Harmonic analysis of signed Ruelle transfer operators

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Abstract

Motivated by wavelet analysis, we prove that there is a one-to-one correspondence between the following data:

  (i) Solutions to $R(h) = h$ where R is a certain non-positive Ruelle transfer operator;
 (ii) Operators that intertwine a certain class of representations of the C^*-algebra $\mathfrak{A}_N$ on two unitary generators U, V subject to the relation

$$UVU^{-1} = V^N.$$

This correspondence enables us to give a criterion for the biorthogonality of a pair of scaling functions and calculate all solutions of the equation $R(h) = h$ in some concrete cases.

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1. Introduction

The multiresolution wavelet theory establishes a close interconnection between two operators: M—the cascade refinement operator and R—the transfer operator,

also called Ruelle operator (see [6,7]). Our present approach stresses representation theory and intertwining operators.

In this paper we show how to get wavelets from representations and we compare representations which yield different wavelets. Examples are given in Section 4.

We recall that M operates on $L^2(\mathbb{R})$ by

$$M\psi(x) = \sqrt{N} \sum_{k \in \mathbb{Z}} a_k \psi(Nx - k) \quad (x \in \mathbb{R}),$$

or, equivalently, in Fourier space

$$\widehat{M\psi}(x) = \frac{m_0(x/N)}{\sqrt{N}} \hat{\psi}\left(\frac{x}{N}\right) \quad (x \in \mathbb{R}),$$

where $N \geqslant 2$ is an integer—the scale, $m_0(z) = \sum_{k \in \mathbb{Z}} a_k z^k$ for $z \in \mathbb{T}$, $\mathbb{T}$ being the unit circle, and $\hat{\psi}$ denotes the Fourier transform

$$\hat{\psi}(x) = \int_{\mathbb{R}} \psi(t) e^{-itx} \, dt.$$

The Ruelle transfer operator is defined on $L^1(\mathbb{T})$ by

$$Rf(z) = \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 f(w) \quad (z \in \mathbb{T}).$$

On $\mathbb{T}$, we consider μ, the normalized Haar measure.

It is the equation

$$M\varphi = \varphi, \tag{1.1}$$

or, equivalently,

$$\varphi(x) = \sqrt{N} \sum_{k \in \mathbb{Z}} a_k \varphi(Nx - k) \quad (x \in \mathbb{R}),$$

which generates the wavelets. It is called the refinement (or scaling) equation.

The orthogonality properties of the integer translates of the scaling function $\varphi \in L^2(\mathbb{R})$, $M\varphi = \varphi$ are closely connected to the problem of finding a positive eigenvector for R

$$h \in L^1(\mathbb{T}), \quad h \geqslant 0, \quad Rh = h \tag{1.2}$$

(see [2–4] where a correspondence is established between the non-zero $L^2(\mathbb{R})$-solutions φ to (1.1) and the non-zero solutions h to (1.2). In general, solutions need not exist.) A necessary condition for the orthogonality of the translates of the scaling function is the quadrature mirror filter restriction:

$$\frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 = 1 \quad (z \in \mathbb{T})$$

which, in terms of the Ruelle operator, can be rewritten as

$$R\mathbb{1} = \mathbb{1}.$$

Lawton [8] gave a necessary and sufficient condition formulated also in terms of the Ruelle operator: the translates of the scaling function are orthogonal if and only if the constant function $\mathbb{1}$ is the only continuous solution of (1.2) (up to a multiplicative constant).

The scaling equation (1.1) can be reformulated in a C^*-algebra setting.

Consider $\mathfrak{A}_N$, the C^*-algebra generated by two unitary operators U and V, satisfying the relation $UVU^{-1} = V^N$. It has a representation on $L^2(\mathbb{R})$ given by

$$U : \psi \mapsto \frac{1}{\sqrt{N}} \psi\left(\frac{x}{N}\right), \quad V : \psi \mapsto \psi(x-1) \quad (x \in \mathbb{R})$$

$V = \pi(z)$ where π is the representation of $L^\infty(\mathbb{T})$ given by

$$\left(\pi(f)\psi\right)^{\hat{}} = f\hat{\psi} \quad \left(f \in L^\infty(\mathbb{T})\right).$$

The scaling equation (1.1) becomes

$$U\varphi = \pi(m_0)\varphi.$$

The system $(U, \pi, L^2(\mathbb{R}), \varphi, m_0)$ is called the wavelet representation with scaling function φ (see [7]).

If a wavelet representation is given with scaling function φ then it produces a solution for (1.2):

$$h_\varphi(z) = \sum_{n \in \mathbb{Z}} z^n \langle \pi(z^n)\varphi | \varphi \rangle = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\omega + 2k\pi)|^2 \quad (z = e^{-i\omega}).$$

In [7] it is proved that a converse also holds. Any solution $h \geqslant 0$ to $Rh = h$ arises in this way, as $h = h_\varphi$ for some representation π of $\mathfrak{A}_N$.

