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The singular linear preservers of non-singular matrices

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ABSTRACT

Given an arbitrary field $\mathbb{K}$, we reduce the determination of the singular endomorphisms f of $\mathrm{M}_n(\mathbb{K})$ such that $f(\mathrm{GL}_n(\mathbb{K})) \subset \mathrm{GL}_n(\mathbb{K})$ to the classification of n-dimensional division algebras over $\mathbb{K}$. Our method, which is based upon Dieudonné's theorem on singular subspaces of $\mathrm{M}_n(\mathbb{K})$, also yields a proof for the classical non-singular case.

1. Introduction

Here, $\mathbb{K}$ will denote an arbitrary field and n a positive integer. We let $\mathrm{M}_{n,p}(\mathbb{K})$ denote the set of matrices with n rows, p columns and entries in $\mathbb{K}$, and $\mathrm{GL}_n(\mathbb{K})$ the set of non-singular matrices in the algebra $\mathrm{M}_n(\mathbb{K})$ of square matrices of order n. The columns of a matrix $M \in \mathrm{M}_n(\mathbb{K})$ will be written $C_1(M), C_2(M), \dots, C_n(M)$, so that

$$M = \begin{bmatrix} C_1(M) & C_2(M) & \cdots & C_n(M) \end{bmatrix}.$$

Given a vector space V, we let $\mathcal{L}(V)$ denote the algebra of endomorphisms of V. For non-singular P and Q in $\mathrm{GL}_n(\mathbb{K})$, we define

$$u_{P,Q} : \begin{cases} \mathrm{M}_n(\mathbb{K}) \longrightarrow \mathrm{M}_n(\mathbb{K}) \\ M \longmapsto P\,M\,Q \end{cases} \quad \text{and} \quad v_{P,Q} : \begin{cases} \mathrm{M}_n(\mathbb{K}) \longrightarrow \mathrm{M}_n(\mathbb{K}) \\ M \longmapsto P\,M^t\,Q. \end{cases}$$

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Clearly, these are non-singular endomorphisms of the vector space $\mathrm{M}_n(\mathbb{K})$ which map $\mathrm{GL}_n(\mathbb{K})$ onto itself, and the subset

$$\mathcal{G}_n(\mathbb{K}) := \left\{u_{P,Q} | (P,Q) \in \mathrm{GL}_n(\mathbb{K})^2\right\} \cup \left\{v_{P,Q} | (P,Q) \in \mathrm{GL}_n(\mathbb{K})^2\right\}$$

is clearly a subgroup of $\mathrm{GL}(\mathrm{M}_n(\mathbb{K}))$, which we will call the *Frobenius group*.

Determining the endomorphisms of the vector space $\mathrm{M}_n(\mathbb{K})$ which preserve non-singularity has historically been one of the first successful linear preserver problem, dating back to Frobenius [6], who classified the linear preservers of the determinant, and Dieudonné [4], who classified the non-singular linear preservers of the general linear group. Some improvements have been made later on the issue (cf. [9,2]). The following theorem is now folklore and essentially sums up what was known to this date:

Theorem 1

(i) *The group $\mathcal{G}_n(\mathbb{K})$ consists of all the endomorphisms f of $\mathrm{M}_n(\mathbb{K})$ such that $f(\mathrm{GL}_n(\mathbb{K})) = \mathrm{GL}_n(\mathbb{K})$.*
(ii) *The group $\mathcal{G}_n(\mathbb{K})$ consists of all the endomorphisms f of $\mathrm{M}_n(\mathbb{K})$ such that $f^{-1}(\mathrm{GL}_n(\mathbb{K})) = \mathrm{GL}_n(\mathbb{K})$.*
(iii) *The group $\mathcal{G}_n(\mathbb{K})$ consists of all the non-singular endomorphisms f of $\mathrm{M}_n(\mathbb{K})$ such that $f(\mathrm{GL}_n(\mathbb{K})) \subset \mathrm{GL}_n(\mathbb{K})$.*
(iv) *If $\mathbb{K}$ is algebraically closed, then $\mathcal{G}_n$ consists of all the endomorphisms f of $\mathrm{M}_n(\mathbb{K})$ such that $f(\mathrm{GL}_n(\mathbb{K})) \subset \mathrm{GL}_n(\mathbb{K})$.*

Our main interest here is finding all the endomorphisms f of $\mathrm{M}_n(\mathbb{K})$ which stabilize $\mathrm{GL}_n(\mathbb{K})$, i.e. $f(\mathrm{GL}_n(\mathbb{K})) \subset \mathrm{GL}_n(\mathbb{K})$. The issue here is the existence of non-singular ones. Here are a few examples:

Example 1. In $\mathrm{M}_2(\mathbb{R})$, the endomorphism

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is singular and stabilizes $\mathrm{GL}_2(\mathbb{R})$. Indeed, if $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$, then $(a,b) \neq (0,0)$ hence $\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2 > 0$.

