JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **221,** 349–363 (1998)
ARTICLE NO. AY975901

Interpolation of Level Sets for Equimeasurable Functions

Loris Faina*

Dipartimento di Matematica, Via L. Vanvitelli, 1, 06123 Perugia, Italy

Received September 29, 1997

The author proves a new theoretical property for the family of rearrangements $R(f)$ of a given measurable function f. Given a finite number of equimeasurable functions $f_1, \ldots, f_n \in R(f)$, it is possible to construct a family of equimeasurable functions $(\eta_\lambda)_{\lambda \in [0,1]} \subset R(f)$ which interpolates the functions $f_1, \ldots, f_n$ in a convex-like way. However, as an application, this interpolation result yields a compact fixed point property. © 1998 Academic Press

1. INTRODUCTION

Let $(T, \mathscr{F}, \mu)$ be a measure space with a σ-algebra $\mathscr{F}$ and a positive measure μ. Two $\mathscr{F}$-measurable functions f and g are called *rearrangements* of each other if

$$\mu\big(f^{-1}([\alpha, +\infty))\big) = \mu\big(g^{-1}([\alpha, +\infty))\big) \quad \text{for every } \alpha \in \mathbf{R}.$$

The common feature of all rearrangements is that a given function f is transformed into a new function f^*, which has some desired properties, like monotonicity or symmetries. This is done by rearranging the level sets of f, $\{t \in T: f(t) \geq \alpha\}$, and then by reconstructing f^* from the rearranged level sets. Several authors pointed out the relevance of rearrangements (Kawoll [8], Riesz [10], Hardy *et al*. [5]). For instance this concept was used as a starting point for new directions in functional analysis and inequalities, or in problems of mathematical physics where natural symmetries arise.

* E-mail address: faina@unipg.it.

0022-247X/98 $25.00

On the other side, several efforts were made in order to obtain topological properties of particular subsets of $L^1(T)$: in this direction we quote Cellina *et al*. [3] and Bressan and Colombo [2] for decomposable sets and Bressan *et al*. [1] for suitable classes of selections to differential inclusions.

The aim of this article is to derive similar properties for the set $R(f)$ of all the functions $g: T \to \mathbf{R}$ which are a rearrangement of f. In order to do this we first obtain an interpolation result for a finite subset $\{f_1, \dots, f_n\} \subset R(f)$. Roughly speaking, this is done by first finding a family $\mathscr{E}$ of measurable sets which interpolates simultaneously all the level sets of $f_1, \dots, f_n$, and then using this family $\mathscr{E}$ to build new functions in $R(f)$ which *interpolate* the original n-tuple $\{f_1, \dots, f_n\}$.

The interpolation result is here obtained under the assumption that f has a continuous distribution with respect to μ. In a forthcoming article we will develop a suitable technique to treat the jumps of the distribution, and therefore to extend the result to the more general case. However, as a consequence of the main result of this article—Theorem 2.2—we get a compact fixed point property.

This article originates from an idea contained in the Magister thesis of the author, developed under the supervision of Professor Arrigo Cellina.

2. THE INTERPOLATION RESULT

From now on, $(T, \mathscr{F}, \mu)$ will denote a finite measure space, where μ is a positive and nonatomic measure; furthermore, let $f \in L^\infty(T)$ be such that $f(t) \geq 0$ for every $t \in T$, and assume that the function,

$$t \to \mu(\{s \in T: f(s) \geq t\}) \tag{2.1}$$

is continuous.

A family $(A_\theta)_{\theta \in [0,1]}$, $A_\theta \in \mathscr{F}$ is called *increasing* if $A_{\alpha_1} \subset A_{\alpha_2}$ when $\alpha_1 < \alpha_2$. An increasing family is called *refining* $C \in \mathscr{F}$ with respect to a measure ν, if $A_0 = \emptyset$, $A_1 = C$, and

$$\nu(A_\alpha) = \alpha \nu(C) \quad \text{for every } \alpha \in [0,1].$$

The following lemma is a fundamental tool.

LEMMA 2.1 (See [9]). *Let $f_1, f_2, \dots, f_m \in L(T)$, and let ν be the vector measure whose components ν_i are the measures $f_i \mu$. Then, there exists a family $(A_\theta)_{\theta \in [0,1]}$ refining T with respect to (ν, μ).*

The interpolation result is described by the following theorem.

THEOREM 2.2. *Let* $f_1, f_2, \ldots, f_m \in R(f)$. *Then, for every* $\varepsilon > 0$ *there exists a family of functions* $(\eta_\lambda^\epsilon)_\lambda$, *where* $\lambda \in S^m = \{\lambda \in \mathbf{R}^m\colon \lambda_i \geq 0$ *and* $\sum_{i=1}^m \lambda_i = 1\}$, *with the properties*:

(a_1) $\eta_\lambda^\varepsilon \in R(f)$ *for every* $\lambda \in S^m$;

(a_2) *there is a family of numbers* $(t_n)_n$ *with* $\lim_n t_n = \|f\|_\infty$, *such that*

$$\{s \in T\colon \eta_{\lambda^j}^\varepsilon(s) \geq t_n\} = \{s \in T\colon f_j(s) \geq t_n\},$$

for every $j = 1, 2, \ldots, m$, *where* λ^j *is the* m*-tuple with entries* $(\delta_j^i;\ i = 1, \ldots, m)$;

(a_3) $\lambda \to \eta_\lambda^\varepsilon$ *is continuous from* S^m *into* $L^1(T)$;

(a_4) $\|\eta_\lambda^\varepsilon - f_i\|_1 \leq (2\|f\|_\infty + \mu(T))\varepsilon + \sup_{j=1,2,\ldots,m,\ \lambda_j \neq 0} \|f_j - f_i\|_1.$

Proof. What follows is an abridged proof of the full one. The complete proof can be found in [7].