Thus, the analysis of orthogonal wavelets is closely related to the study of the positive Ruelle operator R and this operator is linked to the representations of the algebra $\mathfrak{A}_N$.

For an analysis of biorthogonal wavelets, it turns out that we have to consider non-positive Ruelle operators. They correspond to a pair of filters $m_0, m_0' \in L^\infty(\mathbb{T})$ and are defined by

$$R_{m_0, m_0'} f(z) = \frac{1}{N} \sum_{w^N = z} \overline{m_0(w)} m_0'(w) f(w) \quad \left(f \in L^1(\mathbb{T}), \ z \in \mathbb{T}\right).$$

The condition corresponding to the quadrature mirror filter condition, and necessary for the biorthogonality of wavelets, is

$$\frac{1}{N} \sum_{w^N = z} \overline{m_0(w)} m_0'(w) = 1 \quad (z \in \mathbb{T})$$

which rewrites as

$$R_{m_0,m_0'} \mathbb{1} = \mathbb{1}.$$

If two scaling functions φ, φ' are given, with $U\varphi = \pi(m_0)\varphi$, $U\varphi' = \pi(m_0')\varphi'$, then

$$h_{\varphi,\varphi'}(z) = \sum_{n\in\mathbb{Z}} z^n \langle \pi(z^n)\varphi | \varphi' \rangle = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \bar{\hat{\varphi}} \hat{\varphi}'(\omega + 2k\pi) \quad (z = e^{-i\omega})$$

satisfies

$$R_{m_0,m_0'} h_{\varphi,\varphi'} = h_{\varphi,\varphi'}.$$

For more background on wavelets we refer the reader to [6].

We will see in this paper that solutions to $R_{m_0,m_0'} h = h$ correspond to operators that intertwine the representations of $\mathfrak{A}_N$ introduced in [7] arising from m_0 and m_0', respectively. In Section 2 we establish this correspondence and in Section 3 we give a criterion for the biorthogonality of two given scaling functions in terms of the eigenspace of the non-positive Ruelle transfer operator $R_{m_0,m_0'}$ associated to the eigenvalue 1. In Section 4 we consider some concrete examples of filters and give complete solutions for the equation $Rh = h$.

2. Main results

In this section we prove our main theorems on wavelets and representations: Theorem 2.4 and Theorem 2.7. These results prove the bijective correspondence between two sets: operators that intertwine the cyclic representations presented in [7] and solutions to $R_{m_0,m_0'} h = h$.

We begin with some properties of the Ruelle operator. We will denote by $R = R_{m_0,m_0'}$, $m_0, m_0' \in L^\infty(\mathbb{T})$.

Lemma 2.1. *For $f \in L^1(\mathbb{T})$*

(i) $$\int_{\mathbb{T}} Rf(z)\, d\mu = \int_{\mathbb{T}} \overline{m_0(z)} m_0'(z) f(z)\, d\mu.$$

(ii) $$\int_{\mathbb{T}} g(z) Rf(z)\, d\mu = \int_{\mathbb{T}} g(z^N) \overline{m_0(z)} m_0'(z) f(z)\, d\mu.$$

(iii) $$R\big(g(z^N) f(z)\big) = g(z) Rf(z), \qquad R^n\big(g(z^{N^n}) f(z)\big) = g(z) R^n f(z).$$

(iv) $$\int_{\mathbb{T}} R^n f(z)\, d\mu = \int_{\mathbb{T}} \overline{m_0^{(n)}(z)} m_0'^{(n)} f(z)\, d\mu.$$

where $m_0^{(n)}(z) = m_0(z) m_0(z^N)\ldots m_0(z^{N^{n-1}})$.

Proof. (i)

$$\int_{\mathbb{T}} Rf(z)\, d\mu = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2\pi} \int_{0}^{2\pi} \overline{m_0\left(\frac{\theta + 2k\pi}{N}\right)} m_0'\left(\frac{\theta + 2k\pi}{N}\right)$$

$$\times f\left(\frac{\theta + 2k\pi}{N}\right) d\theta$$

$$= \sum_{k=0}^{N-1} \frac{1}{2\pi} \int_{2k\pi/N}^{2(k+1)\pi/N} \overline{m_0(\theta)} m_0'(\theta) f(\theta)\, d\theta$$

$$= \int_{\mathbb{T}} \overline{m_0(z)} m_0'(z) f(z)\, d\mu.$$

(iii) Clear.
(ii) Follows from (i) and (iii).
(iv) Proof by induction. For $n = 1$ it is (i).