Example 2. In $\mathrm{M}_3(\mathbb{Q})$, consider the companion matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since the minimal polynomial $X^3 - 2$ of A is irreducible over $\mathbb{Q}$, the subalgebra $\mathbb{Q}[A]$ is a field. The singular endomorphism

$$M \longmapsto m_{1,1}.I_3 + m_{2,1}.A + m_{3,1}.A^2$$

then clearly maps $\mathrm{GL}_3(\mathbb{Q})$ into $\mathbb{Q}[A]\backslash\{0\}$ hence stabilizes $\mathrm{GL}_3(\mathbb{Q})$.

All those examples can be described in a normalized way. We will need a few definitions first.

Definition 1. A linear subspace V of $\mathrm{M}_n(\mathbb{K})$ will be called *non-singular* when $V\backslash\{0\} \subset \mathrm{GL}_n(\mathbb{K})$, and *full non-singular* when in addition $\dim V = n$.

Let V be a full non-singular subspace of $\mathrm{M}_n(\mathbb{K})$, with $n \geqslant 2$. The projection onto the first column

$$\pi : \begin{cases} V \longrightarrow \mathrm{M}_{n,1}(\mathbb{K}) \\ M \longmapsto C_1(M) \end{cases}$$

is then a linear isomorphism. It follows that

$$\psi : \begin{cases} \mathrm{M}_n(\mathbb{K}) \longrightarrow \mathrm{M}_n(\mathbb{K}) \\ M \longmapsto \pi^{-1}(C_1(M)) \end{cases}$$

is a singular linear map which maps every non-singular matrix to a non-singular matrix. More generally, given a non-zero vector $X \in \mathbb{K}^n$ and an isomorphism $\alpha : \mathbb{K}^n \overset{\simeq}{\rightarrow} V$, the linear maps $M \mapsto \alpha(MX)$ and $M \mapsto \alpha(M^t X)$ are singular endomorphisms of $\mathrm{M}_n(\mathbb{K})$ that stabilize $\mathrm{GL}_n(\mathbb{K})$.

In this article, we will prove that the aforementioned maps are the only singular preservers of $\mathrm{GL}_n(\mathbb{K})$:

Theorem 2 (Main theorem)**.** *Let $n \geqslant 2$. Let f be a linear endomorphism of $\mathrm{M}_n(\mathbb{K})$ such that $f(\mathrm{GL}_n(\mathbb{K})) \subset \mathrm{GL}_n(\mathbb{K})$. Then*:

(i) *either f is bijective and then $f \in \mathcal{G}_n(\mathbb{K})$;*

(ii) *or there exists a full non-singular subspace V of $\mathrm{M}_n(\mathbb{K})$, an isomorphism $\alpha : \mathbb{K}^n \overset{\simeq}{\rightarrow} V$ and a column $X \in \mathbb{K}^n \backslash \{0\}$ such that*:

$$\forall M \in \mathrm{M}_n(\mathbb{K}),\ f(M) = \alpha(MX) \quad or \quad \forall M \in \mathrm{M}_n(\mathbb{K}),\ f(M) = \alpha(M^t X).$$

As a consequence, if f is singular, then $\mathrm{Im} f$ is a full non-singular subspace of $\mathrm{M}_n(\mathbb{K})$.

The rest of the paper is laid out as follows:

- we will first easily derive Theorem 1 from Theorem 2;
- afterwards, we will prove Theorem 2 by using a theorem of Dieudonné on the singular subspaces of $\mathrm{M}_n(\mathbb{K})$;
- in the last section, we will explain how the existence of full non-singular subspaces of $\mathrm{M}_n(\mathbb{K})$ is linked to the existence of n-dimensional division algebras over $\mathbb{K}$. This will prove fruitful in the case $\mathbb{K} = \mathbb{R}$.

