathscr{G}$ and $\mathscr{H}$. Thus, v is always a solution to (2.5) only if

$$\frac{\partial F(\phi, u)}{\partial u} = 0.$$

Hence, the Bäcklund transformation must have the form

$$u \mapsto v = F(\phi), \tag{2.7}$$

where $F' \neq 0$. Then, (2.6) and (2.7) yield

$$\begin{aligned} v_{xxx} &= \mathscr{I}(\phi, u, u_x) + F'(\phi) g(u) u_{xx}, \\ bG(v) + g(v) v_t &= \mathscr{J}(\phi, u, u_x) + F'(\phi) g(v) u_{xx} \end{aligned}$$

for some smooth functions $\mathscr{I}$ and $\mathscr{J}$, which shows that v cannot be a solution to (2.5) in general. ∎

Next, consider a Bäcklund transformation

$$u \mapsto v = F(\phi, u, u_x, u_{xx}) \tag{2.8}$$

for a nonlinear partial differential equation

$$u_{xxx} = \mathscr{F}(u, u_x, u_t) \tag{2.9}$$

defined via an associated integrable system

$$\begin{aligned} \phi_x &= \Omega(\phi, u, u_x, u_{xx}), \\ \phi_t &= \Theta(\phi, u, u_x, u_{xx}), \end{aligned} \tag{2.10}$$

where $(\partial/\partial\phi) F \neq 0$ and $(\partial/\partial u_{xx})\Theta \neq 0$. This setup is more general than that of Theorem 2.1. However, Theorem 2.1 is included to illustrate the method.

LEMMA 2.11. (2.10) *is an integrable system associated with* (2.9) *if and only if* Ω *does not depend on* u_x *and* u_{xx},

$$\mathscr{F}(u, u_x, u_t) = p(u) + q(u) u_x + r(u) u_x^2 + s(u) u_x^3 + Q(u) u_t, \tag{2.12}$$

$$\Theta(\phi, u, u_x) = \tilde{\Theta}(\phi, u) + \left(\hat{\Theta}\frac{\partial \Omega}{\partial \phi} - \Omega \frac{\partial \hat{\Theta}}{\partial \phi}\right) u_x - \frac{1}{2}\frac{\partial \hat{\Theta}}{\partial u} u_x^2 + \hat{\Theta}(\phi, u) u_{xx} \tag{2.13}$$

for some smooth functions p, q, r, s, Q, $\tilde{\Theta}$, and $\hat{\Theta}$ satisfying

$$\frac{\partial \Omega}{\partial u} = Q\hat{\Theta}, \tag{2.14}$$

$$\frac{\partial^2 \hat{\Theta}}{\partial u^2} = 2s\hat{\Theta}, \tag{2.15}$$

$$3\frac{\partial \Omega}{\partial \phi}\frac{\partial \hat{\Theta}}{\partial u} = 3\Omega \frac{\partial^2 \hat{\Theta}}{\partial \phi \partial u} + 2\frac{\partial \Omega}{\partial u}\frac{\partial \hat{\Theta}}{\partial \phi} - 2\left(\frac{\partial^2 \Omega}{\partial \phi \partial u} + r\right)\hat{\Theta}, \tag{2.16}$$

$$\left(\hat{\Theta}\frac{\partial \Omega}{\partial \phi} - \Omega\frac{\partial \hat{\Theta}}{\partial \phi}\right)\frac{\partial \Omega}{\partial \phi} = \Omega\left(\hat{\Theta}\frac{\partial^2 \Omega}{\partial \phi^2} - \Omega\frac{\partial^2 \hat{\Theta}}{\partial \phi^2}\right) + \frac{\partial \tilde{\Theta}}{\partial u} + q\hat{\Theta}, \tag{2.17}$$

$$\tilde{\Theta}\frac{\partial \Omega}{\partial \phi} = \Omega\frac{\partial \tilde{\Theta}}{\partial \phi} + p\hat{\Theta}. \tag{2.18}$$

Proof. From (2.9) and (2.10) we have

$$\phi_{xt} = \Theta\frac{\partial \Omega}{\partial \phi} + u_t\frac{\partial \Omega}{\partial u} + u_{xt}\frac{\partial \Omega}{\partial u_x} + u_{xxt}\frac{\partial \Omega}{\partial u_{xx}},$$

$$\phi_{tx} = \Omega\frac{\partial \Theta}{\partial \phi} + u_x\frac{\partial \Theta}{\partial u} + u_{xx}\frac{\partial \Theta}{\partial u_x} + \mathcal{F}\frac{\partial \Theta}{\partial u_{xx}}.$$

So, $\phi_{xt} = \phi_{tx}$ implies that Ω does not depend on u_x and u_{xx} and

$$\frac{\partial \Omega(\phi, u)}{\partial u} = \frac{\partial \mathcal{F}(u, u_x, u_t)}{\partial u_t}\frac{\partial \Theta(\phi, u, u_x, u_{xx})}{\partial u_{xx}},$$

which implies that $\Theta(\phi, u, u_x, u_{xx}) = \check{\Theta}(\phi, u, u_x) + \hat{\Theta}(\phi, u, u_x)u_{xx}$ and $\mathcal{F}(u, u_x, u_t) = P(u, u_x) + Q(u, u_x)u_t$ for some functions $\check{\Theta}$, $\hat{\Theta}$, P, and Q with $\hat{\Theta} \neq 0$ and $Q \neq 0$. Thus, (2.10) is an integrable system associated with (2.9) if and only if $\hat{\Theta}$ does not depend on u_x,

$$\frac{\partial \Omega(\phi, u)}{\partial u} = Q(u, u_x)\hat{\Theta}(\phi, u), \tag{2.19}$$

$$\hat{\Theta}\frac{\partial \Omega}{\partial \phi} = \Omega\frac{\partial \hat{\Theta}}{\partial \phi} + u_x\frac{\partial \hat{\Theta}}{\partial u} + \frac{\partial \check{\Theta}}{\partial u_x}, \tag{2.20}$$

$$\check{\Theta}\frac{\partial \Omega}{\partial \phi} = \Omega\frac{\partial \check{\Theta}}{\partial \phi} + u_x\frac{\partial \check{\Theta}}{\partial u} + P\hat{\Theta}. \tag{2.21}$$

From (2.19) we see that Q does not depend on u_x and (2.14) holds. (2.20) is equivalent to

$$\check{\Theta}(\phi, u, u_x) = \tilde{\Theta}(\phi, u) + \left(\hat{\Theta}\frac{\partial\Omega}{\partial\phi} - \Omega\frac{\partial\hat{\Theta}}{\partial\phi}\right)u_x - \frac{1}{2}\frac{\partial\hat{\Theta}}{\partial u}u_x^2$$

for some smooth function $\tilde{\Theta}$, and hence (2.13) holds. Then, (2.21) implies that $P(u, u_x) = p(u) + q(u)u_x + r(u)u_x^2 + s(u)u_x^3$ for some smooth functions p, q, r, and s. Therefore, (2.12) holds, and (2.21) is equivalent to (2.15)–(2.18). ∎

COROLLARY 2.22. *If* (2.10) *is an integrable system associated with a nonlinear partial differential equation* $u_{xxx} = p(u) + q(u)u_x + r(u)u_x^2 + u_t$, *then* $\Omega(\phi, u, u_x) = \omega(\phi) + \sigma(\phi)u + \tau(\phi)u^2$ *for some smooth functions* ω, σ, *and* τ. *Moreover, if* (2.10) *is an integrable system associated with* $u_{xxx} = p(u) + q(u)u_x + u_t$, *then either* $\Omega(\phi, u, u_x) = \tau(\phi)(\omega_0 + \sigma_0 u + u^2)$ *for some smooth function* τ *and constants* ω_0 *and* σ_0 *with* $\tau \neq 0$, *or* $\Omega(\phi, u, u_x) = \omega(\phi) + \sigma(\phi)u$ *for some smooth functions* ω *and* σ *with* $\sigma \neq 0$.

