Calculation of two-loop $\beta$-function for general $N=1$ supersymmetric Yang–Mills theory with the higher covariant derivative regularization

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1. Introduction

It is well known that most quantum field theory models are divergent in the ultraviolet region. In order to deal with the divergent expressions, it is necessary to regularize a theory. Although physical results do not depend on regularization, a proper choice of the regularization can considerably simplify calculations or reveal some features of quantum corrections. Most calculations in the quantum field theory where made with the dimensional regularization [1]. However, the dimensional regularization is not convenient for calculations in supersymmetric theories, because it breaks the supersymmetry. That is why in supersymmetric theories one usually uses its modification, called the dimensional reduction [2]. There are a lot of calculations, made in supersymmetric theories with the dimensional reduction, see e.g. [3]. However, it is well known that the dimensional reduction is not self-consistent [4]. Ways, allowing to avoid such problems, are discussed in the literature [5]. Other regularizations are sometimes applied for calculations in supersymmetric theories. For example, in Ref. [6] two-loop $\beta$-function of the $N=1$ supersymmetric Yang–Mills theory was calculated with the differential renormalization [7].

A self-consistent regularization, which does not break the supersymmetry, is the higher covariant derivative regularization [8], which was generalized to the supersymmetric case in Ref. [9] (another variant was proposed in Ref. [10]). However, using this regularization is rather technically complicated. The first calculation of quantum corrections for the (non-supersymmetric) Yang–Mills theory was made in Ref. [11]. Taking into account corrections, made in subsequent papers [12], the result for the $\beta$-function appeared to be the same as the well-known result, obtained with the dimensional regularization [13]. In principle, it is possible to prove that in the one-loop approximation calculations with the higher covariant derivative regularization always agree with the results of calculations with the dimensional regularization [14]. Some calculations in the one-loop and two-loop approximations were made for various theories [15, 16] with a variant of the higher covariant derivative regularization, proposed in [17]. The structure of the corresponding integrals was discussed in Ref. [16].

Application of the higher covariant derivative regularization to calculation of quantum corrections in the $N=1$ supersymmetric electrodynamics in two and three loops [18,19] reveals an interesting feature of quantum corrections: all integrals, defining the $\beta$-function appear to be integrals of total derivatives and can be easily calculated. This makes possible analytical multiloop calculations with the higher covariant derivative regularization in supersymmetric theories and allows to explain the origin of the NSVZ $\beta$-function, which relates the $\beta$-function in $n$-th loop with the $\beta$-function and the anomalous dimensions in the previous loops. Due to this, application of this regularization is sometimes very convenient in the supersymmetric case. The fact that the integrals, appearing with the higher covariant derivative regularization, in the limit of zero external momentum become integrals of total derivatives, seems to be a general feature of all regularizations.

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supersymmetric theories. Nevertheless, with the higher derivative regularization even the two-loop $\beta$-function has not yet been calculated for a general $N = 1$ supersymmetric Yang–Mills theory. This is made in this Letter. Note that in order to do this calculation, it is necessary to introduce higher covariant derivative terms not only for the gauge field, but also for the matter superfields.

The Letter is organized as follows:

In Section 2 we introduce the notation and recall basic information about the higher covariant derivative regularization. The $\beta$-function for the considered theory is calculated in Section 3. The result is briefly discussed in Section 4.

2. $N = 1$ supersymmetric Yang–Mills theory and the higher covariant derivative regularization

In this Letter we calculate $\beta$-function for a general renormalizable $N = 1$ supersymmetric Yang–Mills theory. In the massless case this theory is described by the action

$$S = \frac{1}{2e^2} \text{Re} \text{tr} \int d^4x d^2\theta \ W_a C^{ab} W_b + \frac{1}{4} \int d^4x d^4\theta \ (\phi^*)^i \left(e^{2V}\right)_{ij} \phi_j + \left(\frac{1}{6} \int d^4x d^2\theta \ \lambda^{ijk} \phi_i \phi_j \phi_k + \text{h.c.}\right),$$

(1)

where $\phi$ are chiral matter superfields in the representation $R$, which is in general reducible. $V$ is a real scalar gauge superfield. The superfield $W_A$ is a supersymmetric gauge field stress tensor, which is defined by

$$W_a = \frac{1}{8} \tilde{D}^2 (e^{-2V} D_a e^{2V}).$$

(2)

In our notation $D_a$ and $\tilde{D}_a$ are the right and left supersymmetric covariant derivatives respectively, $V = e^{V} T^A$, and the generators of the fundamental representation are normalized by the condition

$$\text{tr}(\epsilon^A \epsilon^B) = \frac{1}{2} \delta^{AB}.$$  

(3)