$$\int_{\mathbb{T}} R^{n+1} f\, d\mu = \int_{\mathbb{T}} R(R^n f)\, d\mu = \int_{\mathbb{T}} \overline{m_0(z)} m_0'(z) R^n f(z)\, d\mu$$

$$= \int_{\mathbb{T}} R^n \left(\overline{m_0(z^{N^n})} m_0'(z^{N^n}) f(z) \right) d\mu$$

$$= \int_{\mathbb{T}} \overline{m_0^{(n)}(z)} m_0'^{(n)}(z) \overline{m_0(z^{N^n})} m_0'(z^{N^n}) f(z)\, d\mu$$

$$= \int_{\mathbb{T}} \overline{m_0^{(n+1)}(z)} m_0'^{(n+1)}(z) f(z)\, d\mu. \quad \square$$

From [7, Theorem 2.4] we know that, given $m_0 \in L^\infty(\mathbb{T})$ which is non-singular (i.e., does not vanish on a subset of positive measure), there is a 1–1 correspondence between

(a) $h \in L^1(\mathbb{T})$, $h \geqslant 0$, $R(h) = h$ (here $R = R_{m_0, m_0}$)

and

(b) $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$, $\varphi \in \mathcal{H}$,

with the unitary U from $\tilde{\pi}$ satisfying

$$U\varphi = \pi(m_0)\varphi.$$

Rep($\mathfrak{A}_N$, $\mathcal{H}$) is the set of normal representations of the algebra $\mathfrak{A}_N$. These representations are in fact generated by a unitary U on $\mathcal{H}$ and a representation π of $L^\infty(\mathbb{T})$ on $\mathcal{H}$, with the property that

$$U\pi\big(f(z)\big)U^{-1} = \pi\big(f(z^N)\big) \quad \big(f \in L^\infty(\mathbb{T})\big).$$

Here is again Theorem 2.4 from [7]:

Theorem 2.2.

(i) *Let $m_0 \in L^\infty(\mathbb{T})$, and suppose m_0 does not vanish on a subset of $\mathbb{T}$ of positive measure. Let*

$$(Rf)(z) = \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 f(w), \quad f \in L^1(\mathbb{T}). \tag{2.1}$$

Then there is a one-to-one correspondence between the data (a) *and* (b) *below, where* (b) *is understood as equivalence classes under unitary equivalence*:
(a) *$h \in L^1(\mathbb{T})$, $h \geq 0$, and*

$$R(h) = h. \tag{2.2}$$

(b) *$\tilde{\pi} \in \text{Rep}(\mathfrak{A}_N, \mathcal{H})$, $\varphi \in \mathcal{H}$, and the unitary U from $\tilde{\pi}$ satisfying*

$$U\varphi = \pi(m_0)\varphi. \tag{2.3}$$

(ii) *From* (a)$\to$(b), *the correspondence is given by*

$$\langle \varphi | \pi(f)\varphi \rangle_{\mathcal{H}} = \int_{\mathbb{T}} f h \, d\mu, \tag{2.4}$$

where μ denotes the normalized Haar measure on $\mathbb{T}$.
From (a)$\to$(b), *the correspondence is given by*

$$h(z) = h_\varphi(z) = \sum_{n \in \mathbb{Z}} z^n \langle \pi(e_n)\varphi | \varphi \rangle_{\mathcal{H}}. \tag{2.5}$$

(iii) *When* (a) *is given to hold for some h, and $\tilde{\pi} \in \text{Rep}(\mathfrak{A}_N, \mathcal{H})$ is the corresponding cyclic representation with $U\varphi = \pi(m_0)\varphi$, then the representation is unique from h and* (2.4) *up to unitary equivalence: that is, if $\tilde{\pi}' \in \text{Rep}(\mathfrak{A}_N, \mathcal{H}')$, $\varphi' \in \mathcal{H}'$ also cyclic and satisfying*

$$\langle \varphi' | \pi'(f)\varphi' \rangle = \int_{\mathbb{T}} f h \, d\mu$$

and

$$U'\varphi' = \pi'(m_0)\varphi',$$

then there is a unitary isomorphism W of $\mathcal{H}$ onto $\mathcal{H}'$ such that $W\pi(f) = \pi'(f)W$, for $f \in L^\infty(\mathbb{T})$, $WU = U'W$ and $W\varphi = \varphi'$.

Definition 2.3. Given h as in Theorem 2.2, call $(\pi, U, \mathcal{H}, \varphi)$ the cyclic representation of $\mathfrak{A}_N$ associated to h.

The next theorem shows how solutions of $R_{m_0, m_0'} h_0 = h_0$ induce operators that intertwine these cyclic representations.