Proof. The first statement is a direct consequence of (2.14) and (2.15) since $s = 0$ and $Q = 1$ now.

To prove the second statement, we only need to show that if $\Omega(\phi, u) = \omega(\phi) + \sigma(\phi)u + u^2$, then ω and σ are constant, which is a direct consequence of (2.16), since $r = 0$. ∎

LEMMA 2.23. (2.8) *is a Bäcklund transformation for the nonlinear partial differential equation* (2.9) *defined via an associated integrable system of the form* (2.10) *if and only if* Ω *does not depend on* u_x *and* u_{xx} *and, up to a rescaling of the variable* t,

$$F(\phi, u, u_x) = f(\phi) + \alpha u, \tag{2.24}$$

$$\mathcal{F}(u, u_x, u_t) = p(u) + q(u)u_x + r(u)u_x^2 + s(u)u_x^3 + u_t, \tag{2.25}$$

$$\Theta(\phi, u, u_x) = \tilde{\Theta}(\phi, u) + \left(\frac{\partial\Omega}{\partial\phi}\frac{\partial\Omega}{\partial u} - \Omega\frac{\partial^2\Omega}{\partial\phi\partial u}\right)u_x - \frac{1}{2}\frac{\partial^2\Omega}{\partial u^2}u_x^2 + \frac{\partial\Omega}{\partial u}u_{xx} \tag{2.26}$$

for some constant α *and smooth functions* f, p, q, r, s, *and* $\tilde{\Theta}$ *satisfying*

$$\frac{\partial^3\Omega}{\partial u^3} = 2s\frac{\partial\Omega}{\partial u}, \tag{2.27}$$

$$3\frac{\partial\Omega}{\partial\phi}\frac{\partial^2\Omega}{\partial u^2} = 3\Omega\frac{\partial^3\Omega}{\partial\phi\partial u^2} - 2r\frac{\partial\Omega}{\partial u}, \tag{2.28}$$

$$\left(\frac{\partial\Omega}{\partial\phi}\frac{\partial\Omega}{\partial u} - \Omega\frac{\partial^2\Omega}{\partial\phi\partial u}\right)\frac{\partial\Omega}{\partial\phi} = \Omega\left(\frac{\partial^2\Omega\partial\Omega}{\partial\phi^2\partial u} - \Omega\frac{\partial^3\Omega}{\partial\phi^2\partial u}\right) + \frac{\partial\tilde{\Theta}}{\partial u} + q\frac{\partial\Omega}{\partial u}, \tag{2.29}$$

$$\tilde{\Theta}\,\frac{\partial \Omega}{\partial \phi} = \Omega\,\frac{\partial \tilde{\Theta}}{\partial \phi} + p\,\frac{\partial \Omega}{\partial u}, \tag{2.30}$$

$$\alpha s(u) = \alpha^3 s(v), \tag{2.31}$$

$$\frac{3}{2} f'\,\frac{\partial^2 \Omega}{\partial u^2} + \alpha r(u) = \alpha^2 r(v) + 3\alpha^2 f' s(v)\Omega, \tag{2.32}$$

$$\begin{aligned} 3f''\Omega\,\frac{\partial \Omega}{\partial u} + 3f'\Omega\,\frac{\partial^2 \Omega}{\partial \phi\,\partial u} + \alpha q(u) \\ = \alpha q(v) + 2\alpha f' r(v)\Omega + 3\alpha (f')^2 s(v)\Omega^2, \end{aligned} \tag{2.33}$$

$$\begin{aligned} f'''\Omega^3 + 3f''\Omega^2\,\frac{\partial \Omega}{\partial \phi} + f'\Omega\left(\frac{\partial \Omega}{\partial \phi}\right)^2 + f'\Omega^2\,\frac{\partial^2 \Omega}{\partial \phi^2} + \alpha p \\ = p(v) + f' q(v)\Omega + (f')^2 r(v)\Omega^2 \\ + (f')^3 s(v)\Omega^3 + f'\tilde{\Theta}. \end{aligned} \tag{2.34}$$

Proof. First, we want to show that Q is constant. Assume that Q is not constant. Using Lemma 2.11, from (2.8)–(2.10) we obtain

$$v_{xxx} = \mathcal{A}(\phi, u, u_x, u_{xx}, u_t, u_{xt}) + Q(u)\,\frac{\partial F(\phi, u, u_x, u_{xx})}{\partial u_{xx}}\,u_{xxt},$$

$$\mathcal{F}(v, v_x, v_t) = \mathcal{B}(\phi, u, u_x, u_{xx}, u_t, u_{xt}) + Q(v)\,\frac{\partial F(\phi, u, u_x, u_{xx})}{\partial u_{xx}}\,u_{xxt}$$

for some smooth functions $\mathcal{A}$ and $\mathcal{B}$. Thus,

$$Q(u)\,\frac{\partial F(\phi, u, u_x, u_{xx})}{\partial u_{xx}} = Q(v)\,\frac{\partial F(\phi, u, u_x, u_{xx})}{\partial u_{xx}},$$

and hence F does not depend on u_{xx}. Similarly, F does not depend on u_x and u. Thus, $v = F(\phi)$. Then, by (2.14) and direct calculations,

$$v_{xxx} = \mathcal{C}(\phi, u, u_x) + F'(\phi)Q(u)\hat{\Theta}(\phi, u)u_{xx},$$

$$\mathcal{F}(v, v_x, v_t) = \mathcal{D}(\phi, u, u_x) + Q(v)F'(\phi)\hat{\Theta}(\phi, u)u_{xx}$$

for some smooth functions $\mathcal{C}$ and $\mathcal{D}$. This shows that v cannot always be a solution to (2.9). Therefore, Q is a non-zero constant, which will be set to 1 from now on; i.e., (2.25) holds.

Now, (2.14) is equivalent to $\hat{\Theta} = (\partial/\partial u)\,\Omega$. Thus, (2.13) can be rewritten as (2.26), and (2.15)–(2.18) become (2.27)–(2.30), respectively.

