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# Equations of motion of test particles for solving the spin-dependent Boltzmann–Vlasov equation

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## ABSTRACT

A consistent derivation of the equations of motion (EOMs) of test particles for solving the spin-dependent Boltzmann–Vlasov equation is presented. The resulting EOMs in phase space are similar to the canonical equations in Hamiltonian dynamics, and the EOM of spin is the same as that in the Heisenberg picture of quantum mechanics. Considering further the quantum nature of spin and choosing the direction of total angular momentum in heavy-ion reactions as a reference of measuring nucleon spin, the EOMs of spin-up and spin-down nucleons are given separately. The key elements affecting the spin dynamics in heavy-ion collisions are identified. The resulting EOMs provide a solid foundation for using the test-particle approach in studying spin dynamics in heavy-ion collisions at intermediate energies. Future comparisons of model simulations with experimental data will help to constrain the poorly known in-medium nucleon spin–orbit coupling relevant for understanding properties of rare isotopes and their astrophysical impacts.

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## 1. Introduction

The importance of nucleon spin degree of freedom was first recognized more than 50 years ago when Mayer and Jensen introduced the spin–orbit interaction and used it to explain successfully the magic numbers and shell structure of nuclei [1,2]. Subsequently, the nuclear spin–orbit interaction was found responsible for many interesting phenomena in nuclear structure [3–8]. It also affects some features of nuclear reactions, such as the fusion threshold [9], the polarization measured in terms of the analyzing power in pick-up or removal reactions [10–13], and the spin dependence of nucleon collective flow [14,15] in heavy-ion collisions (HICs). However, the role of nucleon spin is much less known in nuclear reactions than structures. In HICs at intermediate energies, a central issue is the density and isospin dependence of the spin–orbit coupling in neutron-rich medium, see, e.g., Ref. [16] for a recent review. It is also interesting to mention that the study of spin-dependent structure functions of nucleons and nuclei has been at the forefronts of nuclear and particle physics [17]. This study will be boosted by future experiments at the proposed electron–ion collider using polarized beams [18]. In this work,

we derive for the first time equations of motion (EOMs) of nucleon test particles [19] for solving the spin-dependent Boltzmann (Vlasov)–Uehling–Uhlenbeck (BUU or VUU) equation. These EOMs provide the physics foundation for simulating spin transport for not only nucleons in heavy-ion reactions but also electrons for understanding many interesting phenomena, such as the spin wave [20–23], the spin-Hall effect [24–26], etc.

Considering the spin degree of freedom, the Wigner function in phase space becomes a  $2 \times 2$  matrix [27]. Its time evolution is governed by the Boltzmann–Vlasov (BV) equation obtained by a Wigner transformation of the Liouville equation for the density matrix [20,28,29]

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t} + \frac{i}{\hbar} [\hat{\varepsilon}, \hat{f}] + \frac{1}{2} \left( \frac{\partial \hat{\varepsilon}}{\partial \vec{p}} \cdot \frac{\partial \hat{f}}{\partial \vec{r}} + \frac{\partial \hat{f}}{\partial \vec{r}} \cdot \frac{\partial \hat{\varepsilon}}{\partial \vec{p}} \right) \\ - \frac{1}{2} \left( \frac{\partial \hat{\varepsilon}}{\partial \vec{r}} \cdot \frac{\partial \hat{f}}{\partial \vec{p}} + \frac{\partial \hat{f}}{\partial \vec{p}} \cdot \frac{\partial \hat{\varepsilon}}{\partial \vec{r}} \right) = 0, \end{aligned} \quad (1)$$

where  $\hat{\varepsilon}$  and  $\hat{f}$  are from the Wigner transformation of the energy and phase-space density matrix, respectively, and they can be decomposed into their scalar and vector parts, i.e.,

$$\hat{\varepsilon}(\vec{r}, \vec{p}) = \varepsilon(\vec{r}, \vec{p})\hat{1} + \vec{h}(\vec{r}, \vec{p}) \cdot \vec{\sigma}, \quad (2)$$