Because the function $t \to H(t) = \mu(\{s \in T\colon f(s) \geq t\})$ is nonincreasing, continuous, $H(0) = \mu(T)$, and $\lim_{t \to \|f\|_\infty} H(t) = 0$, for every $\alpha \in [0, \mu(T)]$ there exists $t_\alpha \in [0, \|f\|_\infty]$ such that $H(t_\alpha) = \alpha$.

Let $\varepsilon > 0$ and $i \in \{1, 2, \ldots, m\}$ be fixed. Let $t_h^1, h \in \mathbf{N}$, $t_0^1 = 0$, be a sequence, let h^* be an integer such that

$$t_h^1 < t_{h+1}^1,$$

$$\lim_h t_h^1 = \|f\|_\infty,$$

$$t_{h+1}^1 - t_h^1 < \varepsilon \quad \text{for every } h \leq h^*,$$

and, setting $I_{1,0}^i = T$, $I_{1,h}^i = \{s \in T\colon f_i(s) \geq t_h^1\}$, such that

$$\mu(I_{1,h^*}^i) < \varepsilon, \tag{2.2}$$

$$\mu(I_{1,h}^i) = \tfrac{1}{2}\mu(I_{1,h-1}^i) \quad \text{for every } h > h^*. \tag{2.3}$$

Our first aim will be constructing families of measurable sets which, in some sense, interpolate the level sets $(I_{1,h}^i)_{i=1,\ldots,m,\ h \in \mathbf{N}}$.

Let $(F_1(\theta))_{\theta \in [0,1]}$ be an increasing family of measurable sets refining the set T with respect to the measure μ, and the measures generated by the densities of the form,

$$1_{I_{1,h}^i \setminus I_{1,h+1}^i}|f_j - f_k| \quad \text{for every } i, j, k = 1, 2, \ldots, m \text{ and } h = 1, 2, \ldots, h^*,$$

and those of the form,

$$1_{I^i_{1,h}} \quad \text{for every } i = 1, 2, \dots, m \text{ and } h = 1, 2, \dots, h^*.$$

Set, for $h = 1, 2, \dots, h^*$ and $\lambda = (\lambda_k)_{i=1,\dots,m} \in S^m$,

$$\tilde{E}_{1,h}(\lambda) = \bigcup_{j=1}^{m} \left[F_1\big(p_j(\lambda)\big) \setminus F_1\big(p_{j-1}(\lambda)\big) \right] \cap I^j_{1,h},$$

where $p_j(\lambda) = \sum_{i=1}^{j} \lambda_i$ ($p_0(\lambda) = 0$). Clearly,

$$\tilde{E}_{1,h+1}(\lambda) \subset \tilde{E}_{1,h}(\lambda) \quad \text{for } h = 1, 2, \dots, h^* - 1 \text{ and } \lambda \in S^m.$$

Then we can verify that

$$\begin{aligned} \mu\big(\tilde{E}_{1,h}(\lambda)\big) &= \sum_{j=1}^{m} \mu\Big(\big[F_1\big(p_j(\lambda)\big) \setminus F_1\big(p_{j-1}(\lambda)\big)\big] \cap I^j_{1,h}\Big) \\ &= \sum_{j=1}^{m} \left[p_j(\lambda) - p_{j-1}(\lambda) \right] \mu\big(I^j_{1,h}\big) = \sum_{j=1}^{m} \lambda_j\, \mu\big(I^j_{1,h}\big) = \mu\big(I^i_{1,h}\big), \end{aligned}$$

for every $i = 1, 2, \dots, m$. Furthermore, for $\lambda, \lambda' \in S^m$, one easily shows that

$$\mu\big(\tilde{E}_{1,h}(\lambda)\, \Delta\, \tilde{E}_{1,h}(\lambda')\big) \le 2\mu\big(I^i_{1,h}\big)\|\lambda - \lambda'\|.$$

Now let $C(m,k)$ be the set of all k-tuples $\sigma_k = (c_1, c_2, \dots, c_k)$ with $c_i \in \{1, 2, \dots, m\}$ and $c_j < c_{j+1}$ for $j = 1, 2, \dots, k-1$. Given $\sigma_k \in C(m,k)$, we write $i \in \sigma_k$ if there exists a j, $1 \le j \le k$, with $c_j = i$.

Order the set $\bigcup_{k=1}^{m} C(m,k)$ in the following way,

$$\sigma_k \le \sigma_{k'},$$

if $k > k'$ or $k = k'$ and $\sigma_k \le \sigma_{k'}$ with respect to the lexicographic order. Then, we shall write the set $\bigcup_{k=1}^{m} C(m,k)$ as $\{d_1, d_2, \dots, d_M\}$ with $d_1 < d_2 < \cdots < d_M$.

Let $(F^{d_k}_{1,h}(\theta))_{\theta \in [0,1]}$ be an increasing family of measurable sets refining the set,

$$D^{d_k}_{1,h} = \left(\bigcap_{i \in d_k} I^i_{1,h} \right) \setminus \left(\bigcup_{i \notin d_k} I^i_{1,h} \right) \qquad h > h^*,$$

with respect to the measure μ.

Set

$$\tilde{E}^{h^*+1}_{1,h^*+1}(\lambda) = \bigcup_{k=1}^{M} F^{d_k}_{1,h^*+1}\big(l_{d_k}(\lambda)\big), \qquad \lambda \in S^m,$$

where $l_{d_k}(\lambda) = \sum_{i \in d_k} \lambda_i$. Note that, for $k \neq k'$ $D^{d_k}_{1,h} \cap D^{d_{k'}}_{1,h} = \varnothing$; thus, $F^{d_k}_{1,h^*+1}(\theta) \cap F^{d_{k'}}_{1,h^*+1}(\theta) = \varnothing$.