2. Some consequences of the main theorem

Let us assume Theorem 2 holds, and use it to prove the various statements in Theorem 1. The case $n = 1$ is trivial so we assume $n \geqslant 2$. Remark first that every $f \in \mathcal{G}_n(\mathbb{K})$ is an automorphism of $\mathrm{M}_n(\mathbb{K})$ and satisfies all the conditions $f(\mathrm{GL}_n(\mathbb{K})) \subset \mathrm{GL}_n(\mathbb{K})$, $f(\mathrm{GL}_n(\mathbb{K})) = \mathrm{GL}_n(\mathbb{K})$ and $f^{-1}(\mathrm{GL}_n(\mathbb{K})) = \mathrm{GL}_n(\mathbb{K})$.

Statement (iii) is straightforward by Theorem 2.

Proof of statement (i). Let $f : \mathrm{M}_n(\mathbb{K}) \to \mathrm{M}_n(\mathbb{K})$ be a linear map such that $f(\mathrm{GL}_n(\mathbb{K})) = \mathrm{GL}_n(\mathbb{K})$. By the next lemma, $\mathrm{GL}_n(\mathbb{K})$ generates the vector space $\mathrm{M}_n(\mathbb{K})$, so f must be onto, hence non-singular, and statement (iii) then shows that $f \in \mathcal{G}_n(\mathbb{K})$. $\square$

Lemma 3. *The vector space $\mathrm{M}_n(\mathbb{K})$ is generated by $\mathrm{GL}_n(\mathbb{K})$.*

Proof. The result is obvious when $n = 1$. We now assume $n \geqslant 2$. Set $(E_{i,j})_{1 \leqslant i,j \leqslant n}$ the canonical basis of $\mathrm{M}_n(\mathbb{K})$. Then $E_{i,j} = (I_n + E_{i,j}) - I_n \in \mathrm{span}(\mathrm{GL}_n(\mathbb{K}))$ for all $i \neq j$.

On the other hand, letting $i \in [[1, n]]$ and choosing arbitrarily $j \in [[1, n]]\backslash\{i\}$, we find that $I_n + E_{i,j} + E_{j,i} - E_{i,i}$ is non-singular, therefore

$$E_{i,i} = I_n - (I_n + E_{i,j} + E_{j,i} - E_{i,i}) + E_{i,j} + E_{j,i} \in \ \mathrm{span}\ \mathrm{GL}_n(\mathbb{K}).$$

This proves that $\mathrm{span}(\mathrm{GL}_n(\mathbb{K})) = \mathrm{M}_n(\mathbb{K})$. $\square$

Proof of statement (ii). Let $f : \mathrm{M}_n(\mathbb{K}) \to \mathrm{M}_n(\mathbb{K})$ be a linear map such that $f^{-1}(\mathrm{GL}_n(\mathbb{K})) = \mathrm{GL}_n(\mathbb{K})$. Assume that f is not injective. Then there would be a non-zero matrix $A \in \mathrm{M}_n(\mathbb{K})$ such that $f(A) = 0$, and it would follow that $A+P$ is non-singular for every non-singular P (since then $f(A+P)=f(P)\in \mathrm{GL}_n(\mathbb{K})$). Then any matrix B equivalent to A would also verify this property, in particular $B:=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, with $r:=\mathrm{rk}A>0$. However $B+(-I_n)$ is singular. This proves that f is one-to-one, hence non-singular, and since $f(\mathrm{GL}_n(\mathbb{K}))\subset \mathrm{GL}_n(\mathbb{K})$, statement (iii) shows that $f\in \mathcal{G}_n(\mathbb{K})$. □

Proof of statement (iv). Assume $\mathbb{K}$ is algebraically closed. Then every non-singular subspace of $\mathrm{M}_n(\mathbb{K})$ has dimension at most 1: indeed, given two non-singular P and Q in $\mathrm{M}_n(\mathbb{K})$, the polynomial $\det(P+xQ)=\det(Q)\,\det(PQ^{-1}+x.I_n)$ is non-constant and must then have a root in $\mathbb{K}$. It follows from Theorem 2 that every linear map $f:\mathrm{M}_n(\mathbb{K})\to \mathrm{M}_n(\mathbb{K})$ which stabilizes $\mathrm{GL}_n(\mathbb{K})$ belongs to $\mathcal{G}_n(\mathbb{K})$. □

3. Proof of the main theorem

The basic idea is to use a theorem of Dieudonné to study the subspace $f^{-1}(V)$ when V is a singular subspace of $\mathrm{M}_n(\mathbb{K})$, i.e. one that is disjoint from $\mathrm{GL}_n(\mathbb{K})$. This is essentially the idea in the original proof of Dieudonné [4] but we will push it to the next level by not assuming that f is one-to-one.