Next, we want to show that F does not depend on u_{xx}. From (2.8)–

(2.10) and (2.25) we obtain

$$
\begin{aligned}
v_{xxx} &= \mathscr{E}(\phi, u, u_x, u_{xx}, u_t) \\
&\quad + \left[3\Omega \frac{\partial^2 F}{\partial\phi\partial u_{xx}} + 3u_x \frac{\partial^2 F}{\partial u \partial u_{xx}} + 3u_{xx} \frac{\partial^2 F}{\partial u_x \partial u_{xx}} + \frac{\partial F}{\partial u_x}\right. \\
&\quad \left. + (2P + 3u_t) \frac{\partial^2 F}{\partial u_{xx}^2}\right] u_{xt} + \frac{\partial F}{\partial u_{xx}} u_{xxt}, \\
\mathscr{F}(v, v_x, v_t) &= \mathscr{G}(\phi, u, u_x, u_{xx}, u_t) + \frac{\partial F}{\partial u_x} u_{xt} + \frac{\partial F}{\partial u_{xx}} u_{xxt}
\end{aligned}
$$

for some smooth functions $\mathscr{E}$ and $\mathscr{G}$. So, if (2.8) is a Bäcklund transformation, then

$$
3\Omega \frac{\partial^2 F}{\partial\phi\partial u_{xx}} + 3u_x \frac{\partial^2 F}{\partial u \partial u_{xx}} + 3u_{xx} \frac{\partial^2 F}{\partial u_x \partial u_{xx}} + (2P + 3u_t) \frac{\partial^2 F}{\partial u_{xx}^2} = 0,
$$

which is equivalent to

$$
F(\phi, u, u_x) = G(\phi, u, u_x) + \beta u_{xx} \tag{2.35}
$$

for some smooth function G and constant β. In order to show $\beta = 0$, we assume that $\beta \neq 0$. From (2.9), (2.10), (2.13), and (2.35) we have

$$
\begin{aligned}
v_{xxx} &= \mathscr{H}(\phi, u, u_x, u_{xx}) \\
&\quad + \left(3\Omega \frac{\partial^2 G}{\partial\phi\partial u_x} + \frac{\partial G}{\partial u} + 3u_x \frac{\partial^2 G}{\partial u \partial u_x} + \beta \frac{\partial P(u, u_x)}{\partial u_x} + 3u_{xx} \frac{\partial^2 G}{\partial u_x^2}\right) u_t \\
&\quad + \frac{\partial G}{\partial u_x} u_{xt} + \beta u_{xxt}, \\
\mathscr{F}(v, v_x, v_t) &= P\left(G + \beta u_{xx}, \Omega \frac{\partial G}{\partial \phi} + u_x \frac{\partial G}{\partial u} + u_{xx} \frac{\partial G}{\partial u_x} + \beta \mathscr{F}\right) \\
&\quad + \mathscr{I}(\phi, u, u_x, u_{xx}) + \frac{\partial G}{\partial u} u_t + \frac{\partial G}{\partial u_x} u_{xt} + \beta u_{xxt}
\end{aligned}
$$

for some smooth functions $\mathscr{H}$ and $\mathscr{I}$. Taking the u_t-derivative of $v_{xxx} = \mathscr{F}(v, v_x, v_t)$ yields

$$
3\Omega \frac{\partial^2 G}{\partial\phi\partial u_x} + 3u_x \frac{\partial^2 G}{\partial u \partial u_x} + \beta \frac{\partial P(u, u_x)}{\partial u_x} + 3u_{xx} \frac{\partial^2 G}{\partial u_x^2} = \beta \frac{\partial P(v, v_x)}{\partial v_x}. \tag{2.36}
$$

Taking the u_t-derivative of (2.36) gives

$$\beta^2 \frac{\partial^2 P(v, v_x)}{\partial v_x^2} = 0,$$

which is equivalent to $r = s = 0$; i.e., $P(v, v_x) = p(v) + q(v)v_x$. Then, (2.36) becomes

$$3\Omega \frac{\partial^2 G}{\partial \phi \partial u} + 3u_x \frac{\partial^2 G}{\partial u \partial u_x} + \beta q(u) + 3u_{xx} \frac{\partial^2 G}{\partial u_x^2} = \beta q(G + \beta u_{xx}). \quad (2.37)$$

Taking the second-order u_{xx}-derivative of (2.37) yields

$$\beta^3 q''(G + \beta u_{xx}) = 0,$$

which implies that $q(v) = q_0 + q_1 v$ for some constants q_0 and q_1. Thus, (2.37) becomes

$$3\Omega \frac{\partial^2 G}{\partial \phi \partial u_x} + 3u_x \frac{\partial^2 G}{\partial u \partial u_x} + \beta q_1 u + 3u_{xx} \frac{\partial^2 G}{\partial u_x^2} = \beta q_1(G + \beta u_{xx}),$$

which is equivalent to

$$3 \frac{\partial^2 G}{\partial u_x^2} = \beta^2 q_1, \quad (2.38)$$

$$3\Omega \frac{\partial^2 G}{\partial \phi \partial u_x} + 3u_x \frac{\partial^2 G}{\partial u \partial u_x} + \beta q_1 u = \beta q_1 G. \quad (2.39)$$

By (2.38), $G(\phi, u) = H(\phi, u) + I(\phi, u)u_x + \frac{1}{6}\beta^2 q_1 u_x^2$ for some smooth functions H and I. (2.39) now is equivalent to $q_1 = 0$ and $I = \gamma$ for some constant γ. Then, $v_{xxx} = \mathcal{F}(v, v_x, v_t)$ implies that

$$3\Omega \frac{\partial^2 H}{\partial \phi \partial u} + \beta p'(u) + 3u_x \frac{\partial^2 H}{\partial u^2} = \beta p'(H + \gamma u_x + \beta u_{xx}),$$

and hence $p'' = 0$. This contradicts the fact that $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ is non-linear.

Third, we want to show that F does not depend on u_x. As in the last section, we easily prove that $F(\phi, u, u_x) = f(\phi) + g(\phi)u + \frac{1}{6}\beta^2 b u^2 + \beta u_x$ and $P(v, v_x) = p(v) + (a + bv)v_x$ for some smooth functions f, g, and p and constants a and b satisfying

$$3g'(\phi)\Omega(\phi, u) + \beta bu = \beta b(f(\phi) + g(\phi)u + \tfrac{1}{6}\beta^2 bu^2).$$