Hence, for every $\lambda \in S^m$ we find

$$\begin{aligned}
\mu\big(\tilde{E}^{h^*+1}_{1,h^*+1}(\lambda)\big) &= \sum_{k=1}^{M} \Big(\sum_{i \in d_k} \lambda_i\Big) \mu\Bigg(\Big[\bigcap_{i \in d_k} I^i_{1,h^*+1}\Big] \diagdown \Big[\bigcup_{i \notin d_k} I^i_{1,h^*+1}\Big]\Bigg) \\
&= \sum_{k=1}^{M} \Big(\sum_{i \in d_k} \lambda_i\Big) \mu\big(D^{d_k}_{1,h^*+1}\big) \qquad (2.4) \\
&= \sum_{j=1}^{M} \lambda_j\, \mu\big(I^j_{1,h^*+1}\big) = H\big(t^1_{h^*+1}\big),
\end{aligned}$$

because the family $\{D^{d_k}_{1,h^*+1};\, i \in d_k,\, k = 1, 2, \dots, M\}$ is a finite decomposition of I^i_{1,h^*+1}. Furthermore, if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_m)$ then

$$\begin{aligned}
\mu\big(\tilde{E}^{h^*+1}_{1,h^*+1}(\lambda)\, \Delta\, \tilde{E}^{h^*+1}_{1,h^*+1}(\lambda')\big) &= \sum_{k=1}^{M} \mu\Big(F^{d_k}_{1,h^*+1}\big(l_{d_k}(\lambda)\big)\, \Delta F^{d_k}_{1,h^*+1}\big(l_{d_k}(\lambda')\big)\Big) \\
&= \sum_{k=1}^{M} \big|l_{d_k}(\lambda) - l_{d_k}(\lambda')\big| \mu\big(D^{d_k}_{1,h^*+1}\big).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\mu\big(\tilde{E}^{h^*+1}_{1,h^*+1}(\lambda)\, \Delta\, \tilde{E}^{h^*+1}_{1,h^*+1}(\lambda')\big) &\le \sum_{i=1}^{m} |\lambda_i - \lambda'_i|\, \mu\big(I^i_{1,h^*+1}\big) \\
&= \mu\big(I^i_{1,h^*+1}\big) \|\lambda - \lambda'\|. \qquad (2.5)
\end{aligned}$$

Suppose, for $h - 1 > h^* + 1$, to have already defined the sets,

$$\tilde{E}^{h-1}_{1,h-1}(\lambda), \tilde{E}^{h-1}_{1,h-2}(\lambda), \dots, \tilde{E}^{h-1}_{1,h^*+1}(\lambda) \qquad \lambda \in S^m;$$

then, we shall define backwardly the sets,

$$\tilde{E}^{h}_{1,h}(\lambda), \tilde{E}^{h}_{1,h-1}(\lambda), \dots, \tilde{E}^{h}_{1,h^*+1}(\lambda) \qquad \lambda \in S^m.$$

Put first

$$\tilde{E}^h_{1,h}(\lambda) = \bigcup_{k=1}^{M} F^{d_k}_{1,h}\big(l_{d_k}(\lambda)\big),$$

and for $l = h-1, h-2, \dots, h^* + 1$,

$$\tilde{E}^h_{1,l}(\lambda) = \tilde{E}^h_{1,l+1}(\lambda) \cup \bigcup_{k=1}^{M} F^{d_k}_{1,l}\big(p^{d_k}_{l,h}(\lambda)\big), \tag{2.6}$$

where

$$p^{d_k}_{l,h}(\lambda) = \max\Bigg\{0 \le p \le l_{d_k}(\lambda): \mu\Bigg(\tilde{E}^h_{1,l+1}(\lambda) \cup \Bigg[\bigcup_{j=1}^{k-1} F^{d_k}_{1,l}\big(p^{d_k}_{l,h}(\lambda)\big) \cup F^{d_k}_{1,l}(p)\Bigg]\Bigg) \le \mu\big(I^i_{1,l}\big)\Bigg\}.$$

Note that $\mu(\tilde{E}^h_{1,h}(\lambda)) < \mu(I^i_{1,h-1})$, hence the set,

$$\Big\{0 \le p \le l_{d_1}(\lambda): \mu\Big(\tilde{E}^h_{1,h}(\lambda) \cup F^{d_1}_{1,h-1}(p)\Big) \le \mu\big(I^i_{1,l}\big)\Big\}$$

contains 0; thus, $p^{d_1}_{h-1,h}(\lambda)$ is well defined. By iterating this procedure, it follows that all the $p^{d_k}_{h-1,h}(\lambda)$ are well defined. Analogously, it can be shown that all the $p^{d_k}_{l,h}(\lambda)$ are well defined for every $l = h-1, \dots, h^* + 1$.