Theorem 2.4. Let $m_0, m_0' \in L^\infty(\mathbb{T})$ be non-singular and $h, h' \in L^1(\mathbb{T})$, $h, h' \geqslant 0$, $R_{m_0, m_0}(h) = h$, $R_{m_0', m_0'}(h') = h'$. Let $(\pi, U, \mathcal{H}, \varphi)$, $(\pi', U', \mathcal{H}', \varphi')$ be the cyclic representations corresponding to h and h', respectively. If $h_0 \in L^1(\mathbb{T})$, $R_{m_0, m_0'}(h_0) = h_0$ and $|h_0|^2 \leqslant chh'$ for some $c > 0$ then there exists a unique operator $S : \mathcal{H}' \to \mathcal{H}$ such that

$$SU' = US, \qquad S\pi'(f) = \pi(f)S \quad \left(f \in L^\infty(\mathbb{T})\right),$$

$$\left\langle \varphi \,\middle|\, \pi(f) S\varphi' \right\rangle = \int_\mathbb{T} f h_0 \, d\mu \quad \left(f \in L^\infty(\mathbb{T})\right).$$

Moreover, $\|S\| \leqslant \sqrt{c}$.

Proof. To simplify the notation let $R_0 := R_{m_0, m_0'}$. Look at the construction of $\tilde{\pi}$ and $\mathcal{H}$ in the proof of Theorem 2.4 in [7]. We reproduce here the main steps of this construction. First, one considers

$$\mathcal{V}_n := \left\{ (\xi, n) \mid \xi \in L^\infty(\mathbb{T}) \right\}$$

and

$$\left\langle (\xi, n) \,\middle|\, (\eta, n) \right\rangle_{\mathcal{H}} = \int_\mathbb{T} R^n(\bar{\xi} \eta h) \, d\mu \quad \text{for } n = 1, 2, \dots.$$

Let $\mathcal{H}_n$ be the completion of $\mathcal{V}_n$ in this scalar product.

When n, k are given, $n \geqslant 0$, $k \geqslant 1$, one constructs the isometry $\mathcal{V}_n \hookrightarrow \mathcal{V}_{n+k}$ by iteration of the one from $\mathcal{V}_n$ to $\mathcal{V}_{n+1}$, i.e.,

$$\mathcal{V}_n \hookrightarrow \mathcal{V}_{n+1} \hookrightarrow \mathcal{V}_{n+2} \hookrightarrow \cdots \hookrightarrow \mathcal{V}_{n+k},$$

where $J : \mathcal{V}_n \to \mathcal{V}_{n+1}$ is defined by

$$J\left((\xi, n)\right) := \left(\xi(z^N), n+1\right).$$

Then $\mathcal{H}$ is defined as the inductive limit of the Hilbert spaces $\mathcal{H}_n$. The set $\bigcup_{n \geqslant 0} \mathcal{V}_n$ is dense in $\mathcal{H}$.

The representation is defined as follows:

$$U(\xi, 0) := (S_0 \xi, 0) = \left(m_0(z)\xi(z^N), 0\right),$$

$$U(\xi, n+1) := \left(m_0(z^{N^n})\xi(z), n\right),$$

and

$$\pi(f)(\xi, n) := \left(f\left(z^{N^n}\right)\xi(z), n\right).$$

The scaling function φ is $(\mathbb{1}, 0)$. Recall also the main property of this representation

$$U\pi(f) = \pi\left(f(z^N)\right)U \quad \left(f \in L^\infty(\mathbb{T})\right).$$

Having these, we return to our proof. Define

$$B\left[(\xi, n)\big|(\xi', n)\right] = \int_{\mathbb{T}} R_0^n(\bar{\xi}\xi' h_0)\, d\mu, \quad \text{for } (\xi, n) \in \mathcal{V}_n,\ (\xi', n) \in \mathcal{V}_n'.$$

Then

$$\left|B\left[(\xi, n)\big|(\xi', n)\right]\right| = \left|\int_{\mathbb{T}} R_0^n(\bar{\xi}\xi' h_0)\, d\mu\right|$$

$$\leqslant \int_{\mathbb{T}} \left|\overline{m_0^{(n)}(z)}m_0'^{(n)}(z)\bar{\xi}\xi' h_0\right| d\mu$$

$$\text{(by (iv) of Lemma 2.1)}$$

$$\leqslant \sqrt{c}\int_{\mathbb{T}} \left|m_0^{(n)}(z)\right|\left|m_0'^{(n)}(z)\right|\left|\xi'\right|\left|\xi\right|\left|h\right|^{1/2}\left|h'\right|^{1/2} d\mu$$

$$\leqslant \sqrt{c}\left(\int_{\mathbb{T}} \left|m_0^{(n)}(z)\right|^2\left|\xi\right|^2 h\, d\mu\right)^{1/2}$$

$$\times \left(\int_{\mathbb{T}} \left|m_0'^{(n)}\right|^2\left|\xi'\right|^2 h'\, d\mu\right)^{1/2}$$

$$= \sqrt{c}\,\|(\xi, n)\|_{\mathcal{H}}\|(\xi', n)\|_{\mathcal{H}'}.$$

Therefore

$$\left|B\left[(\xi, n)\big|(\xi', n)\right]\right| \leqslant \sqrt{c}\,\|(\xi, n)\|_{\mathcal{H}}\|(\xi', n)\|_{\mathcal{H}'}. \tag{2.6}$$