On the other hand, by Corollary 2.22, without loss of generality we can assume that $\Omega(\phi, u) = \omega_0 + \sigma_0 u + u^2$ for some constants ω_0 and σ_0, or $\Omega(\phi, u) = \omega(\phi) + u$ for some smooth function ω. Thus, either $3g'(\phi) = \frac{1}{6}\beta^3 b^2$ and $3\sigma_0 g'(\phi) + \beta b = \beta bg(\phi)$, which together imply that $\beta^4 b^3 = 0$, or $\beta^3 b^2 = 0$. Assume that $\beta \neq 0$. Then we always have $b = 0$, i.e., $P(u, u_x) = p(u) + au_x$, and hence p is not linear and $g = \gamma$ for some constant γ; i.e.,

$$F(\phi, u, u_x) = f(\phi) + \gamma u + \beta u_x.$$

If $\Omega(\phi, u) = \omega(\phi) + u$, then

$$v_{xxx} = \mathcal{M}(\phi, u) + \mathcal{N}(\phi, u)u_x + (f'(\phi) + \beta a)u_{xx} + \gamma u_t + \beta u_{xt}, \tag{2.40}$$

$$\begin{aligned}\mathcal{F}(v, v_x, v_t) = {} & p(f(\phi) + \gamma u + \beta u_x) + \mathcal{O}(\phi, u) + \mathcal{P}(\phi, u)u_x \\ & + (\beta a + f'(\phi))u_{xx} + \gamma u_t + \beta u_{xt}\end{aligned} \tag{2.41}$$

for some smooth functions $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, and $\mathcal{P}$. From (2.40) and (2.41) we deduce that

$$\beta^2 p''(f(\phi) + \gamma u + \beta u_x) = 0,$$

which contradicts the fact that p is not a linear function. Hence, we can assume that $\Omega(\phi, u, u_x) = \omega_0 + \sigma_0 u + u^2$. Then,

$$\begin{aligned}v_{xxx} = {} & \mathcal{Q}(\phi, u) + \mathcal{R}(\phi, u)u_x + 2f'(\phi)u_x^2 \\ & + [f'(\phi)(\sigma_0 + 2u) + \beta a]u_{xx} + \gamma u_t + \beta u_{xt}, \\ \mathcal{F}(v, v_x, v_t) = {} & p(f(\phi) + \gamma u + \beta u_x) + \mathcal{S}(\phi, u) + \mathcal{T}(\phi, u)u_x - f'(\phi)u_x^2 \\ & + [\beta a + f'(\phi)(\sigma_0 + 2u)]u_{xx} + \gamma u_t + \beta u_{xt}\end{aligned}$$

for some smooth functions $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$. Thus, $p''' = 0$; i.e., $p(u) = p_0 + p_1 u + p_2 u^2$ for some constants p_0, p_1, and p_2. On the other hand, from (2.29) one obtains $\tilde{\Theta}(\phi, u) = \theta(\phi) + a(\sigma_0 u + u^2)$ for some smooth function θ. Hence, (2.18) becomes $0 = (\omega_0 + \sigma_0 u + u^2)\theta'(\phi) + p(u)(\sigma_0 + 2u)$, which implies that $\theta(\phi) = \theta_0 + \theta_1\phi$ for some constants θ_0 and θ_1 and

$$p(u) = -\frac{\theta_1(\omega_0 + \sigma_0 u + u^2)}{\sigma_0 + 2u}.$$

So, $p_2 = 0$, which contradicts, again, the fact that p is not a linear function. Therefore, we must have $\beta = 0$.

Finally, consider $u \mapsto v = F(\phi, u)$. If it is a Bäcklund transformation, then we can show as above that

$$F(\phi, u) = f(\phi) + \alpha u \tag{2.42}$$

for some constant α and some smooth function f. Using (2.25), (2.26), and (2.42), direct calculations give

$$\begin{aligned} v_{xxx} &= f'''\Omega^3 + 3f''\Omega^2 \frac{\partial \Omega}{\partial \phi} + f'\Omega \left(\frac{\partial \Omega}{\partial \phi}\right)^2 + f'\Omega^2 \frac{\partial^2 \Omega}{\partial \phi^2} + \alpha p \\ &+ \left(3f''\Omega \frac{\partial \Omega}{\partial u} + f' \frac{\partial \Omega}{\partial \phi}\frac{\partial \Omega}{\partial u} + 2f'\Omega \frac{\partial^2 \Omega}{\partial \phi \partial u} + \alpha q(u)\right) u_x \\ &+ \left(f' \frac{\partial^2 \Omega}{\partial u^2} + \alpha r\right) u_x^2 + \alpha s u_x^3 + f' \frac{\partial \Omega}{\partial u} u_{xx} + \alpha u_t, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{F}(v, v_x, v_t) \\ &\quad = p(v) + f'q(v)\Omega + (f')^2 r(v)\Omega^2 + (f')^3 s(v)\Omega^3 + f'\tilde{\Theta} \\ &\quad + \left[\alpha q(v) + 2\alpha f' r(v)\Omega + 3\alpha (f')^2 s(v)\Omega^2 + f'\left(\frac{\partial \Omega}{\partial \phi}\frac{\partial \Omega}{\partial u} - \Omega \frac{\partial^2 \Omega}{\partial \phi \partial u}\right)\right] u_x \\ &\quad + \left(\alpha^2 r(v) + 3\alpha^2 f' s(v)\Omega - \frac{1}{2} f' \frac{\partial^2 \Omega}{\partial u^2}\right) u_x^2 \\ &\quad + \alpha^3 s(v) u_x^3 + f' \frac{\partial \Omega}{\partial u} u_{xx} + \alpha u_t. \end{aligned}$$

Therefore, $u \mapsto v = F(\phi, u)$ is a Bäcklund transformation for (2.10) if and only if (2.31)–(2.34) hold. ∎

Now, we are ready to prove our main result.

THEOREM 2.43. *A nonlinear partial differential equation* (2.9) *has a Bäcklund transformation* (2.8) *defined via an associated integrable system of the form* (2.10) *if and only if it is equivalent to the KdV equation* $u_{xxx} = -6uu_x + u_t$ *or one of the equations* $u_{xxx} = (q_0 \pm \frac{3}{2}u^2)u_x + u_t$, $u_{xxx} = p_0 + q_0 u_x + 3u_x^2 + u_t$, $u_{xxx} = p_0 + q_0 u_x \pm \frac{1}{2}u_x^3 + u_t$, $u_{xxx} = p_0 + q_0 u_x + \frac{3}{2}u_x^2 - \frac{1}{2}u_x^3 + u_t$ *with* $q_0 > -1$, $u_{xxx} = p_0 + q_0 u_x + \frac{3}{2}u_x^2 + \frac{1}{2}u_x^3 + u_t$ *with* $q_0 < 1$ *and* $u_{xxx} = (q_0 - \frac{3}{2}\sin 2u)u_x - \frac{1}{2}u_x^3 + u_t$ *via a rescaling of the xtu-space and a shift in the u-direction.*

Proof. By Lemma 2.23, Ω does not depend on u_x and u_{xx} and, up to a rescaling of the variable t, (2.24)–(2.34) hold. We first note that if $\alpha = 0$, then (2.32) implies $(\partial^2/\partial u^2)\Omega = 0$, since $f' \neq 0$, and hence (2.27) and (2.28) yield $r = 0$ and $s = 0$, respectively, since $(\partial/\partial u)\Omega \neq 0$.

Second, we assume that $s \neq 0$. Then, $\alpha \neq 0$, and (2.31) implies that $s'(\beta(\phi) + \alpha u)f'(\phi) = 0$; i.e., $s' = 0$. Therefore, we must have $s(u) = s_0$ for some constant $s_0 \neq 0$. Rescaling u if necessary, we can assume $s_0 = \pm\frac{1}{2}$. Thus, (2.31) is equivalent to $\alpha = \pm 1$. By (2.27), without loss of generality, we have

$$\Omega(\phi, u) = \begin{cases} \omega(\phi) + \sin(u + \tau(\phi)) & \text{if } s_0 = -\frac{1}{2}, \\ \omega(\phi) + \sinh(u + \tau(\phi)) & \text{if } s_0 = \frac{1}{2} \end{cases}$$

for some smooth functions ω and τ. We are going to treat these two subcases separately in the next two subsections.