Clearly,

$$\tilde{E}^h_{1,l+1}(\lambda) \subset \tilde{E}^h_{1,l}(\lambda) \quad \text{for } l = h^* + 1, \dots, h-1 \text{ and } \lambda \in S^m;$$

furthermore, analogously to show (2.4) was proved, we obtain

$$\mu\big(\tilde{E}^h_{1,l}(\lambda)\big) = \mu\big(I^i_{1,l}\big) \quad \text{for } l = h^* + 1, \dots, h \text{ and } \lambda \in S^m. \tag{2.7}$$

Note that each time we define a new interpolating set $\tilde{E}^h_{1,h}(\lambda)$ the already defined sets $\tilde{E}^{h-1}_{1,l}(\lambda)$, $l = h^* + 1, \dots, h-1$ are modified, in general, by means of a measurable set, whose measure is, at most, equal to $2\mu(I^i_{1,h})$. This is the key for proving that

$$\mu\big(\tilde{E}^h_{1,l}(\lambda)\,\Delta\,\tilde{E}^{h-1}_{1,l}(\lambda)\big) \le 2\mu\big(\tilde{E}^h_{1,h}(\lambda)\big) \quad \text{for } l = h^* + 1, \dots, h-1. \tag{2.8}$$

The next step is to get the following estimate of the set $\tilde{E}^h_{1,l}(\lambda)\,\Delta\,\tilde{E}^h_{1,l}(\lambda')$,

$$\begin{aligned}\mu\Big(\tilde{E}^h_{1,l}(\lambda)\,\Delta\,\tilde{E}^h_{1,l}(\lambda')\Big) &\le 2\Big[2\mu\Big(\tilde{E}^h_{1,l+2}(\lambda)\,\Delta\,\tilde{E}^h_{1,l+2}(\lambda')\Big) + \mu\Big(I^i_{1,l+1}\Big)\|\lambda-\lambda'\|\Big]\\ &\quad + \mu\Big(I^i_{1,l}\Big)\|\lambda-\lambda'\|.\end{aligned}$$

By iterating this process we get, from (2.3), and because $h, l > h^*$,

$$\begin{aligned}\mu\Big(\tilde{E}^h_{1,l}(\lambda)\,\Delta\,\tilde{E}^h_{1,l}(\lambda')\Big) &\le \left[2^{h-l}\mu\Big(I^i_{1,h}\Big) + \mu\Big(I^i_{1,l}\Big) + 2\left(\sum_{j=l+1}^{h-1}\mu\Big(I^i_{1,j}\Big)\right)\right]\|\lambda-\lambda'\|\\ &= 2\left(\sum_{j=l}^{h-1}\mu\Big(I^i_{1,j}\Big)\right)\|\lambda-\lambda'\|.\end{aligned}$$

Because the series $\sum_{h=h^*+1}^{\infty}\mu(I^i_{1,h})$ converges, from (2.8) and (2.7), the sequence $(1_{\tilde{E}^h_{1,l}(\lambda)})_h$ is a Cauchy sequence in $L^1(T)$, for every $l \ge h^* + 1$. Hence, there are measurable sets $\hat{E}_{1,l}(\lambda), l \ge h^* + 1, \lambda \in S^m$ such that

$$1_{\tilde{E}^h_{1,l}(\lambda)} \overset{L^1}{\to} 1_{\hat{E}_{1,l}(\lambda)}.$$

Setting

$$\tilde{E}_{1,l}(\lambda) = \bigcup_{k\ge l}\hat{E}_{1,k}(\lambda),$$

it is straightforward to verify that

$$\mu\Big(\tilde{E}_{1,l}(\lambda)\Big) = \mu\Big(I^i_{1,l}\Big) \quad \text{for every } l \ge h^* + 1,\ \lambda \in S^m; \tag{2.9}$$

$$\tilde{E}_{1,l+1}(\lambda) \subset \tilde{E}_{1,l}(\lambda) \quad \text{for every } l \ge h^* + 1,\ \lambda \in S^m; \tag{2.10}$$

$$\mu\Big(\tilde{E}_{1,l}(\lambda)\,\Delta\,\tilde{E}_{1,l}(\lambda')\Big) \le 4\mu\Big(I^i_{1,l}\Big)\|\lambda-\lambda'\| \quad \text{for every } l \ge h^*+1,\ \lambda, \lambda' \in S^m. \tag{2.11}$$

At this point we have constructed a family of measurable sets $(\tilde{E}_{1,h}(\lambda))_{\lambda\in S^m, h\in\mathbf{N}}$ which interpolates the level sets $(I^i_{1,h})$ for $i = 1, 2, \ldots, m$, but we cannot still ensure that

$$\tilde{E}_{1,h+1}(\lambda) \subset \tilde{E}_{1,h}(\lambda) \quad \text{for } h \in \mathbf{N};$$

therefore we are going to modify these sets in order to fulfill this fundamental property.

More precisely, we redefine the sets $(\tilde{E}_{1,h}(\lambda))_{\lambda \in S^m,\, h \in \mathbf{N}}$ as follows,

$$E_{1,h}(\lambda) = \tilde{E}_{1,h}(\lambda) \quad \text{if } h > h^*;$$

and

$$E_{1,h}(\lambda) = \tilde{E}_{1,h^*+1}(\lambda) \cup \bigcup_{j=1}^{m} \left[F_1\left(\tilde{p}_j^h(\lambda)\right) \setminus F_1\left(p_{j-1}(\lambda)\right)\right] \cap I^j_{1,h} \qquad \text{if } h \le h^*,$$

where

$$\tilde{p}_j^h(\lambda) = \max\Bigg\{ p_{j-1}(\lambda) \le p \le p_j(\lambda) : \mu\Bigg(\tilde{E}_{1,h^*+1}(\lambda) \cup \left[\bigcup_{k=1}^{j-1} \left[F_1\left(\tilde{p}_k^h(\lambda)\right) \setminus F_1\left(p_{k-1}(\lambda)\right)\right] \cap I^k_{1,h} \right] \cup \left[\left[F_1(p) \setminus F_1\left(p_{j-1}(\lambda)\right)\right] \cap I^j_{1,h} \right] \Bigg) \le \mu\left(I^i_{1,h}\right) \Bigg\}.$$