3.1. A reduction principle

Let $f:\mathrm{M}_n(\mathbb{K})\to \mathrm{M}_n(\mathbb{K})$ be a linear map which stabilizes $\mathrm{GL}_n(\mathbb{K})$, and let $(P,Q)\in \mathrm{GL}_n(\mathbb{K})$. Then any of the maps $u_{P,Q}\circ f, f\circ u_{P,Q}$ and $M\mapsto f(M)^t$ is linear and stabilizes $\mathrm{GL}_n(\mathbb{K})$. Moreover, it is easily checked that if any one of them is of one of the types listed in Theorem 2, then f also is. Our proof will make a great use of that remark.

3.2. A review of Dieudonné's theorem

Definition 2. A linear subspace of a $\mathbb{K}$-algebra is called *singular* when it contains no invertible element.

For example, given an $i\in [[1,n]]$, the subset of matrices $\mathrm{M}_n(\mathbb{K})$ which have null entries on the ith column is an (n^2-n)-dimensional singular subspace.

Definition 3. Let E be a finite-dimensional vector space, H a hyperplane[1] of E and D a line of E. We define:

- $\mathcal{L}_D(E)$ as the set of endomorphisms u of E such that $D\subset \mathrm{Ker}u$;
- $\mathcal{L}^H(E)$ as the set of endomorphisms u of E such that $\mathrm{Im}u\subset H$.

Then $\mathcal{L}_D(E)$ and $\mathcal{L}^H(E)$ are both (n^2-n)-dimensional singular subspaces of $\mathcal{L}(E)$. The singular subspace $\mathcal{L}_D(E)$ will be said to be of *kernel-type*, and the singular subspace $\mathcal{L}^H(E)$ of *image-type*.

The following theorem of Dieudonné [4], later generalized by Flanders [5] and Meshulam [10], will be used throughout our proof:

Theorem 4 (Dieudonné's theorem)**.** *Let E be an n-dimensional vector space over $\mathbb{K}$, and V a singular subspace of $\mathcal{L}(E)$. Then*:

(a) *one has* $\dim V\leqslant n^2-n$;
(b) *if* $\dim V=n^2-n$*, then we are in one of the mutually exclusive situations*:

- *there is one (and only one) hyperplane H of E such that $V=\mathcal{L}^H(E)$;*
- *there is one (and only one) line D of E such that $V=\mathcal{L}_D(E)$.*

[1] Here, by a hyperplane (resp. a line), we mean a *linear* subspace of codimension one (resp. of dimension one). When we will exceptionally have to deal with affine subspaces, we will always specify it.

3.3. Inverse image of a singular subspace of kernel-type

In what follows, the algebra $\mathrm{M}_n(\mathbb{K})$ will be canonically identified with the algebra $\mathcal{L}(\mathbb{K}^n)$ of endomorphisms of $E := \mathbb{K}^n$. Let $f : \mathrm{M}_n(\mathbb{K}) \to \mathrm{M}_n(\mathbb{K})$ be an endomorphism which stabilizes $\mathrm{GL}_n(\mathbb{K})$. Notice that, given a line D of E and a non-zero vector $X \in D$, the singular subspace $\mathcal{L}_D(E)$ is actually the kernel of the linear map $M \mapsto MX$ on $\mathrm{M}_n(\mathbb{K})$.

Lemma 5. *Let $X \in \mathbb{K}^n \setminus \{0\}$ and set $D := \mathrm{span}(X)$. Then:*

- *either there is an hyperplane H of E such that $f^{-1}(\mathcal{L}_D(E)) = \mathcal{L}^H(E)$;*
- *or there is a line D' of E such that $f^{-1}(\mathcal{L}_D(E)) = \mathcal{L}_{D'}(E)$.*

Moreover, the linear map $M \mapsto f(M)X$ from $\mathrm{M}_n(\mathbb{K})$ to $\mathbb{K}^n$ is onto.