Action (1) should be invariant under the gauge transformations

$$\phi \rightarrow e^{iA} \phi, \quad e^{2V} \rightarrow e^{iA^+} e^{2V} e^{-iA},$$

(4)

where $A$ is an arbitrary chiral superfield. As a consequence, the coefficient $\lambda^{ijk}$ should satisfy the condition

$$(T^A)^m_{mj} \lambda^{mk} + (T^A)^m_{jm} \lambda^{im} + (T^A)^m_{km} \lambda^{jm} = 0.$$  

(5)

For calculation of quantum corrections it is convenient to use the background field method. In the supersymmetric case it can be formulated as follows [20]: Let us make the substitution

$$e^{2V} \rightarrow e^{2V'} \equiv e^{Q^+} e^{2V} e^{Q^-},$$

(6)

in action (1), where $Q$ is a background superfield. Then the theory is invariant under the background gauge transformations

$$\phi \rightarrow e^{iK} \phi, \quad V \rightarrow e^{iK} V e^{-iK}, \quad e^{Q^+} \rightarrow e^{iK} e^{Q^+} e^{-iK},$$

(7)

where $K$ is an arbitrary real superfield, and $A$ is a background-chiral superfield. This invariance allows to set $Q = Q^+ = V$.

It is convenient to choose a regularization and gauge fixing so that invariance (7) is unbroken. First, we fix a gauge by adding

$$S_{gf} = -\frac{1}{32e^2} \text{tr} \int d^4x d^2\theta \ (V D^2 \tilde{D}^2 V + \tilde{V} D^2 D^2 V)$$

(8)

to the action. The corresponding Faddeev–Popov and Nielsen–Kallosh ghost Lagrangians are constructed by the standard way.

For regularization we add the terms

$$S_\Lambda = \frac{1}{2e^2} \text{Re} \text{tr} \int d^4x d^2\theta \ \left(\frac{D^2_{\Phi^+}}{\Lambda^{2n}} + V + \frac{1}{8} \int d^4x d^2\theta \left[(\phi^*)^i \left(e^{Q^+} e^{2V} \frac{D^2_{\Phi^+}}{\Lambda^{2m}} e^{Q^-}\right)_{ij} \phi_j + (\phi^*)^i \left(e^{Q^+} e^{2V} \frac{D^2_{\Phi^+}}{\Lambda^{2m}} e^{Q^-}\right)_{ij} \phi_j\right]ight) ,$$

(9)

where $D_{\Phi^+}$ is the background covariant derivative and we assume that $m < n.$ (Because the considered theory contains a nontrivial superpotential, it is also necessary to introduce the higher covariant derivative term for the matter superfields.)

The regularized theory is evidently invariant under the background gauge transformations. The regularization, described above, is rather simple, but breaks the BRST-invariance of the action. That is why it is necessary to use a special subtraction scheme, which restore the Slavnov–Taylor identities in each order of the perturbation theory [21]. For the supersymmetric case such a scheme was constructed in Ref. [22].

It is well known [23] that the higher covariant derivative term does not remove divergences in the one-loop approximation. In order to cancel the remaining one-loop divergences, it is necessary to introduce into the generating functional the Pauli–Villars determinants

$$\prod_i \left(\int D\Phi^+ D\phi \text{e}^{iS_\Lambda}\right)^{-\epsilon_i},$$

(10)

$^1$ Other choices of the higher derivative terms are also possible.
where $S_I$ is the action for the Pauli–Villars fields,²

$$S_I = \frac{1}{8} \int d^4x d^4\theta \left[ (\phi_1^i)^\dagger \left[ e^{\alpha^2} e^{2\nu} \left( 1 + \frac{(D_\nu^2)^m}{A^{2m}} \right) e^{\alpha^2} \right] (\phi_1)_j + (\phi_1^i)^\dagger \left[ e^{\alpha^2} \left( 1 + \frac{(D_\nu^2)^m}{A^{2m}} \right) e^{2\nu} e^{\alpha^2} \right] (\phi_1)_j \right] + \left( \frac{1}{4} \int d^4x d^2\theta \left[ M_I^{ij}(\phi_1)_i (\phi_1)_j + \text{h.c.} \right] \right).$$

(11)

The masses of the Pauli–Villars fields are proportional to the parameter $A$:

$$M_I^{ij} = a_I^{ij} A.$$  

(12)

This means that $A$ is the only dimensionful parameter of the regularized theory. We assume that the mass term does not break the gauge invariance. Also we will choose the masses so that

$$M_I^{ij} = \delta_I^{ij}.$$  

(13)

The generating functional for connected Green functions and the effective action are defined by the standard way.