Clearly,

$$E_{1,l+1}(\lambda) \subset E_{1,l}(\lambda) \quad \text{for } l \in \mathbf{N} \text{ and } \lambda \in S^m,$$

and

$$\mu\left(E_{1,l}(\lambda)\right) = \mu\left(I^i_{1,l}\right) \quad \text{for } l \in \mathbf{N} \text{ and } \lambda \in S^m;$$

furthermore, for $h = 1, 2, \ldots, h^*$, we have

$$\begin{aligned} &\mu\left(E_{1,h}(\lambda)\, \Delta E_{1,h}(\lambda')\right) \\ &\quad \le 2\mu\left(E_{1,h^*+1}(\lambda)\, \Delta E_{1,h^*+1}(\lambda')\right) + \mu\left(\tilde{E}_{1,h}(\lambda)\, \Delta \tilde{E}_{1,h}(\lambda')\right) \\ &\quad \le \left(8\mu\left(I^i_{1,h^*+1}\right) + 2\mu\left(I^i_{1,h}\right)\right)\|\lambda - \lambda'\|. \end{aligned}$$

The family of measurable sets $(E_{1,h}(\lambda))_{\lambda \in S^m, h \in \mathbf{N}}$, will be the skeleton upon which the remaining interpolating level sets will be constructed.

We define a family of measurable sets $(I^i_{n,h})^{i=1,\ldots,m}_{n \ge 2,\, h \ge 1}$ with the following properties: choose a $t^n_h > 0$, $h \in \mathbf{N}$ odd, such that

$$\mu\left(f_i^{-1}\left(\left[t^{n-1}_{(h-1)/2}, t^n_h\right)\right)\right) = \mu\left(f_i^{-1}\left(\left[t^n_h, t^{n-1}_{(h+1)/2}\right)\right)\right),$$

and put:

- $t_0^n = 0$ for every $n \geq 1$;
- $t_h^n = t_{h/2}^{n-1}$ if $h \in \mathbf{N}$ is even, $n \geq 2$;
- $I_{n,h}^i = f_i^{-1}([t_{(h-1)/2}^{n-1}, t_h^n))$, $h \in \mathbf{N}$ odd;
- $I_{n,h}^i = f_i^{-1}([t_{h-1}^n, t_{h/2}^{n-1}))$, $h \in \mathbf{N}$ even.

Let $(A(\theta))_{\theta \in [0,1]}$ be an increasing family of measurable sets refining the set T with respect to the measure μ.

For $h \in \mathbf{N}$ odd, define

$$E_{2,h}(\lambda) = \left[E_{1,(h-1)/2}(\lambda) \setminus E_{1,(h+1)/2}(\lambda)\right] \cap A\left(p_h^2(\lambda)\right),$$
$$E_{2,h-1}(\lambda) = \left[E_{1,(h-1)/2}(\lambda) \setminus E_{1,(h+1)/2}(\lambda)\right] \setminus E_{2,h}(\lambda),$$

where

$$p_h^2(\lambda) = \max\Big\{0 \leq p \leq 1\colon \mu\Big(\left[E_{1,(h-1)/2}(\lambda) \setminus E_{1,(h+1)/2}(\lambda)\right] \cap A(p)\Big) \leq \tfrac{1}{2}\mu\Big(E_{1,(h-1)/2}(\lambda) \setminus E_{1,(h+1)/2}(\lambda)\Big)\Big\}.$$

Then, for fixed $n \in \mathbf{N}$, $n \geq 3$ and for $h \in \mathbf{N}$ odd, define

$$E_{n,h}(\lambda) = E_{n-1,(h-1)/2}(\lambda) \cap A\left(p_h^n(\lambda)\right),$$
$$E_{n,h-1}(\lambda) = E_{n-1,(h-1)/2}(\lambda) \setminus E_{n,h}(\lambda),$$

where

$$p_h^n(\lambda) = \max\Big\{0 \leq p \leq 1\colon \mu\Big(E_{n-1,(h-1)/2}(\lambda) \cap A(p)\Big) \leq \tfrac{1}{2}\mu\Big(E_{n-1,(h-1)/2}(\lambda)\Big)\Big\}. \quad (2.12)$$

By construction we derive that

$$E_{n,h}(\lambda) \cap E_{n,l}(\lambda) = \varnothing \quad \text{if } h \neq l,\ n > 1; \quad (2.13)$$

$$\mu\left(\bigcup_{j=h}^{\infty} E_{n,j}(\lambda)\right) = \mu\left(I_{n,h}^i\right) \quad \text{for every } n, h \in \mathbf{N}; \quad (2.14)$$

$$\mu\left(E_{n,h}(\lambda)\,\Delta E_{n,h}(\lambda')\right) \leq 2^{n-1}\Big(\mu\Big(E_{1,j(h)}(\lambda)\,\Delta E_{1,j(h)}(\lambda')\Big) + \mu\Big(E_{1,j(h)-1}(\lambda)\,\Delta E_{1,j(h)}(\lambda')\Big)\Big)\|\lambda - \lambda'\|, \quad (2.15)$$

where $j(h)$ is the first integer such that

$$2^{(n-1)(j(h)-1)} \leq h \leq 2^{(n-1)j(h)}.$$

Now, we define a sequence of functions $(\eta_\lambda^n)_{n\in\mathbf{N},\,\lambda\in S^m}$ as the formal series,

$$\eta_\lambda^n(t) = \sum_{k=1}^{\infty} t_k^n 1_{E_{n,k}(\lambda)}.$$

By virtue of (2.14) and (2.15) we get

$$\|\eta_\lambda^n - \eta_{\lambda'}^n\|_1 \leq C(n)\|\lambda - \lambda'\|, \tag{2.16}$$

where $C(n)$ is a suitable constant depending on n.