Proof. Since the subspace $\mathcal{L}_D(E)$ contains no non-singular matrix, the assumption on f guarantees that $f^{-1}(\mathcal{L}_D(E))$ is a singular subspace of $\mathrm{M}_n(\mathbb{K})$. Since $f^{-1}(\mathcal{L}_D(E))$ is the kernel of $\alpha : M \mapsto f(M)X$, the rank theorem shows that $\dim f^{-1}(\mathcal{L}_D(E)) \geqslant n^2 - n$. Theorem 4 then shows our first statement, hence another use of the rank theorem proves that $\dim f^{-1}(\mathcal{L}_D(E)) = n^2 - n$ and α is onto. □

We will now show that the type of $f^{-1}(\mathcal{L}_D(E))$ (kernel or image) is actually independent of the given line D. This will prove a lot harder than in Dieudonné's original proof [4] because f is not assumed one-to-one.

Proposition 6. *Let D_1 and D_2 denote two distinct lines in $\mathbb{K}^n$. Then the singular subspaces $f^{-1}(\mathcal{L}_{D_1}(E))$ and $f^{-1}(\mathcal{L}_{D_2}(E))$ are either both of kernel-type or both of image-type.*

Proof. We will use a *reductio ad absurdum* by assuming there is a line D and an hyperplane H of E such that $f^{-1}(\mathcal{L}_{D_1}(E)) = \mathcal{L}_D(E)$ and $f^{-1}(\mathcal{L}_{D_2}(E)) = \mathcal{L}^H(E)$. By right-composing f with $u_{P,Q}$ for some well-chosen non-singular P and Q, and then left-composing $u_{I_n,R}$ for some well-chosen non-singular R, we are reduced to the case $D_1 = D = \mathrm{span}(e_1), D_2 = \mathrm{span}(e_2)$ and $H = \mathrm{span}(e_2, \dots, e_n)$, where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{K}^n$. Then f has the following properties:

- Any matrix with first column 0 is mapped by f to a matrix with first column 0, and $M \mapsto C_1(f(M))$ is onto.
- Any matrix with first line 0 is mapped by f to a matrix with second column 0, and $M \mapsto C_2(f(M))$ is onto.

By the factorization theorem for linear maps [7, Proposition I, p.45], we deduce that there are two isomorphisms $\alpha : \mathrm{M}_{n,1}(\mathbb{K}) \xrightarrow{\simeq} \mathrm{M}_{n,1}(\mathbb{K})$ and $\beta : \mathrm{M}_{1,n}(\mathbb{K}) \xrightarrow{\simeq} \mathrm{M}_{n,1}(\mathbb{K})$ such that, for every

$$M = \begin{bmatrix} C & \cdots \end{bmatrix} = \begin{bmatrix} L \\ \vdots \end{bmatrix} \quad \text{with } C \in \mathrm{M}_{n,1}(\mathbb{K}) \text{ and } L \in \mathrm{M}_{1,n}(\mathbb{K}),$$

one has

$$f(M) = \begin{bmatrix} \alpha(C) & \beta(L) & \cdots \end{bmatrix}.$$

Set now $C_1 := \alpha \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $C_2 := \beta \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$. We then recover two injective linear maps $\alpha' : \mathrm{M}_{n-1,1}(\mathbb{K}) \hookrightarrow \mathrm{M}_{n,1}(\mathbb{K})$ and $\beta' : \mathrm{M}_{1,n-1}(\mathbb{K}) \hookrightarrow \mathrm{M}_{n,1}(\mathbb{K})$ such that for every $M = \begin{bmatrix} 1 & L \\ C & ? \end{bmatrix} \in \mathrm{M}_n(\mathbb{K})$ with first coefficient 1, one has