In this Letter we will calculate the $\beta$-function. We use the following notation. Terms in the effective action, corresponding to the renormalized two-point Green function of the gauge superfield, are written as

$$I^{(2)}_V = -\frac{1}{8\pi} \text{tr} \int \frac{d^4p}{(2\pi)^4} d^4\theta \mathbf{V}(-p) \partial^2 \Sigma_{1/2} \mathbf{V}(p) d^{-1}(\alpha, \lambda, \mu/p),$$

(15)

where $\alpha$ is a renormalized coupling constant. We calculate

$$\frac{d}{d\ln A} \left[ d^{-1}(\alpha_0, \lambda_0, A/p) - \alpha_0^{-1} \right]_{p=0} = -\frac{d\alpha_0^{-1}}{d\ln A} = \frac{\beta(\alpha_0)}{\alpha_0^2}.$$  

(16)

The anomalous dimension is defined similarly. First we consider the two-point Green function for the matter superfield in the massless limit:

$$I^{(2)}_\phi = \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} d^4\theta \left[ (\phi_1^i)^\dagger (-p, \theta) \phi_j(p, \theta) (Z_G)_{ij}(\alpha, \lambda, \mu/p),$$

(17)

where $Z$ denotes the renormalization constant for the matter superfield. Then the anomalous dimensions is defined by

$$\gamma_{ij}(\alpha_0, \lambda, \mu) = -\frac{\partial}{\partial \ln A} \left[ \ln Z(\alpha_0, \lambda, \mu) \right]_{ij}.$$  

(18)

3. Two-loop $\beta$-function

After calculation of the supergraphs, we have obtained the following result for the two-loop $\beta$-function:

$$\beta_2(\alpha) = \frac{3\alpha^2}{2\pi} C_2 + \alpha^3 T(R) I_0 + \alpha^3 C_2^2 I_1 + \frac{\alpha^3}{r} C(R) C(R)_{ij} I_2 + \alpha^3 T(R) C_2 I_3 + \alpha^2 C(R) \left\{ \sum_{ijkl} \lambda_{ijkl} A^{ijkl} \right\} I_4,$$  

(19)

where the following notation is used:

$$\text{tr} \left[ T^A T^B \right] = T(R) \delta^{AB}, \quad (T^A)_{ij} (T^A)_{kl} = C(R)_{ij}, \quad f^{ACD} f^{BCE} = C_2 \delta^{AB}, \quad r = \delta_{AA}.$$  

(20)

(Note that $T(R) = C(R)_{ij}/r$.) Here

$$I_i = I_i(0) - \sum_l c_l I_l(M_l) \quad \text{for } i = 0, 2, 3,$$  

(21)

and the integrals $I_0(M), I_1, I_2(M), I_3(M)$ and $I_4$ are given by

$$I_0(M) = 4\pi \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4\theta}{4!} \frac{d}{d\ln A} \left[ \ln(\theta^2(1 + q^{2m}/A^{2m}))^2 + M^2 \right] + \frac{M^2}{2q^2(1 + q^{2m}/A^{2m})^2 + M^2} \left[ \frac{2m q^{2m}/A^{2m} q^2(1 + q^{2m}/A^{2m})}{q^2(1 + q^{2m}/A^{2m})^2 + M^2} \right].$$  

(22)

² Note that this action differs from the one, used in [18], because here the quotient of the coefficients in the kinetic term and in the mass term does not contain the factor $Z$. Using terminology of Ref. [24], one can say that here we calculate the canonical coupling $\alpha$, while in Ref. [18] we calculated the holomorphic coupling $\alpha$. Certainly, after the renormalization the effective action does not depend on the definitions. However, the definitions used here are much more convenient.
\[I_1 = 96\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{d\ln A} \frac{d}{k^2} \frac{d}{dk^2} \left[ \frac{1}{q^2(q+k)^2(1+q^2/\Lambda^2)(1+(q+k)^2/\Lambda^2)} \right],\]

\[I_2(M) = -16\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{d\ln A} \frac{d}{q^2} \left( (q+k)^2(1+(q+k)^2/\Lambda^2)^2 + M^2 \right) \times \frac{1}{k^2(1+q^2/\Lambda^2)} \left[ \frac{q^4(2+(q+k)^2/\Lambda^2)^2 + q^2(1+q^2/\Lambda^2)^2}{q^2(1+q^2/\Lambda^2)^2 + M^2} \right.\]