Also $(\eta_\lambda^n)_n$ converges in $L^1(T)$ uniformly with respect to $\lambda \in S^m$. Denote by $(\eta_\lambda^\varepsilon)_{\lambda\in S^m}$ the L^1-limits of these sequences, that is $\eta_\lambda^\varepsilon = L^1 - \lim_n \eta_\lambda^n$. We are now going to prove (a_3), that is the continuity of the function $\lambda \to \eta_\lambda^\varepsilon$. To this end, for every $\tau > 0$ there is a $\bar{n} \in \mathbf{N}$ such that for every $n \geq \bar{n}$ we have

$$\|\eta_\lambda^n - \eta_\lambda^\varepsilon\|_1 \leq \frac{\tau}{3} \quad \text{for every } \lambda \in S^m.$$

Furthermore, from (2.16), for every $\tau > 0$ there is a $\delta = \delta(\tau, \bar{n})$ such that for every λ, with $\|\lambda - \lambda_0\| < \delta$, we have

$$\|\eta_\lambda^{\bar{n}} - \eta_{\lambda_0}^{\bar{n}}\|_1 < \frac{\tau}{3}.$$

Then, for $\|\lambda - \lambda_0\| < \delta$ we get $\|\eta_\lambda^\varepsilon - \eta_{\lambda_0}^\varepsilon\|_1 \leq \tau$. Note that there exists a dense and countable set $D \supset \{t_n^1\}_n$ in $f(T)$ such that

$$\mu\big(\{t \in T\colon f(t) \geq d\}\big) = \mu\big(\{t \in T\colon \eta_\lambda^\varepsilon(t) \geq d\}\big) \quad \text{for every } d \in D.$$

Thus, from (2.1), the functions η_λ^ε also satisfy the properties (a_1) and (a_2). Furthermore, by construction we note that, except for a set Z of measure less than ε, we have

$$\min_{i\in\{1,2,\ldots,m\}} \left|\eta_\lambda^\varepsilon(t) - f_i\right| \leq \varepsilon \quad \text{for every } t \in T \setminus Z.$$

Thus,

$$\|\eta_\lambda^\varepsilon - f_i\|_1 \leq (2\|f\|_\infty + \mu(T))\,\varepsilon + \sup_{j\in\{1,2,\ldots,m\},\ \lambda_j\neq 0} \int_T |f_j - f_i|\, d\mu,$$

and this concludes the proof. ∎

3. THE COMPACT FIXED POINT PROPERTY

Any continuous function which maps a closed subset A of a metric space X into a totally bounded set of a normed space E can be extended to the whole space X keeping the value into a totally bounded set (see [4]). Indeed the range of the extension is contained in the convex hull of a totally bounded set of a normed space, which is still totally bounded.

An analogous result was proved by [6] for maps with values in the space of Lebesgue integrable functions, by using the concept of decomposable hull instead of that of convex hull.

The purpose of this section is to prove a similar result for maps with values in $L^1(T)$ using the concept of rearrangement. However, as a consequence, we get a compact fixed point property for the space $R(f)$.

THEOREM 3.1 (Compact extension result). *Let A be a closed subset of a metric space (X, d), and let $h\colon A \to R(f)$ be a continuous map whose image is relatively compact. Then, there exists a totally bounded set B, with $h(A) \subset B \subset R(f)$, and a continuous function $\tilde{h}\colon X \to B$ such that $\tilde{h}_{|A} = h$.*

Proof. Let $(A_n)_{n \geq 1}$ be the open sets defined by

$$A_1 = \{x \in X\colon d(x, A) > 1\},$$
$$\cdots$$
$$A_n = \left\{x \in X\colon \frac{1}{2^{n-1}} < d(x, A) < \frac{3}{2^{n-1}}\right\}, \qquad n \in \mathbf{N}.$$

We have: $X \setminus A = \bigcup_{n \geq 1} A_n$. Set $\varepsilon_n = 1/2^n$, $n \geq 1$, and let $N_n = \{g_1^n, \ldots, g_{j_n}^n\}$ be an ε_n-net of $h(A)$. Let $\pi_n\colon X \to A$ be a function such that $d(x, \pi x) - d(x, A) \leq \varepsilon_n$. Put

$$\mathscr{U}_j^n = A_n \cap B\Big(\pi_n^{-1}\Big(h^{-1}\big(g_j^n + \varepsilon_n B_1\big)\Big); \varepsilon_n\Big).$$

Consider the pairs (n, j); $n \geq 1$, $j = 1, \ldots, j_n$, in the lexicographic order; the pair (n, j) is identified with a natural l by the relation $l = \sum_{i=1}^{n-1} j_i + j$ ($j_0 = 0$). If l corresponds to the pair (n, j), g_l and $\mathscr{U}^l$ will denote, respectively, g_j^n and $\mathscr{U}_j^n$.

Let $\{q^l(x)\}$ be a continuous partition of unity subordinate to $\{\mathscr{U}^l\}$. Apply Theorem 2.1 to the functions of the set $N_1 \cup N_2$, namely, for

$$f_i^1 = g_i \qquad i = 1, \ldots, p = j_1 + j_2,$$

and for $\varepsilon = 1$, and denote by $(\eta_\lambda^\varepsilon)_\lambda$ the corresponding interpolating functions.

Define a continuous function $\tilde{h}_1$ on A_1 by setting $\tilde{h}_1(x) = \eta^\varepsilon_{\lambda(x)}$, where $\lambda(x) = (\lambda_1(x), \ldots, \lambda_p(x))$, $\lambda_i(x) = q^i(x)$.