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$$\hat{f}(\vec{r}, \vec{p}) = f_0(\vec{r}, \vec{p})\hat{1} + \vec{g}(\vec{r}, \vec{p}) \cdot \vec{\sigma}, \quad (3)$$

where  $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  and  $\hat{1}$  are respectively the Pauli matrices and the  $2 \times 2$  unit matrix,  $\varepsilon$  and  $f_0$  are the scalar part of the effective single-particle energy  $\hat{\varepsilon}$  and phase-space density  $\hat{f}$ , respectively, and  $\vec{h}$  and  $\vec{g}$  are the corresponding vector distributions. Adding the Uehling–Uhlenbeck collision term, the resulting spin-dependent BUU equation can be used to describe the spin-dependent dynamics in various systems.

While the spin-dependent BUU equation can be taken as the starting point of investigating spin dynamics in HICs, a consistent derivation of the EOMs of nucleon test particles for simulations is still lacking. The situation is quite different for electron spin transport in solid state physics. To our best knowledge, the treatment of electron spin transport relevant to the present study mostly follows two approaches. One way (Method I) is to start from a model Hamiltonian and use the canonical EOMs for the time evolution of the electron's coordinate and momentum, while the time evolution of the electron's spin is given by its commutation relation with the model Hamiltonian as in the Heisenberg picture of quantum mechanics [24,30,31]. Moreover, an adiabatic approximation is often used so that the time evolution of spin can be solved first. Inserting the solution for spin evolution into the EOMs for coordinate and momentum then leads to the Berry curvature terms [32–34]. Another method (Method II) frequently used is to linearize the spin-dependent BUU equation for spin-up and spin-down particles separately through the relaxation time approximation [25,35–37]. In nuclear physics, EOMs of nucleon test particles should be derived consistently from the BUU transport equation used to model HICs. It is well known that the spin-independent BV equation can be solved numerically by using the test-particle method [38,39]. In particular, it was shown that the EOMs of test particles are identical to the canonical EOMs if only the lowest order term in expanding the Wigner function is considered (see Ref. [19] and comments in Ref. [40]). Applying the test-particle approach to solving the spin-dependent BUU equation for the first time, we found that the EOMs of nucleon test particles are similar to those for electrons obtained within Method I described above, albeit with different forms of interactions.

## 2. Decomposition of the spin-dependent phase-space distribution function and its evolution

To avoid confusion, we begin by first commenting on the two approaches of deriving the BV equation with different definitions of the spinor Wigner distribution function often used in the literature. Equation (1) was derived by means of the density matrix method and taking the semiclassical limit as outlined by Smith and Jensen [20]. It can be separated into two equations governing the scalar and vector distributions, respectively,

$$\frac{\partial f_0}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_0}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_0}{\partial \vec{p}} + \frac{\partial \vec{h}}{\partial \vec{p}} \cdot \frac{\partial \vec{g}}{\partial \vec{r}} - \frac{\partial \vec{h}}{\partial \vec{r}} \cdot \frac{\partial \vec{g}}{\partial \vec{p}} = 0, \quad (4)$$

$$\begin{aligned} \frac{\partial \vec{g}}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial \vec{g}}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial \vec{g}}{\partial \vec{p}} + \frac{\partial f_0}{\partial \vec{r}} \cdot \frac{\partial \vec{h}}{\partial \vec{p}} \\ - \frac{\partial f_0}{\partial \vec{p}} \cdot \frac{\partial \vec{h}}{\partial \vec{r}} + \frac{2\vec{g} \times \vec{h}}{\hbar} = 0. \end{aligned} \quad (5)$$