Equation (2.6) implies also that B can be extended from $\mathcal{V}_n \times \mathcal{V}_n'$ to $\mathcal{H}_n \times \mathcal{H}_n'$ such that (2.6) remains valid.

Next we prove that B is compatible with the inductive limit structure that defines the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$.

$$B\left[J(\xi, n)\big|J'(\xi', n)\right] = B\left[(\xi(z^N), n+1)\big|(\xi'(z^N), n+1)\right]$$

$$= \int_{\mathbb{T}} R_0^{n+1}\left(\overline{\xi(z^N)}\xi'(z^N)h_0(z)\right) d\mu$$

$$= \int_{\mathbb{T}} R_0^n \big(R_0 \big(\overline{\xi(z^N)} \xi'(z^N) h_0(z) \big) \big) \, d\mu$$

$$= \int_{\mathbb{T}} R_0^n \big(\overline{\xi(z)} \xi'(z) R_0(h_0) \big) \, d\mu$$

$$= \int_{\mathbb{T}} R_0^n \big(\overline{\xi(z)} \xi'(z) h_0 \big) \, d\mu = B\big[(\xi, n) \big| (\xi', n) \big].$$

The compatibility with the inductive limit entails the existence of a sesquilinear extension of B to $\mathcal{H} \times \mathcal{H}'$ with

$$\big| B[\xi | \xi'] \big| \leqslant \sqrt{c} \|\xi\|_{\mathcal{H}} \|\xi'\|_{\mathcal{H}'}. \tag{2.7}$$

There are some commuting properties between B and $(\tilde{\pi}, \tilde{\pi}')$ as follows:

$$B\big[U(\xi, n+1) \big| U(\xi', n+1) \big] = B\big[\big(m_0(z^{N^n}) \xi(z), n \big) \big| \big(m_0'(z^{N^n}) \xi'(z), n \big) \big]$$

$$= \int_{\mathbb{T}} R_0^n \big(\overline{m_0(z^{N^n})} m_0'(z^{N^n}) \overline{\xi(z)} \xi'(z) h_0(z) \big) \, d\mu$$

$$= \int_{\mathbb{T}} \overline{m_0(z)} m_0'(z) R_0^n \big(\overline{\xi(z)} \xi'(z) h_0(z) \big) \, d\mu$$

$$= \int_{\mathbb{T}} R_0 \big(R_0^n \big(\overline{\xi(z)} \xi'(z) h_0(z) \big) \big) \, d\mu = B\big[(\xi, n+1) \big| (\xi', n+1) \big].$$

So, by density

$$B[U\xi | U'\xi'] = B[\xi | \xi'] \quad (\xi \in \mathcal{H}, \ \xi' \in \mathcal{H}'). \tag{2.8}$$

For $f \in L^\infty(\mathbb{T})$,

$$B\big[\pi(f)(\xi, n) \big| (\xi', n) \big] = B\big[\big(f(z^{N^n}) \xi(z), n \big) \big| (\xi', n) \big]$$

$$= \int_{\mathbb{T}} R_0^n \big(\overline{f}(z^{N^n}) \overline{\xi}(z) \xi'(z) h_0(z) \big) \, d\mu$$

$$= B\big[(\xi, n) \big| \big(\overline{f}(z^{N^n}) \xi'(z), n \big) \big] = B\big[(\xi, n) \big| \pi'(\overline{f})(\xi', n) \big]$$

and, again by density,

$$B\big[\pi(f)\xi | \xi' \big] = B\big[\xi | \pi'(\overline{f})\xi' \big] \quad (\xi \in \mathcal{H}, \ \xi' \in \mathcal{H}'). \tag{2.9}$$

As $\varphi = (\mathbb{1}, 0) = \varphi'$ we obtain also

$$B\big[\varphi | \pi'(f)\varphi' \big] = B\big[(\mathbb{1}, 0) \big| \pi'(f)(\mathbb{1}, 0) \big] = \int_{\mathbb{T}} f(z) h_0(z) \, d\mu. \tag{2.10}$$