Let $R_1 = \{b_m^1\}_{m=0,\dots,m_1}$ be an ε_3-net of the totally bounded set $\tilde{h}_1(A_1)$. Let θ_1 be a function mapping each x belonging to A_1 into an element of R_1, whose distance from $\tilde{h}_1(x)$ is less than ε_3. From the continuity of $\tilde{h}_1$, for every $\varepsilon > 0$ and for every $x \in A_1$ there is a $\delta_\varepsilon^x > 0$ such that if $y \in A_1$ is such that $d(x, y) < \delta_\varepsilon^x$ then $|\tilde{h}_1(x) - \tilde{h}_1(y)| < \varepsilon$.

Define the open sets

$$\mathscr{V}_{j,m}^1 = \mathscr{U}_j^1 \cap \bigcup_{x \in \theta_1^{-1}(b_m^1)} B\left(x, \delta_{\varepsilon_3}^x\right), \qquad j = 1, \dots, j_1;\ m = 0, \dots, m_1,$$

$$\mathscr{V}_{j,0}^n = \mathscr{U}_j^n, \qquad j = 1, \dots, j_n;\ n \geq 2.$$

Consider the triples (n, j, m); $n \geq 1$, $j = 1, \dots, j_n$, $m = 0, \dots, m_n$ (set $m_n = 0$ if $n \neq 1$), in the lexicographic order; the triple (n, j, m) is identified with a natural l by the relation $l = \sum_{i=1}^{n-1} j_i(m_i + 1) + j \cdot (m + 1)$. Denote with l_n the index corresponding to the triple (n, j_n, m_n). If l corresponds to the triple (n, j, m); $g_l, \mathscr{V}^l$ will denote, respectively, g_j^n and $\mathscr{V}_{j,m}^n$.

Let $\{q^l(x)\}$ be a continuous partition of unity subordinate to $\{\mathscr{V}^l\}$. Apply Theorem 2.1 to the functions of the set $R_1 \cup N_1 \cup N_2 \cup N_3$, i.e.,

$$f_i^2 = b_i^1 \qquad i = 0, \dots, m_1;$$

$$f_i^2 = g_{\tau(i)} \qquad i = m_1 + 2, \dots, \sum_{k=1}^{3} j_k + m_1 + 1,$$

where $\tau(i) = i - (m_1 + 1)$ and for $\varepsilon = \varepsilon_3/(2\|f\|_\infty + \mu(T))$, and denote with $(\eta_\lambda^\varepsilon)_\lambda$ the corresponding interpolating functions.

Define a continuous function $\tilde{h}_2$ on $A_1 \cup A_2$ by setting $\tilde{h}_2 = \eta_{\lambda(x)}^\varepsilon$, where $\lambda_i(x) = q^i(x)$. From the definition of R_1, for every $x \in A_1 \setminus \overline{A}_2$ there are a $\bar{x} \in A_1 \setminus \overline{A}_2$ and a $b\frac{1}{m} \in R_1$ such that

$$\theta_1(\bar{x}) = b\frac{1}{m},$$

$$d(x, \bar{x}) < \delta_{\varepsilon_3},$$

$$\left\| b\frac{1}{m} - \tilde{h}_1(\bar{x}) \right\|_1 < \varepsilon_3,$$

$$\left\| b\frac{1}{m} - \tilde{h}_1(x) \right\|_1 < 2\varepsilon_3.$$

Furthermore, if $q^i(x) \neq 0$ then there are a $\bar{x}_i \in A_1 \setminus \overline{A}_2$ and a $b^1_{m(i)} \in R_1$ such that

$$\theta_1(\bar{x}_i) = b^1_{m(i)},$$

$$d(x, \bar{x}_i) < \delta_{\varepsilon_3},$$

$$\left\| b^1_{m(i)} - \tilde{h}_1(\bar{x}_i) \right\|_1 < \varepsilon_3,$$

$$\left\| b^1_{m(i)} - \tilde{h}_1(x) \right\|_1 < 2\varepsilon_3.$$

Thus,

$$\begin{aligned} \left\| \tilde{h}_1(x) - \tilde{h}_2(x) \right\|_1 &\leq \left\| \tilde{h}_1(x) - b\frac{1}{m} \right\|_1 + \left\| b\frac{1}{m} - \tilde{h}_2(x) \right\|_1 \\ &\leq \left\| \tilde{h}_1(x) - b\frac{1}{m} \right\|_1 + \varepsilon_3 + \sup_{q^i(x)\neq 0} \left\| b\frac{1}{m} - b^1_{m(i)} \right\|_1 \\ &\leq 2\varepsilon_3 + \varepsilon_3 + 4\varepsilon_3 = 7\varepsilon_3. \end{aligned}$$

With a technique analogous to that adopted in the first step, one shows by induction that if

$$\left\| \tilde{h}_j(x) - \tilde{h}_{j+1}(x) \right\|_1 \leq 7\varepsilon_{j+2} \quad \text{on} \left(\bigcup_{l \leq j} A_l \right) \setminus \overline{A}_{j+1} \text{ for } j = 2, \dots, n \tag{@}$$

is true, then there exists $\tilde{h}_{n+1}$ such that (@) holds for $j = n + 1$, and that again from the definition of R_n, for every $x \in \bigcup_{j=1}^n A_j \setminus \overline{A}_{n+1}$ there are a $\bar{x} \in \bigcup_{j=1}^n A_j \setminus \overline{A}_{n+1}$ and a $b\frac{n}{m} \in R_n$ such that

$$\theta_n(\bar{x}) = b\frac{n}{m},$$

$$d(x, \bar{x}) < \delta_{\varepsilon_3},$$

$$\left\| b\frac{n}{m} - \tilde{h}_n(\bar{x}) \right\|_1 < \varepsilon_{n+2},$$

$$\left\| b\frac{n}{m} - \tilde{h}_n(x) \right\|_1 < 2\varepsilon_{n+2}.$$

Thus, following the same reasoning used in the case $n = 1$, we have that

$$\left\| \tilde{h}_n(x) - \tilde{h}_{n+1}(x) \right\|_1 \leq \left\| \tilde{h}_n(x) - b\frac{n}{m} \right\|_1 + \left\| b\frac{n}{m} - \tilde{h}_{n+1}(x) \right\|_1 \leq 7\varepsilon_{n+2}.$$