Another way to get the spin-dependent BV equation is to start from the time-dependent Hartree–Fock equations for the one-body density matrix with spin degree of freedom and rewrite the equations with the help of the Wigner transformation, see, e.g., Refs. [28,29]. In this way, one will get four coupled equations which describe the time evolution of the four-component Wigner

phase-space densities from the  $2 \times 2$  density matrix with spin. The definition of the Wigner function of particles with spin-1/2 was suggested in Refs. [27,41] as

$$f_{\sigma, \sigma'}(\vec{r}, \vec{p}, t) = \int d^3s e^{-i\vec{p} \cdot \vec{s} / \hbar} \psi_{\sigma'}^*(\vec{r} - \frac{\vec{s}}{2}, t) \psi_{\sigma}(\vec{r} + \frac{\vec{s}}{2}, t), \quad (6)$$

$$f(\vec{r}, \vec{p}, t, 0) = f_{1,1}(\vec{r}, \vec{p}, t) + f_{-1,-1}(\vec{r}, \vec{p}, t), \quad (7)$$

$$\tau(\vec{r}, \vec{p}, t, x) = f_{-1,1}(\vec{r}, \vec{p}, t) + f_{1,-1}(\vec{r}, \vec{p}, t), \quad (8)$$

$$\tau(\vec{r}, \vec{p}, t, y) = -i[f_{-1,1}(\vec{r}, \vec{p}, t) - f_{1,-1}(\vec{r}, \vec{p}, t)], \quad (9)$$

$$\tau(\vec{r}, \vec{p}, t, z) = f_{1,1}(\vec{r}, \vec{p}, t) - f_{-1,-1}(\vec{r}, \vec{p}, t), \quad (10)$$

with  $\sigma(\sigma') = 1$  for spin up and  $-1$  for spin down. The above definitions are convenient in treating the expectation values of the spin components. Equation (6) gives the matrix components of the Wigner function with spin degree of freedom.  $f(\vec{r}, \vec{p}, t, 0)$  is the ordinary Wigner phase-space density irrespective of the particle spin, while  $\tau(\vec{r}, \vec{p}, t, x)$ ,  $\tau(\vec{r}, \vec{p}, t, y)$ , and  $\tau(\vec{r}, \vec{p}, t, z)$ , representing the three components of the spin Wigner density  $\vec{\tau}(\vec{r}, \vec{p}, t)$ , are the probabilities of the spin projection on the  $x$ ,  $y$ , and  $z$  directions, respectively. With the above definitions, the Wigner density  $f(\vec{r}, \vec{p}, t)$  in Eq. (7) and the spin Wigner density  $\vec{\tau}(\vec{r}, \vec{p}, t)$  (Eqs. (8)–(10)) can be expressed in terms of the  $f_0(\vec{r}, \vec{p}, t)$  and  $\vec{g}(\vec{r}, \vec{p}, t)$  in Eq. (3) as [41]

$$f(\vec{r}, \vec{p}, t) = 2f_0(\vec{r}, \vec{p}, t), \quad (11)$$

$$\vec{\tau}(\vec{r}, \vec{p}, t) = 2\vec{g}(\vec{r}, \vec{p}, t). \quad (12)$$

In this way, the two approaches using two different definitions of the spinor Wigner function lead to exactly the same spin-dependent BV equation.

## 3. Single-particle energy with spin–orbit interaction

While our derivation is general, to be specific, for the spin-dependent part of the single-particle Hamiltonian in Eq. (2) we take the Skyrme-type effective two-body interaction including the spin–orbit coupling [42,43]

$$\begin{aligned} \hat{h}_q^{so} = & -\frac{1}{2}W_0\nabla \cdot (\vec{J} + \vec{J}_q) + \vec{\sigma} \cdot [-\frac{1}{2}W_0\nabla \times (\vec{J} + \vec{J}_q)] \\ & + \frac{1}{4i}W_0[(\nabla \times \vec{\sigma}) \cdot \nabla(\rho + \rho_q) + \nabla(\rho + \rho_q) \cdot (\nabla \times \vec{\sigma})] \\ & - \frac{1}{4i}W_0[\nabla \cdot (\nabla \times (\vec{s} + \vec{s}_q)) + (\nabla \times (\vec{s} + \vec{s}_q)) \cdot \nabla], \end{aligned} \quad (13)$$