Subcase $s_0 = -\frac{1}{2}$. (2.32) now becomes

$$\alpha r(u) = r(f(\phi) + \alpha u) - \tfrac{3}{2} f'(\phi)\omega(\phi).$$

Taking the u-derivative of it yields $r'(u) = r'(f(\phi) + \alpha u)$, and hence $r''(f(\phi) + \alpha u)f'(\phi) = 0$; i.e.,

$$r(u) = r_0 + r_1 u$$

for some constants r_0 and r_1. Then, (2.28) implies that $r_1 = 0$ and $\omega(\phi) = \omega_0$ for some constant ω_0. So, (2.28) and (2.32) are equivalent to

$$3\omega_0\tau'(\phi) + 2r_0 = 0, \qquad 2(\alpha - 1)r_0 + 3\omega_0 f'(\phi) = 0, \tag{2.44}$$

respectively. Since $f' \neq 0$, from these two equations we see that $r_0 \neq 0$ if and only if $\omega_0 \neq 0$ and $\alpha = -1$. Assume that $r_0 \neq 0$. Rescaling x and t if necessary, we can actually assume that $r_0 = \frac{3}{2}$. Then, (2.44) becomes

$$\omega_0\tau'(\phi) = -1 \qquad \text{and} \qquad \omega_0 f'(\phi) = 2.$$

Thus, (2.33) is equivalent to $q(u) - q(f(\phi) - u) = 0$; i.e., $q(u) = q_0$ for some constant q_0. (2.29) is equivalent to

$$\tilde{\Theta}(\phi, u) = \theta(\phi) + \left(\frac{1}{\omega_0^2} - 1 - q_0\right) \sin(u + \tau(\phi))$$

for some smooth function θ. (2.30) implies that $\theta(\phi) = \theta_0$ and $p(u) = p_0$ for some constants θ_0 and p_0, and hence it is equivalent to $\theta_0 = -\omega_0 p_0$, and

(2.34) is equivalent to $1 = (1 + q_0)\omega_0^2$. Therefore, the equation is equivalent to

$$u_{xxx} = p_0 + q_0 u_x + \tfrac{3}{2} u_x^2 - \tfrac{1}{2} u_x^3 + u_t$$

with $q_0 > -1$, and the only Bäcklund transformation obtained is

$$u \mapsto f_0 \pm 2\sqrt{1 + q_0}\,\phi - u$$

with ϕ defined by

$$\phi_x = \frac{\pm 1}{\sqrt{1 + q_0}} + \sin(u + \tau_0 \mp \sqrt{1 + q_0}\,\phi),$$

$$\phi_t = \frac{\mp p_0}{\sqrt{1 + q_0}} - (\pm\sqrt{1 + q_0} + \sin(u + \tau_0 \mp \sqrt{1 + q_0}\,\phi))u_x$$
$$+ \tfrac{1}{2}\sin(u + \tau_0 \mp \sqrt{1 + q_0}\,\phi)u_x^2 + \cos(u + \tau_0 \mp \sqrt{1 + q_0}\,\phi)u_{xx},$$

where f_0 and τ_0 are arbitrary constants. Assume that $r_0 = 0$. Then $\omega = 0$, i.e., $\Omega(\phi, u) = \sin(u + \tau(\phi))$. (2.33) can be rewritten as

$$\alpha(q(f + \alpha u) - q(u)) = \frac{3}{2}\left(\frac{\alpha}{2}f' - \tau'\right)f' + \frac{3}{2}f'' \sin 2(u + \tau)$$
$$- \frac{3}{2}\left(\frac{\alpha}{2}f' - \tau'\right)f' \cos 2(u + \tau),$$

which implies that

$$\alpha f'(\phi)q'(f(\phi) + \alpha u) = h(\phi) - 2l(\phi)\sin 2(u + \tau(\phi))$$
$$+ 2k(\phi)\cos 2(u + \tau(\phi))$$

for some smooth functions h, k, and l; i.e.,

$$f'(\phi)q(f(\phi) + \alpha u) = g(\phi) + h(\phi)u + k(\phi)\sin 2(u + \tau(\phi))$$
$$+ l(\phi)\cos 2(u + \tau(\phi))$$

for some smooth function g. Since $f' \neq 0$ and $\alpha = \pm 1$, we must have $q(u) = q_0 + q_1 u + q_2 \sin 2u + q_3 \cos 2u$ for some constants q_0, q_1, q_2, and q_3. Replacing u by $u - u_0$ for some constant u_0 if necessary, we can assume that $q_3 = 0$; i.e.,

$$q(u) = q_0 + q_1 u + q_2 \sin 2u. \tag{2.45}$$

Thus, (2.33) is equivalent to

$$\alpha q_1 f = \frac{3}{2}\left(\frac{\alpha}{2} f' - \tau'\right) f', \tag{2.46}$$

$$(\alpha - 1) q_1 = 0, \tag{2.47}$$

$$q_2(\cos 2f - \alpha) = \frac{3}{2} f'' \cos 2\tau + \frac{3}{2}\left(\frac{\alpha}{2} f' - \tau'\right) f' \sin 2\tau, \tag{2.48}$$

$$\alpha q_2 \sin 2f = \frac{3}{2} f'' \sin 2\tau - \frac{3}{2}\left(\frac{\alpha}{2} f' - \tau'\right) f' \cos 2\tau. \tag{2.49}$$

By (2.45) and direct calculations, (2.29) is equivalent to

$$\begin{aligned}\tilde{\Theta}(\phi, u) = \theta(\phi) &+ (\tau'(\phi)^2 - q_0) \sin(u + \tau(\phi)) \\ &+ (\tau''(\phi) - q_1) \cos(u + \tau(\phi)) + \frac{q_2}{2} \cos(u - \tau(\phi)) \\ &- q_1 u \sin(u + \tau(\phi)) + \frac{q_2}{6} \cos(3u + \tau(\phi))\end{aligned} \tag{2.50}$$

for some smooth function θ. From (2.30) we obtain

$$\begin{aligned}&\cos(u + \tau) \frac{\partial}{\partial \phi}\left(\tilde{\Theta}\tau' \cos(u + \tau) - \sin(u + \tau) \frac{\partial \tilde{\Theta}}{\partial \phi}\right) \\ &\quad = -\left(\tilde{\Theta}\tau' \cos(u + \tau) - \sin(u + \tau) \frac{\partial \tilde{\Theta}}{\partial \phi}\right) \tau' \sin(u + \tau),\end{aligned}$$

which together with (2.50) implies that

$$\theta'' = 0, \tag{2.51}$$

$$\theta'\tau' = 0, \tag{2.52}$$

$$\theta\tau'' = 0, \tag{2.53}$$

$$\left(q_1 + \frac{2}{3} q_2 \cos 2\tau + \tau''\right) \tau'^2 = 0, \tag{2.54}$$

$$\left(-q_1 + \frac{2}{3} q_2 \cos 2\tau + \tau''\right) \tau'' = 0, \tag{2.55}$$

$$2(\tau')^2\tau'' + \frac{1}{2}\tau'''' + q_1(\tau')^2 - \frac{2}{3} q_2\tau'' \sin 2\tau = 0, \tag{2.56}$$

$$\frac{2}{3} q_2(\tau')^2 \sin 2\tau - q_1\tau'' - \frac{1}{2}\tau'\tau''' = 0. \tag{2.57}$$