Define a function $\tilde{h}: X \to X$ by setting, for every $x \in A_n$,

$$\tilde{h}(x) = \lim_{m \geq n} \tilde{h}_m(x),$$

and $\tilde{h}(x) = h(x)$ for every $x \in A$. Because the image of each $\tilde{h}_m$ is contained in $R(f)$ and $R(f)$ is closed, then it is $\tilde{h}(X) \subset R(f)$ also.

From the relation,

$$\left\| \tilde{h}_p(x) - \tilde{h}_q(x) \right\|_1 \leq 7 \sum_{j=p}^{q} \varepsilon_{j+2}, \qquad p \leq q,$$

$$x \in \bigcup_{h=1}^{p} A_h \setminus \overline{A}_{p+1},$$

it is easy to verify that $\tilde{h}$ is continuous on $X \setminus A$.

To prove the continuity on A one shows that if $a \in A$, and $x \in X \setminus A$, with x sufficiently close to a, by (a_4) of Theorem 2.2, it follows that

$$\left\| \tilde{h}_n(x) - \tilde{h}(a) \right\|_1 \leq \frac{\varepsilon}{2}.$$

Because of the relation,

$$\left\| \tilde{h}(x) - \tilde{h}_n(x) \right\|_1 < \sum_{j=n}^{\infty} \varepsilon_j \leq \varepsilon_n < \frac{\varepsilon}{2}, \tag{3.17}$$

for n sufficiently large, for every $x \in \bigcup_{l=1}^{n} A_l \setminus \overline{A}_{n+1}$ we have

$$\left\| \tilde{h}(x) - \tilde{h}(a) \right\|_1 < \varepsilon \quad \text{for every } x \in X \text{ with } x \text{ sufficiently close to } a.$$

It remains to show that $\tilde{h}(X)$ is totally bounded.

Fix $\varepsilon > 0$; because $\tilde{h}$ is continuous, and $\overline{h(A)}$ is compact, there exists $\delta > 0$ such that $\tilde{h}(A + \delta B_1) \subset \tilde{h}(A) + (\varepsilon/2)B_1$. Because $h(A)$ is totally bounded, then $\tilde{h}(A + \delta B_1)$ can be covered by a finite number of balls of radius ε. Choose k so that $\{A_j \colon j = 1, \dots, k\}$ covers $X \setminus [A + \delta B_1]$ while A_{k+1} has empty intersection with it. Because each $\tilde{h}_j(\bigcup_{l=1}^{k} A_l)$, $j \geq k$ is totally bounded, and (3.17) holds, we have that whenever j satisfies $\varepsilon_j < \varepsilon/2$, an $\varepsilon/2$-net of $\tilde{h}_j(\bigcup_{l=1}^{k} A_l)$ is also an ε-net of $\tilde{h}(\bigcup_{l=1}^{k} A_l)$.

Hence we have found a finite ε-net for the set $\tilde{h}(X)$. ∎

However, as a consequence of Theorem 3.1, the set $R(f)$ has a relatively compact retract property. Another consequence of Theorem 3.1, is the following compact fixed point property.

COROLLARY 3.2. *Let F: $R(f) \to R(f)$ be a continuous function with $F(R(f))$ totally bounded. Then F has a fixed point in $R(f)$.*

ACKNOWLEDGMENTS

I thank Professor Anna Martellotti for the help that she gave me. She had the patience to read several versions of the proof of Theorem 2.2, a proof that seemed not to have an end, giving me a great deal of advice. But the most important thing is that she never lost confidence in my idea of the proof, and this was fundamental for gaining the final result.

REFERENCES

1. A. Bressan, A. Cellina, and A. Fryszkowski, A class of absolute retracts in spaces of integrable functions, *Proc. Amer. Math. Soc.* **112** (1991), 413–418.
2. A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69–86.
3. A. Cellina, G. Colombo, and A. Fonda, Approximate selection and fixed points for upper semicontinuous maps with decomposable values, *Proc. Amer. Math. Soc.* **98** (1986), 1–4.
4. J. Dugundji, "Topology," Allyn & Bacon, Needam Height, MA, 1966.
5. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," Cambridge Univ. Press, Cambridge, U.K., 1934.
6. L. Faina, Extensions of compact continuous maps into decomposable sets, *Rend. Sem. Mat. Univ. Padova* **81** (1991), 27–33.
7. L. Faina, Topological properties of equimeasurable functions, *Atti Sem. Mat. Fis. Univ. Modena*, to appear.
8. B. Kawoll, "Rearrangements," No. 1150, Lecture Notes in Mathematics, Springer-Verlag, Berlin/New York, 1980.
9. A. Liapounov, Sur les fonctions-vecteurs complément additives, *Izv. Akad. Nauk. SSSR Ser. Mat.* **4** (1940), 465–478.
10. F. Riesz, Sur une inègalite intégrale, *J. London Math. Soc.* **5** (1930), 162–168.