where  $q = n$  or  $p$  is the isospin index, and  $\rho$ ,  $\vec{s}$ ,  $\vec{J}$ , and  $\vec{J}$  are the number, spin, momentum, and spin-current densities, respectively. According to the definition of the Wigner function in Eq. (6), these densities can be directly expressed as [42,43]

$$\rho(\vec{r}) = \int d^3p f(\vec{r}, \vec{p}), \quad (14)$$

$$\vec{s}(\vec{r}) = \int d^3p p \vec{\tau}(\vec{r}, \vec{p}), \quad (15)$$

$$\vec{J}(\vec{r}) = \int d^3p p \frac{\vec{p}}{\hbar} f(\vec{r}, \vec{p}), \quad (16)$$

$$\vec{J}(\vec{r}) = \int d^3p p \frac{\vec{p}}{\hbar} \times \vec{\tau}(\vec{r}, \vec{p}). \quad (17)$$

After a Wigner transformation, the Eq. (13) can be readily expressed in terms of the above densities as

$$h_q^{so}(\vec{r}, \vec{p}) = h_1 + h_4 + (\vec{h}_2 + \vec{h}_3) \cdot \vec{\sigma} \quad (18)$$

with  $h_1, \vec{h}_2, \vec{h}_3$ , and  $h_4$  given by

$$h_1 = -\frac{W_0}{2} \nabla_{\vec{r}} \cdot [\vec{J}(\vec{r}) + \vec{J}_q(\vec{r})], \quad (19)$$

$$\vec{h}_2 = -\frac{W_0}{2} \nabla_{\vec{r}} \times [\vec{J}(\vec{r}) + \vec{J}_q(\vec{r})], \quad (20)$$

$$\vec{h}_3 = \frac{W_0}{2} \nabla_{\vec{r}} [\rho(\vec{r}) + \rho_q(\vec{r})] \times \vec{p}, \quad (21)$$

$$h_4 = -\frac{W_0}{2} \nabla_{\vec{r}} \times [\vec{s}(\vec{r}) + \vec{s}_q(\vec{r})] \cdot \vec{p}. \quad (22)$$

Comparing with Eq. (2), the effective single-particle energy  $\hat{\varepsilon}$  can be written as

$$\varepsilon_q(\vec{r}, \vec{p}) = \frac{p^2}{2m} + U_q + h_1 + h_4, \quad (23)$$

$$\vec{h}_q(\vec{r}, \vec{p}) = \vec{h}_2 + \vec{h}_3, \quad (24)$$

where  $U_q$  is the spin-independent mean-field potential. The nuclear tensor force can be implemented in a similar way if needed, once the scalar and the vector components are decomposed from the corresponding energy-density functional.

#### 4. Spin-dependent EOMs of test particles

We now derive the EOMs from the decoupled spin-dependent BV equation [Eqs. (4) and (5)] by using the test-particle method [19,38]. The vector part  $\vec{g}(\vec{r}, \vec{p})$  of the spinor Wigner function distribution in Eq. (3) can be represented by a real unit vector  $\vec{n}$  times a scalar function  $f_1(\vec{r}, \vec{p})$ , i.e.,

$$\vec{g}(\vec{r}, \vec{p}) = \vec{n} f_1(\vec{r}, \vec{p}). \quad (25)$$

Here we assume that  $\vec{n}$  is independent of  $\vec{r}$  and  $\vec{p}$ , which is valid if  $\vec{n}$  evolves much faster than the phase-space coordinates  $\vec{r}$  and  $\vec{p}$  or if  $\vec{n}$  is a global constant. Under this assumption and by substituting Eq. (25) into Eqs. (4) and (5), we obtain