$$f(M) = \begin{bmatrix} C_1 + \alpha'(C) & C_2 + \beta'(L) & ? \end{bmatrix}.$$

Let $(L, C) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K})$. Notice then that there exists an $N \in \mathrm{M}_{n-1}(\mathbb{K})$ such that $M = \begin{bmatrix} 1 & L \\ C & N \end{bmatrix}$ is non-singular. Indeed, the matrix $N := CL + I_{n-1}$ fits this condition (remark that $\begin{bmatrix} 1 & L \\ C & CL+I_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & I_{n-1} \end{bmatrix}$). For any such M, the matrix $f(M)$ must then be non-singular, which proves that $C_1 + \alpha'(C)$ and $C_2 + \beta'(L)$ are linearly independent.
However, this has to hold for every pair $(L, C) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K})$. Therefore no vector in the affine hyperplane $\mathcal{H}_1 := C_1 + \mathrm{Im}\alpha'$ is colinear to a vector in the affine hyperplane $\mathcal{H}_2 := C_2 + \mathrm{Im}\beta'$. There finally lies a contradiction: indeed, should we choose a vector x_0 in $E \setminus (\mathrm{Im}\alpha' \cup \mathrm{Im}\beta')$ (classically, such a vector exists because E is never the union of two strict linear subspaces), then the line $\mathrm{span}(x_0)$ would have to intersect both hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$. □

We may actually assume there is some line D such that $f^{-1}(\mathcal{L}_D(E))$ has kernel-type, because, if not, we may replace f with $M \mapsto f(M^t)$. Therefore we may now assume, without loss of generality:

For every line D of E, there is a line D' of E such that $f^{-1}(\mathcal{L}_D(E)) = \mathcal{L}_{D'}(E)$.

3.4. Reducing the problem further

We let here $(e_1, \ldots, e_n)$ denote the canonical basis of $E = \mathbb{K}^n$ and set $D_i := \mathrm{span}(e_i)$ for every $i \in [[1, n]]$. We now have n lines $D'_1, \ldots, D'_n$ in E such that $\forall i \in [[1, n]],\ f^{-1}(\mathcal{L}_{D_i}(E)) = \mathcal{L}_{D'_i}(E)$. In every line D'_i, we choose a non-zero vector x_i.

Set $F := \mathrm{span}(x_1, \ldots, x_n)$ and $p := \dim F$. From $(x_1, \ldots, x_n)$ can be extracted a basis of F.

- Replacing f with $M \mapsto f(M)P$ for some suitable permutation matrix P, we may assume $(x_1, \ldots, x_p)$ is a basis of F.
- Replacing f with $M \mapsto f(MP)$ for some non-singular $P \in \mathrm{GL}_n(\mathbb{K})$, we may finally assume $(x_1, \ldots, x_p) = (e_1, \ldots, e_p)$, so that $F = \mathrm{span}(e_1, \ldots, e_p)$.

After these reductions, let us restate some of the assumptions on f: for every $i \in [[1, p]]$ and every $M \in \mathrm{M}_n(\mathbb{K})$, if the ith column of M is 0, then the ith column of $f(M)$ is also 0, and $N \mapsto C_i(f(N))$ is onto (from $\mathrm{M}_n(\mathbb{K})$ to $\mathrm{M}_{n,1}(\mathbb{K})$). By the factorization theorem for linear maps, we recover p automorphisms $\alpha_1, \ldots, \alpha_p$ of $\mathrm{M}_{n,1}(\mathbb{K})$ such that, for every $M = \begin{bmatrix} C_1 & C_2 & \cdots & C_p & ? \end{bmatrix}$ in $\mathrm{M}_n(\mathbb{K})$, one has:

$$f(M) = \begin{bmatrix} \alpha_1(C_1) & \alpha_2(C_2) & \cdots & \alpha_p(C_p) & ? \end{bmatrix}.$$

We will now reduce the previous situation to the case $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \mathrm{id}$.

Lemma 7. *Under the previous assumptions, let $(C_1, \ldots, C_p) \in \mathrm{M}_{n,1}(\mathbb{K})^p$ be a linearly independent p-tuple. Then $(\alpha_1(C_1), \ldots, \alpha_p(C_p))$ is linearly independent.*

Proof. Indeed, $(C_1, \ldots, C_p)$ can be extended into a basis $(C_1, \ldots, C_n)$ of $\mathrm{M}_{n,1}(\mathbb{K})$. Since $M := \begin{bmatrix} C_1 \cdots C_n \end{bmatrix}$ is non-singular, $f(M)$ also is, which proves our claim. □