Since B is sesquilinear and bounded, there exists $S : \mathcal{H}' \to \mathcal{H}$, a bounded linear operator with $\|S\| \leqslant \sqrt{c}$ such that

$$B[\xi|\xi'] = \langle \xi | S\xi' \rangle \quad (\xi \in \mathcal{H}, \ \xi' \in \mathcal{H}').$$

Rewriting (2.8) in terms of S, one obtains

$$SU' = US,$$

(2.9) gives

$$\pi(f)S = S\pi'(f) \quad \big(f \in L^\infty(\mathbb{T})\big),$$

and (2.10) yields

$$\langle \varphi | \pi(f)S\varphi' \rangle = \int_{\mathbb{T}} f h_0 \, d\mu.$$

For the uniqueness part we will use a lemma which will be useful outside this context too.

Lemma 2.5. *If n_1, n_2 are integers and $f_1, f_2 \in L^\infty(\mathbb{T})$ then*

$$\langle U^{-n_1} \pi(f_1)\varphi \,|\, SU'^{-n_2} \pi'(f_2)\varphi \rangle$$
$$= \begin{cases} \int_{\mathbb{T}} \overline{f_1}(z^{N^{n_2-n_1}}) \overline{m}_0^{(n_2-n_1)}(z) f_2(z) h_0(z) \, d\mu & \text{for } n_2 \geqslant n_1, \\ \int_{\mathbb{T}} \overline{f_1}(z) m_0'^{(n_1-n_2)}(z) f_2(z^{N^{n_1-n_2}}) h_0(z) \, d\mu & \text{for } n_1 \geqslant n_2. \end{cases}$$

Proof. For $n_2 \geqslant n_1$

$$\langle U^{-n_1} \pi(f_1)\varphi \,|\, SU'^{-n_2} \pi'(f_2)\varphi \rangle$$
$$= \langle U^{n_2} U^{-n_1} \pi(f_1)\varphi \,|\, U^{n_2} SU'^{-n_2} \pi'(f_2)\varphi \rangle$$
$$= \langle U^{n_2-n_1} \pi(f_1)\varphi \,|\, S\pi'(f_2)\varphi \rangle = \langle \pi\big(f_1(z^{N^{n_2-n_1}})\big) U^{n_2-n_1}\varphi \,|\, S\pi'(f_2)\varphi' \rangle$$

and using $U^n \varphi = \pi(m_0^{(n)})\varphi$

$$= \langle \pi\big(f_1(z^{N^{n_2-n_1}})\big) \pi\big(m_0^{(n_2-n_1)}\big)\varphi \,|\, S\pi'(f_2)\varphi' \rangle$$
$$= \langle \varphi \,|\, S\pi'\big(\overline{f_1}(z^{N^{n_2-n_1}}) \overline{m}_0^{(n_2-n_1)}(z) f_2(z)\big)\varphi' \rangle$$
$$= \int_{\mathbb{T}} \overline{f_1}(z^{N^{n_2-n_1}}) \overline{m}_0^{(n_2-n_1)}(z) f_2(z) h_0(z) \, d\mu.$$

For $n_1 \geqslant n_2$ the computation is completely analogous. $\square$

The set

$$\big\{ U^{-n} \pi(f)\varphi \,|\, n \in \mathbb{N}, \ f \in L^\infty(\mathbb{T}) \big\}$$

is dense in $\mathcal{H}$ and similarly for $\mathcal{H}'$, therefore Lemma 2.5 implies the uniqueness of S. $\quad\square$

Even more uniqueness holds. In Theorem 2.4 in [7] it is proved that if $(\pi_1, U_1, \mathcal{H}_1, \varphi_1)$ and $(\pi_2, U_2, \mathcal{H}_2, \varphi_2)$ are two cyclic representations of $\mathfrak{A}_N$ corresponding to the same h then there exists a unique unitary isomorphism $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$W\pi_1(f) = \pi_2(f)W, \quad f \in L^\infty(\mathbb{T}), \qquad WU_1 = U_2W, \qquad W\varphi_1 = \varphi_2.$$

Theorem 2.6. *Let m_0, m_0', h, h', h_0 be as in Theorem 2.4. Suppose $(\pi_i, U_i, \mathcal{H}_i, \varphi_i)$, $i = 1, 2$, are cyclic representations corresponding to h, $(\pi_i', U_i', \mathcal{H}_i', \varphi_i')$, $i = 1, 2$, are cyclic representations corresponding to h' and $S_i : \mathcal{H}_i' \to \mathcal{H}_i$, $i = 1, 2$, are bounded operators such that*

$$S_i\pi_i'(f) = \pi_i(f)S_i \quad \left(f \in L^\infty(\mathbb{T})\right), \qquad S_iU_i' = U_iS_i \quad (i = 1, 2),$$

$$\left\langle\varphi_i \middle| S_i\pi_i'(f)\varphi_i'\right\rangle = \int_{\mathbb{T}} fh_0\, d\mu \quad (i = 1, 2).$$