$$\begin{aligned} \frac{\partial f_0}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_0}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_0}{\partial \vec{p}} + \left( \frac{\partial \vec{h}}{\partial \vec{p}} \cdot \vec{n} \right) \cdot \frac{\partial f_1}{\partial \vec{r}} \\ - \left( \frac{\partial \vec{h}}{\partial \vec{r}} \cdot \vec{n} \right) \cdot \frac{\partial f_1}{\partial \vec{p}} \approx 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial f_1}{\partial t} \vec{n} + \left( \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_1}{\partial \vec{r}} \right) \vec{n} - \left( \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_1}{\partial \vec{p}} \right) \vec{n} + \frac{\partial f_0}{\partial \vec{r}} \cdot \frac{\partial \vec{h}}{\partial \vec{p}} \\ - \frac{\partial f_0}{\partial \vec{p}} \cdot \frac{\partial \vec{h}}{\partial \vec{r}} + \left( \frac{2\vec{n} \times \vec{h}}{\hbar} + \frac{\partial \vec{n}}{\partial t} \right) f_1 \approx 0. \end{aligned} \quad (27)$$

Generally, the magnitude of the Poisson bracket  $\{f_0, \vec{h}\}$ , i.e.,  $(\partial f_0 / \partial \vec{r}) \cdot (\partial \vec{h} / \partial \vec{p}) - (\partial f_0 / \partial \vec{p}) \cdot (\partial \vec{h} / \partial \vec{r})$ , is much smaller than that of  $\hbar$  or  $\{f_0, \varepsilon\}$ . In this approximation and by separating components parallel and perpendicular to  $\vec{n}$ , Eq. (27) can be divided into two parts, i.e.,

$$\begin{aligned} \frac{\partial f_1}{\partial t} \vec{n} + \left( \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_1}{\partial \vec{r}} \right) \vec{n} - \left( \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_1}{\partial \vec{p}} \right) \vec{n} + \frac{\partial f_0}{\partial \vec{r}} \cdot \frac{\partial \vec{h}}{\partial \vec{p}} \\ - \frac{\partial f_0}{\partial \vec{p}} \cdot \frac{\partial \vec{h}}{\partial \vec{r}} = 0, \end{aligned} \quad (28)$$

$$\frac{\partial \vec{n}}{\partial t} \approx \frac{2\vec{h} \times \vec{n}}{\hbar}. \quad (29)$$

As  $\vec{n}$  is a unit vector, we have  $\vec{n} \cdot \vec{n} = 1$ . By taking the inner product of  $\vec{n}$  with Eq. (28) (or Eq. (27)) on both sides, one obtains

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_1}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_1}{\partial \vec{p}} + \frac{\partial f_0}{\partial \vec{r}} \cdot \left( \frac{\partial \vec{h}}{\partial \vec{p}} \cdot \vec{n} \right) \\ - \frac{\partial f_0}{\partial \vec{p}} \cdot \left( \frac{\partial \vec{h}}{\partial \vec{r}} \cdot \vec{n} \right) = 0. \end{aligned} \quad (30)$$

Adding and subtracting Eqs. (26) and (30), we get two equations for two types of particles with phase-space distribution functions  $f^\pm = f_0 \pm f_1$ , i.e.,

$$\frac{\partial f^+}{\partial t} + \left( \frac{\partial \varepsilon}{\partial \vec{p}} + \frac{\partial V_{hn}}{\partial \vec{p}} \right) \cdot \frac{\partial f^+}{\partial \vec{r}} - \left( \frac{\partial \varepsilon}{\partial \vec{r}} + \frac{\partial V_{hn}}{\partial \vec{r}} \right) \cdot \frac{\partial f^+}{\partial \vec{p}} = 0, \quad (31)$$

$$\frac{\partial f^-}{\partial t} + \left( \frac{\partial \varepsilon}{\partial \vec{p}} - \frac{\partial V_{hn}}{\partial \vec{p}} \right) \cdot \frac{\partial f^-}{\partial \vec{r}} - \left( \frac{\partial \varepsilon}{\partial \vec{r}} - \frac{\partial V_{hn}}{\partial \vec{r}} \right) \cdot \frac{\partial f^-}{\partial \vec{p}} = 0, \quad (32)$$

with  $V_{hn} \equiv \vec{h} \cdot \vec{n}$ . It can be understood from Eqs. (3) and (25) that  $f^+$  and  $f^-$  are the eigenfunctions of  $\hat{f}$ , representing the phase-space distributions of particles with their spin in  $+\vec{n}$  and  $-\vec{n}$  directions, respectively, i.e., spin-up and spin-down particles.