Define then $P \in \mathrm{GL}_n(\mathbb{K})$ as the matrix canonically associated to α_1. Then we may replace f with $f \circ u_{P^{-1}, I_n}$, which changes no previous assumption. In this case, $\alpha_1 = \mathrm{id}_{\mathrm{M}_{n,1}(\mathbb{K})}$. We claim then that $\alpha_2, \ldots, \alpha_p$ are scalar multiples of the identity. Consider α_2 for example. Since any linearly independent pair (C_1, C_2) in $\mathrm{M}_{n,1}(\mathbb{K})$ can be extended into a linearly independent p-tuple in $\mathrm{M}_n(\mathbb{K})$, Lemma 7 shows $(C_1, \alpha_2(C_2))$ must be linearly independent. It follows that for every $C \in \mathrm{M}_{n,1}(\mathbb{K})$, the matrices C and $(\alpha_2)^{-1}(C)$ must be linearly dependent. Classically, this proves $(\alpha_2)^{-1}$ is a scalar multiple of id, hence α_2 also is. The same line of reasoning also shows that this is true of $\alpha_3, \ldots, \alpha_p$.

We thus find non-zero scalars $\lambda_2, \ldots, \lambda_p$ such that, for every $M = \begin{bmatrix} C_1 & C_2 & \cdots & C_p & ? \end{bmatrix}$ in $\mathrm{M}_n(\mathbb{K})$, one has $f(M) = \begin{bmatrix} C_1 & \lambda_2.C_2 & \cdots & \lambda_p.C_p & ? \end{bmatrix}$.

By replacing f with $f \circ u_{I_n,P^{-1}}$ for $P := D(1, \lambda_2, \dots, \lambda_p, 1, \dots, 1)$, we are thus reduced to the following situation:

For every $M = [C_1 \quad C_2 \quad \cdots \quad C_p \quad ?]$ in $\mathrm{M}_n(\mathbb{K})$, one has $f(M) = [C_1 \quad C_2 \quad \cdots \quad C_p \quad ?]$.

3.5. The coup de grâce

- If $p = n$, then we are reduced to the case $f = \mathrm{id}_{\mathcal{M}_n(\mathbb{K})}$, in which $f = u_{I_n,I_n}$.
- Assume $p = 1$.
  Then Ker$f$ is the set of matrices with 0 as first column. Indeed, since $\bigcap_{k=1}^n \mathcal{L}_{D_k}(E) = \{0\}$, we find

$$\mathrm{Ker}f = \bigcap_{k=1}^{n} f^{-1}\left(\mathcal{L}_{D_k}(E)\right) = \bigcap_{k=1}^{n} \mathcal{L}_{D'_k}(E) = \mathcal{L}_{D_1}(E).$$

  By the factorization theorem for linear maps, we find a linear injection $g : \mathbb{K}^n \hookrightarrow \mathrm{M}_n(\mathbb{K})$ such that $\forall M \in \mathcal{M}_n(\mathbb{K}),\ f(M) = g(Me_1)$, where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Notice then that Im$g$ = Imf and Img is an n-dimensional linear subspace of $\mathrm{M}_n(\mathbb{K})$.
  Finally, Img is actually non-singular: indeed, for every $x \in \mathbb{K}^n \setminus \{0\}$, there exists $M \in \mathrm{GL}_n(\mathbb{K})$ such that $Me_1 = x$, hence $g(x) = f(M)$ is non-singular. We have thus proven that f verifies condition (ii) in Theorem 2.

Our proof of Theorem 2 will then be finished should we prove that only the above two cases can arise. Assume then $1 < p < n$ and consider the vector x_{p+1}. Notice that we now simply have $f^{-1}(\mathcal{L}_D(E)) = \mathcal{L}_D(E)$ for any line D of $F = \mathrm{span}(e_1, \dots, e_p)$. Moreover, the situation is left unchanged should we choose a non-singular $P \in \mathrm{GL}_p(\mathbb{K})$, set $Q := \begin{bmatrix} P & 0 \\ 0 & I_{n-p} \end{bmatrix}$ and replace f with $u_{I_n,P^{-1}} \circ f \circ u_{I_n,P}$. It follows that we may actually assume $D'_{p+1} = D_1$ in addition to the previous assumptions (at this point, the reader must check that none of the previous reductions changes the lines $D_{p+1}, \dots, D_n$).