Then there exists unique unitary isomorphisms $W : \mathcal{H}_1 \to \mathcal{H}_2$ and $W' : \mathcal{H}_1' \to \mathcal{H}_2'$ such that

$$W\pi_1(f) = \pi_2(f)W \quad \left(f \in L^\infty(\mathbb{T})\right), \qquad WU_1 = U_2W, \qquad W\varphi_1 = \varphi_2,$$
$$W'\pi_1'(f) = \pi_2'(f)W' \quad \left(f \in L^\infty(\mathbb{T})\right), \qquad W'U_1' = U_2'W',$$
$$W'\varphi_1' = \varphi_2', \qquad S_2W' = WS_1.$$

Proof. Let W, W' be given by Theorem 2.4 in [7]. Consider $\tilde{S}_1 = W^{-1}S_2W' : \mathcal{H}_1' \to \mathcal{H}_1$; we prove that $\tilde{S}_1$ satisfies the same conditions as S_1 so it must be equal to S_1 (see Theorem 2.4). Let $f \in L^\infty(\mathbb{T})$.

$$\begin{aligned}
\tilde{S}_1\pi_1'(f) &= W^{-1}S_2W'\pi_1'(f) = W^{-1}S_2\pi_2'(f)W' \\
&= W^{-1}\pi_2(f)S_2W' = \pi_1(f)W^{-1}S_2W' = \pi_1(f)\tilde{S}_1.
\end{aligned}$$

Similarly $\tilde{S}_1U_1' = U_1\tilde{S}_1$. Also

$$\begin{aligned}
\left\langle\varphi_1\middle|\tilde{S}_1\pi_1'(f)\varphi_1'\right\rangle &= \left\langle\varphi_1\middle|\pi_1(f)W^{-1}S_2W'\varphi_1'\right\rangle = \left\langle\varphi_1\middle|\pi_1(f)W^{-1}S_2\varphi_2'\right\rangle \\
&= \left\langle\varphi_1\middle|W^{-1}\pi_2(f)S_2\varphi_2'\right\rangle = \left\langle W\varphi_1\middle|\pi_2(f)S_2\varphi_2'\right\rangle \\
&= \left\langle\varphi_2\middle|\pi_2(f)S_2\varphi_2'\right\rangle = \int_{\mathbb{T}} fh_0\, d\mu. \quad\square
\end{aligned}$$

Next we approach the converse problem: that is, given non-singular $m_0, m_0' \in L^\infty(\mathbb{T})$ and $h, h' \in L^1(\mathbb{T})$, $h, h' \geqslant 0$, $R_{m_0,m_0}h = h$, $R_{m_0',m_0'}h' = h'$, consider the

cyclic representations $(\pi, U, \mathcal{H}, \varphi)$ and $(\pi', U', \mathcal{H}', \varphi')$. We want to see if an intertwining operator $S : \mathcal{H}' \to \mathcal{H}$ induces an eigenvector $h_0 \in L^1(\mathbb{T})$ with

$$R_{m_0, m_0'} h_0 = h_0$$

and

$$\langle \varphi \mid S\pi'(f)\varphi' \rangle = \int_{\mathbb{T}} f h_0 \, d\mu \quad \left(f \in L^\infty(\mathbb{T}) \right).$$

The answer is positive and is given in the next theorem.

Theorem 2.7. *Let $m_0, m_0', h, h', (\pi, U, \mathcal{H}, \varphi), (\pi', U', \mathcal{H}', \varphi')$ be as in Theorem 2.4. Suppose $S : \mathcal{H}' \to \mathcal{H}$ is a bounded operator that satisfies*

$$S\pi'(f) = \pi(f)S \quad \left(f \in L^\infty(\mathbb{T}) \right), \qquad SU' = US.$$

Then there exists a unique $h_0 \in L^1(\mathbb{T})$ such that

$$R_{m_0, m_0'} h_0 = h_0$$

and

$$\langle \varphi \mid S\pi'(f)\varphi' \rangle = \int_{\mathbb{T}} f h_0 \, d\mu \quad \left(f \in L^\infty(\mathbb{T}) \right).$$

Moreover,

$$|h_0|^2 \leqslant \|S\|^2 h h' \quad \text{almost everywhere on } \mathbb{T}.$$

Proof. We will need the following result:

Lemma 2.8. *If $f_i \in L^\infty(\mathbb{T})$, $i \in \mathbb{N}$, f_i converges pointwise to $f \in L^\infty(\mathbb{T})$ μ-a.e. and $\|f_i\|_\infty \leqslant M < \infty$ for $i \in \mathbb{N}$ then*

$$\pi(f_i)(\xi) \to \pi(f)(\xi) \quad \text{in } \mathcal{H} \quad (\xi \in \mathcal{H}).$$