Following the test-particle method and using an auxiliary variable  $\vec{s}$ , the time evolution of the Wigner function  $f^\pm(\vec{r}, \vec{p})$  can be expressed as [19]

$$\begin{aligned} f^\pm(\vec{r}, \vec{p}, t) = \int \frac{d^3 r_0 d^3 p_0 d^3 s}{(2\pi\hbar)^3} \exp\{i\vec{s} \cdot [\vec{p} - \vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)] / \hbar\} \\ \times \delta[\vec{r} - \vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t)] f^\pm(\vec{r}_0, \vec{p}_0, t_0), \end{aligned} \quad (33)$$

where  $f^\pm(\vec{r}_0, \vec{p}_0, t_0)$  is the Wigner functions at time  $t_0$  with the initial conditions  $\vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t_0) = \vec{r}_0$  and  $\vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t_0) = \vec{p}_0$ . Our main task is now to find the new phase-space coordinates  $\vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t)$  and  $\vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)$  and to obtain the Wigner function at the next time step  $t = t_0 + \Delta t$  with a small increment  $\Delta t$ . By substituting Eq. (33) into Eqs. (31) and (32), we obtain

$$\begin{aligned} \left[ -\frac{\partial \vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \right] \cdot \frac{\partial f^\pm(\vec{r}, \vec{p}, t)}{\partial \vec{r}} \pm \frac{\partial V_{hn}}{\partial \vec{p}} \cdot \frac{\partial f^\pm(\vec{r}, \vec{p}, t)}{\partial \vec{r}} \\ + \int \frac{d^3 r_0 d^3 p_0 d^3 s}{(2\pi\hbar)^3} \{ f^\pm(\vec{r}_0, \vec{p}_0, t_0) \left[ \frac{-i\vec{s}}{\hbar} \cdot \frac{\partial \vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} \right. \\ \left. - \frac{[\varepsilon(\vec{r} - \frac{\vec{s}}{2}, t) - \varepsilon(\vec{r} + \frac{\vec{s}}{2}, t)]}{i\hbar} \right] \\ \mp f^\pm(\vec{r}_0, \vec{p}_0, t_0) \frac{[V_{hn}(\vec{r} - \frac{\vec{s}}{2}, t) - V_{hn}(\vec{r} + \frac{\vec{s}}{2}, t)]}{i\hbar} \} \\ \times \exp\{i\vec{s} \cdot [\vec{p} - \vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)] / \hbar\} \\ \times \delta[\vec{r} - \vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t)] = 0. \end{aligned} \quad (34)$$

Comparing similar terms in the above equation, we get the following equations for  $\vec{R}$  and  $\vec{P}$ , respectively, i.e.,

$$\left[ -\frac{\partial \vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \right] \cdot \frac{\partial f^\pm(\vec{r}, \vec{p}, t)}{\partial \vec{r}} \pm \frac{\partial V_{hn}}{\partial \vec{p}} \cdot \frac{\partial f^\pm(\vec{r}, \vec{p}, t)}{\partial \vec{r}} = 0, \quad (35)$$

and

$$\begin{aligned} f^\pm(\vec{r}_0, \vec{p}_0, t_0) \left\{ \frac{-i\vec{s}}{\hbar} \cdot \frac{\partial \vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} \right. \\ \left. - \frac{[\varepsilon(\vec{r} - \frac{\vec{s}}{2}, t) - \varepsilon(\vec{r} + \frac{\vec{s}}{2}, t)]}{i\hbar} \right\} \mp f^\pm(\vec{r}_0, \vec{p}_0, t_0) \\ \times \left\{ \frac{V_{hn}(\vec{r} - \frac{\vec{s}}{2}, t) - V_{hn}(\vec{r} + \frac{\vec{s}}{2}, t)}{i\hbar} \right\} = 0. \end{aligned} \quad (36)$$