Another use of the factorization theorem then helps us find an endomorphism α of $\mathrm{M}_{n,1}(\mathbb{K})$ such that, for every $M = [C_1\ C_2 \cdots C_p\ ?]$ in $\mathcal{M}_n(\mathbb{K})$, one has $f(M) = [C_1\ C_2 \cdots C_p\ \alpha(C_1)\ ?]$. Borrowing an argument from Section 3.4, we deduce that for any linearly independent pair (C_1, C_2) in $\mathrm{M}_{n,1}(\mathbb{K})$, the triple $(C_1, C_2, \alpha(C_1))$ is also linearly independent (this is where the assumption $1 < p < n$ comes into play). Clearly, this is absurd: indeed, choose C_1 arbitrarily in $\mathrm{M}_{n,1}(\mathbb{K}) \setminus \{0\}$, then $C_2 := \alpha(C_1)$ if $(C_1, \alpha(C_1))$ is linearly independent, and choose arbitrarily C_2 in $\mathrm{M}_{n,1}(\mathbb{K}) \setminus \mathrm{span}(C_1)$ if not (there again, we use $p \geqslant 2$). This contradiction shows $p \in \{1, n\}$, which completes our proof of Theorem 2.

4. A link with division algebras

We will show here how the full non-singular subspaces of $\mathrm{M}_n(\mathbb{K})$ are connected to division algebra over $\mathbb{K}$. Let us recall first a few basic facts about them.

Definition 4. A *division algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space D equipped with a bilinear map $\star : D \times D \to D$ such that $x \mapsto a \star x$ and $x \mapsto x \star a$ are automorphisms of D for every $a \in D \setminus \{0\}$.

Of course, every field extension of $\mathbb{K}$, and more generally every skew-field extension of $\mathbb{K}$ is a division algebra over $\mathbb{K}$. There are however non-associative division algebras, the most famous example being the algebra of octonions (see [3] for an extensive treatment on them).

Remarks 3

(a) Note that associativity is not required on the part of $\star$!

(b) If D is finite-dimensional, then the latter condition in the definition of a division algebra is verified if and only if $x \mapsto a \star x$ is bijective for every $a \in D \backslash \{0\}$. The data of $\star$ is then equivalent to that of a linear map

$$\alpha : D \longrightarrow \mathcal{L}(D)$$

which maps $D \backslash \{0\}$ into $\mathrm{GL}(D)$ (indeed, to such a map α, we naturally associate the pairing $(a, b) \mapsto \alpha(a)[b]$).

The correspondence between full non-singular subspaces of $\mathrm{GL}_n(\mathbb{K})$ and division algebras over $\mathbb{K}$ is now readily explained:

- Let V be a full non-singular subspace V of $\mathrm{GL}_n(\mathbb{K})$. Setting a basis of V, we define an isomorphism $\theta : \mathbb{K}^n \overset{\simeq}{\rightarrow} V$ which induces an isomorphism of algebras $\overline{\theta} : \mathrm{M}_n(\mathbb{K}) \overset{\simeq}{\rightarrow} \mathcal{L}(V)$. Restricting $\overline{\theta}$ to V then gives rise to a division algebra structure on V.
- Conversely, given a division algebra D with structural map $\alpha : D \rightarrow \mathcal{L}(D)$, we can choose a basis of D, which defines an algebra isomorphism $\psi : \mathcal{L}(D) \overset{\simeq}{\rightarrow} \mathrm{M}_n(\mathbb{K})$, and then associate to D the full non-singular subspace $\psi(\alpha(D))$ of $\mathrm{M}_n(\mathbb{K})$.

Working with the canonical basis of $\mathbb{K}^n$, we have just established a bijective correspondence between the set of structures of division algebras on $\mathbb{K}^n$ (which extend its canonical vector space structure), and the set of full non-singular subspaces of $\mathrm{M}_n(\mathbb{K})$.

By combining our main theorem with the Bott–Milnor–Kervaire theorem on division algebras over the real numbers (cf. [1,8]), this yields:

Proposition 8. *Let $n \in \mathbb{N} \backslash \{2, 4, 8\}$. Then every linear endomorphism f of $\mathrm{M}_n(\mathbb{R})$ which stabilizes $\mathrm{GL}_n(\mathbb{R})$ belongs to the Frobenius group $\mathcal{G}_n(\mathbb{R})$.*

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