If $\tau'' \neq 0$, then (2.51)–(2.57) are equivalent to $\theta = 0$, $q_1 = 0$, and $\tau'' = -\frac{2}{3} q_2 \cos 2\tau$, and hence $q_2 \neq 0$. (2.46) becomes $f' = 2\alpha\tau'$, (2.49) is equivalent to $f = n\pi - 2\tau$ for some integer n, and (2.48) says that $\alpha = -1$. Thus, (2.30) yields $p = 0$ and (2.34) is satisfied. Therefore, up to a rescaling of the variables x and t and a shift in the u-direction, the equation is equivalent to

$$u_{xxx} = (q_0 - \tfrac{3}{2} \sin 2u) u_x - \tfrac{1}{2} u_x^3 + u_t,$$

and the Bäcklund transformation has the form

$$u \mapsto n\pi - 2\tau(\phi) - u$$

with ϕ defined by

$$\begin{aligned}
\phi_x &= \sin(u + \tau(\phi)),\\
\phi_t &= (\tau'(\phi)^2 - q_0) \sin(u + \tau(\phi)) + \cos 2\tau(\phi) \cos(u + \tau(\phi)) - \tfrac{3}{4} \cos(u - \tau(\phi))\\
&\quad - \tfrac{1}{4} \cos(3u + \tau(\phi)) + \tau'(\phi) u_x + \tfrac{1}{2} \sin(u + \tau(\phi)) u_x^2 + \cos(u + \tau(\phi)) u_{xx},
\end{aligned}$$

where n is an integer and τ is any function satisfying $\tau'' = \cos 2\tau$, for example,

$$\tau(\phi) = \frac{\pi}{4} - \text{arc sin} \frac{e^{2\sqrt{2}\,\phi} - 1}{e^{2\sqrt{2}\,\phi} + 1}.$$

If $\tau'' = 0$ and $\tau' \neq 0$, i.e., $\tau(\phi) = \tau_0 + \tau_1 \phi$ for some constants τ_0 and τ_1 with $\tau_1 \neq 0$, then $\theta(\phi) = \theta_0$ for some constant θ_0 by (2.52) and $q_1 = q_2 = 0$ by (2.54). (2.46) becomes $f' = 2\alpha\tau'$; i.e., $f(\phi) = f_0 + 2\alpha\tau_1\phi$ for some constant f_0. (2.50) can be rewritten as

$$\tilde{\Theta}(\phi, u) = \theta_0 = (\tau_1^2 - q_0) \sin(u + \tau_0 + \tau_1\phi),$$

and (2.30) is equivalent to $p(u) = p_0$ for some constant p_0 and $\theta_0 = p_0/\tau_1$. Thus, (2.34) amounts to $(1 + \alpha)p_0 = 0$. Therefore, up to a rescaling of the variables x and t, the equation is equivalent to

$$u_{xxx} = p_0 + q_0 u_x - \tfrac{1}{2} u_x^3 + u_t,$$

and the Bäcklund transformation has the form

$$u \mapsto f_0 + 2\alpha\tau_1\phi + \alpha u$$

with ϕ defined by

$$\phi_x = \sin(u + \tau_0 + \tau_1\phi),$$

$$\phi_t = \frac{p_0}{\tau_1} + (\tau_1^2 - q_0)\sin(u + \tau_0 + \tau_1\phi) + \tau_1 u_x + \frac{1}{2}\sin(u + \tau_0 + \tau_1\phi)u_x^2 + \cos(u + \tau_0 + \tau_1\phi)u_{xx},$$

where $\alpha = \pm 1$ if $p_0 = 0$ and $\alpha = -1$ if $p_0 \neq 0$, f_0, τ_0, and τ_1 are arbitrary constants satisfying $\tau_1 \neq 0$. If $\tau' = 0$, i.e., $\tau(\phi) = \tau_0$ for some constant τ_0, then $q_1 \neq 0$ by (2.46) since $f' \neq 0$, and $\alpha = 1$ by (2.47). Rescaling the variables x, t, and u if necessary, we can assume that $q_1 = 3$. Then, (2.46) is equivalent to $f(\phi) = (f_0 + \phi)^2$ for some constant f_0, and (2.48) cannot be satisfied. So, $\tau' = 0$ cannot occur.

Subcase $s_0 = \frac{1}{2}$. Ignoring the reality restriction of the previous subcase, we see that the equation must be equivalent to

$$u_{xxx} = p_0 + q_0 u_x + \tfrac{3}{2}u_x^2 + \tfrac{1}{2}u_x^3 + u_t$$

or

$$u_{xxx} = (q_0 - \tfrac{3}{2}\sinh 2u)u_x + \tfrac{1}{2}u_x^3 + u_t$$

or

$$u_{xxx} = p_0 + q_0 u_x + \tfrac{1}{2}u_x^3 + u_t.$$

Imposing the reality condition to the complex Bäcklund transformations obtained there, we conclude that the only Bäcklund transformation for the first equation is

$$u \mapsto f_0 \pm 2\sqrt{q_0 - 1}\,\phi - u,$$

where $q_0 > 1$ and ϕ is defined by

$$\phi_x = \frac{\pm 1}{\sqrt{q_0 - 1}} + \sinh(u + \tau_0 \mp \sqrt{q_0 - 1}\,\phi),$$

$$\phi_t = \frac{\pm p_0}{\sqrt{q_0 - 1}} - (\pm\sqrt{q_0 - 1}\sinh(u + \tau_0 \mp \sqrt{q_0 - 1}\,\phi))u_x - \tfrac{1}{2}\sinh(u + \tau_0 \mp \sqrt{q_0 - 1}\,\phi)u_x^2 + \cosh(u + \tau_0 \mp \sqrt{q_0 - 1}\,\phi)u_{xx},$$

where f_0 and τ_0 are arbitrary constants, there is no Bäcklund transformation for the second equation, and the only Bäcklund transformation for the third equation is

$$u \mapsto f_0 + 2\alpha\tau_1\phi + \alpha u$$

with ϕ defined by

$$\phi_x = \sinh(u + \tau_0 + \tau_1\phi),$$

$$\phi_t = \frac{p_0}{\tau_1} + (\tau_1^2 - q_0)\sinh(u + \tau_0 + \tau_1\phi) + \tau_1 u_x - \frac{1}{2}\sinh(u + \tau_0 + \tau_1\phi)u_x^2 + \cosh(u + \tau_0 + \tau_1\phi)u_{xx},$$

where $\alpha = \pm 1$ if $p_0 = 0$ and $\alpha = -1$ if $p_0 \neq 0$, f_0, τ_0, and τ_1 are arbitrary constants satisfying $\tau_1 \neq 0$.