\[= \frac{2q^2M^2(2+(q+k)^2/\Lambda^2 + q^2/\Lambda^2)^2}{(q^2(1+q^2/\Lambda^2)^2 + M^2)^2},\]

\[I_3(M) = 4\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{d\ln A} \frac{d}{q^2} \left[ \frac{1}{q^2(k^2(1+k^2/\Lambda^2)^2 + M^2)^2} \right.\]

\[= \frac{1}{(1+q^2/\Lambda^2)^2 + (1+k^2/\Lambda^2)^2},\]

\[I_4 = 64\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d}{q^2} \frac{d}{q^2} \frac{d}{q^2} \frac{d}{q^2} \left[ \frac{1}{k^2q^2(1+k^2/\Lambda^2)^2} \right.\]

\[= \frac{1}{(1+q^2/\Lambda^2)^2 + (1+k^2/\Lambda^2)^2},\]

It is easy to see that all these integrals are integrals of total derivatives, due to the identity

\[\int d^4q \frac{1}{(2\pi)^4} \frac{d}{q^2} f(q^2) = \frac{1}{16\pi^2} f(q^2 = \infty) - f(q^2 = 0),\]

which can be easily proved in the four-dimensional spherical coordinates. Using this identity we find

\[I_0 = \frac{1}{4\pi} \frac{d}{d\ln A} \left( \sum_l c_l \ln M_l^2 \right) = \frac{1}{2\pi},\]

\[I_1 = -6 \int \frac{d^4q}{(2\pi)^4} \frac{d}{q^2} \left[ \frac{1}{q^2(1+q^2/\Lambda^2)^2} \right] = -\frac{3}{4\pi^2},\]

\[I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{d}{k^2} \left[ \frac{1}{k^2(1+k^2/\Lambda^2)^2} \right] = \frac{1}{2\pi^2},\]

\[I_3 = \int \frac{d^4q}{(2\pi)^4} \frac{d}{q^2} \left[ \sum_l c_l \frac{2(1+q^2/\Lambda^2)^2}{(q^2(1+q^2/\Lambda^2)^2 + M_l^2)^2} \right] = \frac{1}{4\pi^2},\]

\[I_4 = -\int \frac{d^4k}{(2\pi)^4} \frac{d}{k^2} \left[ \frac{4}{k^2(1+k^2/\Lambda^2)^2} \right] = -\frac{1}{2\pi^2}.\]

Note that the Pauli–Villars fields nontrivially contributes only to integrals \(I_0\) and \(I_1\), where they are very important. For example, in the two-loop integral \(I_3\) the Pauli–Villars contribution cancels the one-loop subdivergence, produced by the matter superfields.

Thus, in the two-loop approximation

\[\beta(\alpha) = -\frac{\alpha^2}{2\pi} (3C_2 - TR) + \frac{\alpha^3}{(2\pi)^2} \left( -3C_2^2 + TRC_2 + \frac{2}{r} RC(R)_i \frac{C(R)_j}{8\pi^3 r} \right) - \frac{\alpha^2 C(R)_i \delta_{ij} \delta_{kl} \beta_{dil}}{8\pi^3 r} + \ldots.\]
Taking into account that the one-loop anomalous dimension is given by
\[
\gamma_i^j(\alpha) = -\frac{\alpha}{\pi} C(R)^{ij} + \frac{1}{4\pi^2} \lambda^{ikl} \lambda_{jkl} + \cdots,
\]
we see that our result agrees with the exact NSVZ $\beta$-function \[25\]
\[
\beta(\alpha) = -\frac{\alpha^2}{2\pi} \left[ 3C_2 - T(R) + \frac{C(R)^{ij} \gamma_i^j(\alpha)}{\pi} \right].
\]
Up to notation, this result is in agreement with the results of calculations made with the dimensional reduction, see e.g. \[3\].

4. Conclusion

In this Letter we demonstrate, how the two-loop $\beta$-function in $N = 1$ supersymmetric theories can be calculated with the higher covariant derivative regularization. The most interesting feature of this calculation is the factorization of rather complicated integrals into integrals of total derivatives. Partially this fact can be explained substituting solutions of Slavnov-Taylor identities into the Schwinger-Dyson equations. However, a complete proof of this fact has not yet been done. Its origin is also so far unclear. Possibly, this feature appears due to using of the background field method \[26\]. Factorization of integrals, obtained with the higher covariant derivative regularization, into integrals of total derivatives can allow to do a simple derivation of the Novikov, Shifman, Vainshtein, and Zakharov $\beta$-function, which relates $n$-loop contribution to the $\beta$-function with the $\beta$-function and the anomalous dimension in previous loops. In this Letter we have shown how this can be done at the two-loop level.

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