In order to satisfy the above two equations with arbitrary  $f^\pm(\vec{r}, \vec{p}, t)$  and  $f^\pm(\vec{r}_0, \vec{p}_0, t_0)$ , we get the equations of motion for  $\vec{R}$

$$\frac{\partial \vec{R}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} = \frac{\partial \varepsilon}{\partial \vec{p}} \pm \frac{\partial V_{hn}}{\partial \vec{p}}, \quad (37)$$

and for  $\vec{P}$

$$\begin{aligned} \vec{s} \cdot \frac{\partial \vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} &= [\varepsilon(\vec{r} - \frac{\vec{s}}{2}, t) - \varepsilon(\vec{r} + \frac{\vec{s}}{2}, t)] \\ &\pm [V_{hn}(\vec{r} - \frac{\vec{s}}{2}, t) - V_{hn}(\vec{r} + \frac{\vec{s}}{2}, t)]. \end{aligned} \quad (38)$$

Expanding Eq. (38) in  $\vec{s}$  and keeping only the lowest order term, we then obtain

$$\frac{\partial \vec{P}(\vec{r}_0, \vec{p}_0, \vec{s}, t)}{\partial t} = -\frac{\partial \varepsilon}{\partial \vec{r}} \mp \frac{\partial V_{hn}}{\partial \vec{r}}. \quad (39)$$

Considering Eqs. (18)–(24) and combining Eqs. (37), (39), and (29), the EOMs of test particles for solving the spin-dependent BV equation with the spin-orbit interaction are thus

$$\frac{\partial \vec{R}}{\partial t} = \frac{\vec{p}}{m} + \nabla_{\vec{p}}(h_1 + h_4) \pm \nabla_{\vec{p}}(\vec{h}_2 \cdot \vec{n} + \vec{h}_3 \cdot \vec{n}), \quad (40)$$

$$\frac{\partial \vec{P}}{\partial t} = -\nabla_{\vec{r}} U_q - \nabla_{\vec{r}}(h_1 + h_4) \mp \nabla_{\vec{r}}(\vec{h}_2 \cdot \vec{n} + \vec{h}_3 \cdot \vec{n}), \quad (41)$$

$$\frac{\partial \vec{n}}{\partial t} = \frac{2(\vec{h}_2 + \vec{h}_3) \times \vec{n}}{\hbar}, \quad (42)$$

with the upper sign for  $f^+$  and lower sign for  $f^-$ , respectively, and  $h_1$ ,  $h_2$ ,  $h_3$ , and  $h_4$  given by Eqs. (19)–(22). According to the definition of the spinor Wigner phase-space density distribution in Eqs. (3) and (25),  $\vec{n}$  is the direction (unit vector) of the local spin polarization in 3-dimensional coordinate space, which can be expressed by the test-particle method [38,39] as

$$\vec{n} = \frac{\sum_i \vec{n}_i \delta(\vec{r} - \vec{r}_i) \delta(\vec{p} - \vec{p}_i)}{|\sum_i \vec{n}_i \delta(\vec{r} - \vec{r}_i) \delta(\vec{p} - \vec{p}_i)|}, \quad (43)$$

with  $\vec{n}_i$  being the spin expectation direction of the  $i$ th nucleon. Based on Eqs. (7)–(10), the scalar Wigner density distribution  $f(\vec{r}, \vec{p})$  and the vector Wigner density distribution  $\vec{\tau}(\vec{r}, \vec{p})$  can be expressed by the test-particle method as