Third, we assume that $s = 0$ and $r \neq 0$; i.e., $\mathcal{F}(u, u_x, u_t) = p(u) + q(u)u_x + r(u)u_x^2 + u_t$. Then, $\alpha \neq 0$. From (2.27) we obtain

$$\Omega(\phi, u) = a(\phi) + b(\phi)u + c(\phi)u^2$$

for some smooth functions a, b, and c with $b \neq 0$ or $c \neq 0$, and hence, (2.26) becomes

$$\Theta = \tilde{\Theta} + [a'b - ab' + 2(a'c - ac')u + (b'c - bc')u^2]u_x - cu_x^2 + (b + 2cu)u_{xx}.$$

Thus, (2.32) can be rewritten as

$$3c(\phi)f'(\phi) + \alpha r(u) = \alpha^2 r(f(\phi) + \alpha u).$$

As above, this implies that

$$r(u) = r_0 + r_1 u$$

for some constants r_0 and r_1 with $r_0 \neq 0$ or $r_1 \neq 0$. So, (2.28) is equivalent to

$$3a'c = 3ac' - r_0 b, \qquad 3b'c = 3bc' - 2r_0 c - r_1 b, \qquad 0 = r_1 c.$$

They imply that $r_1 = 0$ and $c \neq 0$. Hence, we can assume that $c = 1$. Therefore, (2.28) is satisfied if and only if

$$a(\phi) = a_0 + a_1\phi + \phi^2, \qquad b(\phi) = -a_1 - 2\phi$$

for some constants a_0 and a_1; here without loss of generality we have made the choice that $r_0 = 3$. Then, (2.32) is equivalent to

$$f(\phi) = f_0 + \alpha(\alpha - 1)\phi$$

for some constant f_0. Thus, $\alpha \neq 1$. (2.33) implies that $q(u) = q_0 + q_1 u + q_2 u^2 + q_3 u^3$ for some constants q_0, q_1, q_2, and q_3. Thus, (2.33) also implies that $(1 - \alpha^3)q_3 = 0$; i.e., $q_3 = 0$ since $\alpha \neq 1$. Moreover, from (2.33) we obtain $(\alpha + 1)(q_2 + 6) = 0$ and $\alpha^2 q_2 = 6(\alpha + 1)$; i.e., $\alpha = -1$ and $q_2 = 0$ since α is real. Similarly, (2.33) also yields $q_1 = 0$. Then, direct calculations show that (2.29) is equivalent to

$$\tilde{\Theta}(\phi, u) = \theta(\phi) + (q_0 + 4a_0 - a_1^2)[(a_1 + 2\phi)u - u^2]$$

for some smooth function θ, and (2.30) becomes

$$(a_1 + 2\phi - 2u)p(u) = \sigma(\phi) + \tau(\phi)u + \mu(\phi)u^2, \tag{2.58}$$

where σ, τ, and μ are smooth functions. Taking the third-order u-derivative and then the ϕ-derivative of (2.58) yields $p'''(u) = 0$; that is,

$$p(u) = p_0 + p_1 u + p_2 u^2 \tag{2.59}$$

for some constants p_0, p_1, and p_2. Substituting (2.59) into (2.58) we obtain $p_2 = 0$. So, (2.30) is equivalent to $p_1 = 0$, $\theta(\phi) = -p_0 - (q_0 + 4a_0 - a_1^2)(a_0 + a_1\phi + \phi^2)$. Then, (2.34) is always satisfied. Therefore, the equation is equivalent to

$$u_{xxx} = p_0 + q_0 u_x + 3u_x^2 + u_t,$$

and the only Bäcklund transformation obtained is

$$u \mapsto f_0 + 2\phi - u$$

with ϕ defined by

$$\begin{aligned}
\phi_x &= a_0 + a_1\phi + \phi^2 - (a_1 + 2\phi)u + u^2, \\
\phi_t &= -p_0 - (q_0 + 4a_0 - a_1^2)[a_0 + a_1\phi + \phi^2 - (a_1 + 2\phi)u + u^2] \\
&\quad + [2a_0 - a_1^2 - 2a_1\phi - 2\phi^2 + 2(a_1 + 2\phi)u - 2u^2]u_x \\
&\quad - \tfrac{1}{2} u_x^2 + (-a_1 - 2\phi + 2u)u_{xx},
\end{aligned}$$

where f_0, a_0, and a_1 are arbitrary constants.

Next, assume that $s = r = 0$, i.e., $\mathcal{F}(u, u_x, u_t) = p(u) + q(u)u_x + u_t$. Since $f' \neq 0$, (2.32) is equivalent to

$$\frac{\partial^2 \Omega}{\partial u^2} = 0.$$

Thus, without loss of generality, we can assume that

$$\Omega(\phi, u) = a(\phi) + u$$

for some smooth function a. So, (2.27) and (2.28) are trivially satisfied. We want to show that $\alpha \neq 0$. Assume that $\alpha = 0$. Then, (2.33) implies that $f'' = 0$, i.e.,

$$f(\phi) = f_0 + f_1\phi$$

for some constants f_0 and f_1 with $f_1 \neq 0$. (2.34) is equivalent to

$$\tilde{\Theta}(\phi, u) = \theta(\phi) + \kappa(\phi)u + a''(\phi)u^2,$$

where

$$\theta(\phi) = a(\phi)a'(\phi)^2 + a(\phi)^2a''(\phi) - \frac{1}{f_1}p(f_0 + f_1\phi) - a(\phi)q(f_0 + f_1\phi),$$
$$\kappa(\phi) = a'(\phi)^2 + 2a(\phi)a''(\phi) - q(f_0 + f_1\phi).$$

Thus, (2.29) and (2.30) imply, respectively, that

$$q(u) = q_0 + q_1u, \qquad p(u) = p_0 + p_1u + p_2u^2 + p_3u^3$$

for some constants p_0, p_1, p_2, p_3, q_0, and q_1. Moreover, (2.29) and (2.30) yield

$$q_1 = -3a''(\phi), \qquad p_3 = -a'''(\phi), \qquad p_2 = a'(\phi)a''(\phi) - a'''(\phi)a(\phi) - \kappa'(\phi).$$

So,

$$a(\phi) = a_0 + a_1\phi + a_2\phi^2$$

for some constants a_0, a_1, and a_2,

$$q_1 = -6a_2, \qquad p_3 = 0, \qquad p_2 = -6a_2(a_1 + f_1 + 2a_2\phi).$$

Then, we must have $a_2 = 0$, $p_2 = 0$, and $q_1 = 0$; that is, $\mathcal{F}(u, u_x, u_t) = p_0 + p_1 u + q_0 u_x + u_t$, which contradicts the fact that $u_{xxx} = \mathcal{F}(u, u_x, u_t)$ is nonlinear. Therefore, $\alpha \neq 0$. Since $\alpha \neq 0$ and $f' \neq 0$, taking the second-order u-derivative and then the ϕ-derivative of (2.33) yields $q''' = 0$; i.e.,

$$q(u) = q_0 + q_1 u + q_2 u^2$$

for some constants q_0, q_1, and q_2. So, (2.33) is equivalent to

$$q_2 = \alpha^2 q_2, \tag{2.60}$$

$$3f''(\phi) + \alpha q_1 = \alpha^2(q_1 + 2q_2 f(\phi)), \tag{2.61}$$

$$3a(\phi)f''(\phi) = \alpha(q_1 f(\phi) + q_2 f(\phi)^2). \tag{2.62}$$

Then, (2.29) is equivalent to

$$\tilde{\Theta}(\phi, u) = \theta(\phi) - (q_0 - a'(\phi)^2 + a(\phi)a''(\phi))u - \tfrac{1}{2}(q_1 + a''(\phi))u^2 - \tfrac{1}{3}q_2 u^3$$

for some smooth function θ, and hence, (2.30) implies that

$$p(u) = p_0 + p_1 u + p_2 u^2 + p_3 u^3$$

for some constants p_0, p_1, p_2, and p_3. Thus, (2.30) is equivalent to

$$2q_2 a'(\phi) = 3a'''(\phi) - 6p_3, \tag{2.63}$$

$$a'(\phi)(q_1 + 3a''(\phi)) = 3a(\phi)a'''(\phi) - 2p_2, \tag{2.64}$$

$$a'(\phi)(q_0 - a'(\phi)^2) = a(\phi)(a(\phi)a'''(\phi) - 2a'(\phi)a''(\phi)) - \theta'(\phi) - p_1, \tag{2.65}$$

$$a'(\phi)\theta(\phi) = a(\phi)\theta'(\phi) + p_0. \tag{2.66}$$

Now, we want to show that $q_1 \neq 0$ or $q_2 \neq 0$. Assume that $q_1 = q_2 = 0$. (2.61) implies that $f'' = 0$; i.e., $f(\phi) = f_0 + f_1\phi$ for some constants f_0 and f_1. (2.63) yields

$$a(\phi) = a_0 + a_1\phi + a_2\phi^2 + \frac{p_3}{3}\phi^3$$

for some constants a_0, a_1, and a_2. Thus, (2.64) becomes

$$p_3^2 = 0, \qquad a_2 p_3 = 0, \qquad a_2^2 = 0, \qquad 3a_1 a_2 = 3a_0 p_3 - p_2;$$

that is, $p_3 = a_2 = p_2 = 0$ and, hence, $\mathcal{F}(u, u_x, u_t) = p_0 + p_1 u + u_t$, which contradicts the fact that $u_{xxx} = P(u, u_x) + u_t$ is nonlinear. Therefore, $q_1 \neq 0$ or $q_2 \neq 0$. Assume that $q_2 = 0$; then $q_1 \neq 0$ and, without loss of generality, we can assume further that $q_1 = -6$ and $q_0 = 0$. (2.61) is equivalent to

$$f(\phi) = 2f_0 + 2f_1\phi - \alpha(\alpha - 1)\phi^2$$

for some constants f_0 and f_1. (2.62) implies that $\alpha \neq 1$ and, hence, is equivalent to

$$a(\phi) = \frac{1}{\alpha - 1} f(\phi).$$

Then, (2.63) is equivalent to $p_3 = 0$, (2.64) is equivalent to $\alpha = -1$, and $p_2 = 0$, and (2.65) is equivalent to

$$\theta(\phi) = \theta_0 - p_1\phi + (4f_0 + f_1^2)(-f_1\phi + f^2)$$

for some constant θ_0. So, (2.66) is equivalent to $p_1 = p_0 = 0$, and (2.34) is then always satisfied. Therefore, the equation is equivalent to the KdV equation

$$u_{xxx} = -6uu_x + u_t$$

and the only Bäcklund transformation obtained is

$$u \mapsto 2f_0 + 2f_1\phi - 2\phi^2 - u,$$

with ϕ defined by

$$\begin{aligned} \phi_x &= -f_0 - f_1\phi + \phi^2 + u, \\ \phi_t &= (4f_0 + f_1^2)(-f_0 - f_1\phi + \phi^2) + (2f_0 + f_1^2 - 2f_1\phi + 2\phi^2)u \\ &\quad + 2u^2 + (-f_1 + 2\phi)u_x + u_{xx}, \end{aligned}$$

where f_0 and f_1 are arbitrary constants. Finally, assume that $q_2 \neq 0$. By a translation in the u-direction if necessary, we can assume that $q_1 = 0$. Moreover, rescaling x and t if necessary, we can assume further that $q = \pm\frac{3}{2}$. Now, (2.60)–(2.64) say that $\alpha^2 = 1$,

$$f(\phi) = \begin{cases} f_1 \sinh \phi + f_2 \cosh \phi & \text{if } q = \frac{3}{2} \\ f_1 \sin \phi + f_2 \cos \phi & \text{if } q = -\frac{3}{2} \end{cases} \tag{2.67}$$

for some constants f_1 and f_2, $a(\phi) = (\alpha/2)f(\phi)$, $p_3 = 0$, and $p_2 = 0$, respectively. (2.65) is equivalent to

$$\theta(\phi) = \theta_0 - p_1\phi - \frac{\alpha}{2}\left[q_0 + \frac{1}{4}(-f_1^2 \pm f_2^2)\right]f(\phi)$$

for some constant θ_0. (2.66) is satisfied if and only if $p_1 = \theta_0 = p_0 = 0$; i.e.,

$$\theta(\phi) = -\frac{\alpha}{2}\left[q_0 + \frac{1}{4}(-f_1^2 \pm f_2^2)\right]f(\phi).$$

Then (2.34) always holds. Therefore, the equation is equivalent to

$$u_{xxx} = (q_0 \pm \tfrac{3}{2}u^2)u_x + u_t$$

and the only Bäcklund transformation obtained is

$$u \mapsto f(\phi) + \alpha u,$$

where f is given by (2.67), $\alpha = \pm 1$ and ϕ is defined by

$$\phi_x = \frac{\alpha}{2}f(\phi) + u,$$

$$\phi_t = -\left[q_0 + \frac{1}{4}(-f_1^2 \pm f_2^2)\right]\left(\frac{\alpha}{2}f(\phi) + u\right) \mp \frac{\alpha}{4}f(\phi)u^2$$
$$\mp \frac{1}{2}u^3 + \frac{\alpha}{2}f'(\phi)u_x + u_{xx}.$$

This finishes the proof. ∎

To end this paper, we make some remarks. First, it is clear that the known Bäcklund transformations for the MKdV equation, the potential KdV equation, and the potential MKdV equation are recovered in Theorem 2.43, as mentioned in the introduction. Second, it seems natural that if more than one pseudopotential are used, more Bäcklund transformations may be found. Actually, we do get a new Bäcklund transformation for the KdV equation in this way (see [14]). Finally, some Bäcklund transformations, although not all, can be found on a computer (see, for example, [7]). We will address this equation in a further publication.

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