$$f(\vec{r}, \vec{p}) = \frac{1}{N_{TP}} \sum_i \delta(\vec{r} - \vec{r}_i) \delta(\vec{p} - \vec{p}_i), \quad (44)$$

$$\vec{\tau}(\vec{r}, \vec{p}) = \frac{1}{N_{TP}} \sum_i \vec{n}_i \delta(\vec{r} - \vec{r}_i) \delta(\vec{p} - \vec{p}_i), \quad (45)$$

with  $N_{TP}$  being the number of test particles per nucleon. In this way, the number, spin, momentum, and spin-current densities can also be calculated via

$$\rho(\vec{r}) = \frac{1}{N_{TP}} \sum_i \delta(\vec{r} - \vec{r}_i), \quad (46)$$

$$\vec{s}(\vec{r}) = \frac{1}{N_{TP}} \sum_i \vec{n}_i \delta(\vec{r} - \vec{r}_i), \quad (47)$$

$$\vec{j}(\vec{r}) = \frac{1}{N_{TP}} \sum_i \frac{\vec{p}_i}{\hbar} \delta(\vec{r} - \vec{r}_i), \quad (48)$$

$$\vec{J}(\vec{r}) = \frac{1}{N_{TP}} \sum_i \left( \frac{\vec{p}_i}{\hbar} \times \vec{n}_i \right) \delta(\vec{r} - \vec{r}_i). \quad (49)$$

## 5. Quantum nature of spin

The above EOMs for the phase-space coordinates are the same as the canonical equations in Hamiltonian dynamics, and the EOM of spin is the same as that in the Heisenberg picture of quantum mechanics. These EOMs have been applied in our previous studies [14–16]. As first demonstrated by the Stern–Gerlach experiment [44], the projection of spin onto any reference direction used in measurements is quantized. In non-central HICs, the angular momentum is in the  $y$  direction perpendicular to the reaction plane ( $x$ – $o$ – $z$  plane). It is thus natural to set  $y$  direction as the third (magnetic) spin direction. We then fix  $\vec{n} = \hat{y}$  in Eqs. (40) and (41) and set  $\vec{n}_i = \pm \hat{y}$  in Eqs. (45), (47), and (49) depending on whether the  $i$ th particle is spin-up or spin-down with respect to the  $y$  axis. In this way the time evolution of  $\vec{n}$  (Eq. (42)) is not needed. Thus, as describing the isospin dynamics with separate EOMs for neutrons and protons [45], we now have separate EOMs for spin-up (upper sign) and spin-down (lower sign) particles

$$\frac{\partial \vec{R}}{\partial t} = \frac{\vec{p}}{m} + \nabla_{\vec{p}}(h_1 + h_4) \pm \nabla_{\vec{p}}(h_{2y} + h_{3y}), \quad (50)$$

$$\frac{\partial \vec{P}}{\partial t} = -\nabla_{\vec{r}} U_q - \nabla_{\vec{r}}(h_1 + h_4) \mp \nabla_{\vec{r}}(h_{2y} + h_{3y}). \quad (51)$$

It is seen that the  $h_{2y} \equiv \vec{h}_2 \cdot \vec{n}$  and  $h_{3y} \equiv \vec{h}_3 \cdot \vec{n}$  lead to the spin-dependent motion while the  $h_1$  and  $h_4$  affect the global motion in phase space.

To summarize, the spin-dependent Boltzmann–Vlasov equation can be solved by extending the test-particle method. Considering the quantum nature of spin and choosing the direction of total angular momentum in heavy-ion reactions as a reference of measuring nucleon spin, the EOMs of spin-up and spin-down nucleons are derived. The key elements affecting the spin dynamics in heavy-ion collisions are identified. The derived EOMs of test particles provide the theoretical foundation of simulating spin-dependent dynamics in intermediate-energy heavy-ion collisions. Future comparisons of model simulations with experimental data will help to constrain the poorly known in-medium nucleon spin-orbit coupling.

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