Proof. Consider first $(\xi, n) \in \mathcal{V}_n$ so $\xi \in L^\infty(\mathbb{T})$,

$$\left\| \pi(f_i)(\xi, n) - \pi(f)(\xi, n) \right\|_{\mathcal{H}}^2 = \left\| \left(\left(f_i(z^{N^n}) - f(z^{N^n}) \right) \xi(z), n \right) \right\|^2$$

$$= \int_{\mathbb{T}} R_{m_0, m_0}^n \left(|f_i - f|^2 (z^{N^n}) |\xi|^2(z) h(z) \right) d\mu$$

$$= \int_{\mathbb{T}} |f_i - f|^2(z) R_{m_0, m_0}^n \left(|\xi|^2 h \right) d\mu \to 0$$

by Lebesgue's dominated convergence theorem.

The set

$$\mathcal{V} = \left\{ (\xi, n) \mid \xi \in L^\infty(\mathbb{T}), \ n \in \mathbb{N} \right\}$$

is dense in $\mathcal{H}$. Fix $\epsilon > 0$ and let $a \in \mathcal{H}$. There exists a $b \in \mathcal{V}$ with $\|a - b\|_{\mathcal{H}} < \epsilon/(3M)$,

$$
\begin{aligned}
\left\| \pi(f_i)(a) - \pi(f)(a) \right\|_{\mathcal{H}} &\leqslant \left\| \pi(f_i)(a) - \pi(f_i)(b) \right\|_{\mathcal{H}} \\
&\quad + \left\| \pi(f_i)(b) - \pi(f)(b) \right\|_{\mathcal{H}} \\
&\quad + \left\| \pi(f)(b) - \pi(f)(a) \right\|_{\mathcal{H}} \\
&\leqslant \left\| \pi(f_i) \right\| \|a - b\|_{\mathcal{H}} + \left\| \pi(f_i)(b) - \pi(f)(b) \right\|_{\mathcal{H}} \\
&\quad + \left\| \pi(f) \right\| \|a - b\|_{\mathcal{H}}.
\end{aligned}
$$

There is an n_ϵ such that for $i \geqslant n_\epsilon$ one has $\|\pi(f_i)(b) - \pi(f)(b)\|_{\mathcal{H}} < \epsilon/3$. Then for such indices i,

$$\left\| \pi(f_i)(a) - \pi(f)(a) \right\|_{\mathcal{H}} \leqslant \|f_i\|_\infty \frac{\epsilon}{3M} + \frac{\epsilon}{3} + \|f\|_\infty \frac{\epsilon}{3M} \leqslant \epsilon.$$

This concludes the proof of the lemma. □

Returning to the proof of the theorem, construct the continuous linear functional on $L^\infty(\mathbb{T})$ by

$$\Lambda : f \mapsto \langle \varphi \mid \pi(f) S\varphi' \rangle.$$

Its restriction to the continuous functions on $\mathbb{T}$ is continuous so there is a finite regular Borel measure ν on $\mathbb{T}$ such that

$$\langle \varphi \mid \pi(f) S\varphi' \rangle = \int_{\mathbb{T}} f \, d\nu, \quad f \in C(\mathbb{T}).$$

Now take $f \in L^\infty(\mathbb{T})$. Lusin's theorem provides a sequence of continuous functions f_i on $\mathbb{T}$ that converges to f pointwise $\mu + |\nu|$-a.e. and with $\|f_i\|_\infty \leqslant \|f\|_\infty$ for all $i \in \mathbb{N}$. Using our lemma

$$\langle \varphi \mid \pi(f) S\varphi' \rangle = \lim_{i \to \infty} \langle \varphi \mid \pi(f_i) S\varphi' \rangle = \lim_{i \to \infty} \int_{\mathbb{T}} f_i \, d\nu = \int_{\mathbb{T}} f \, d\nu,$$

the last equality following from Lebesgue's dominated convergence theorem. Hence

$$\langle \varphi \mid \pi(f) S\varphi' \rangle = \int_{\mathbb{T}} f \, d\nu \quad \left(f \in L^\infty(\mathbb{T}) \right).$$

The measure ν in absolutely continuous, because, if E is a Borel set of μ measure zero, then $\pi(\chi_E) = 0$ so $\nu(E) = \int_{\mathbb{T}} \chi_E \, d\nu = 0$. Consequently, there is an $h_0 \in L^1(\mathbb{T})$ such that $d\nu = h_0 \, d\mu$ and we rewrite the previous equation:

$$\langle \varphi | \pi(f) S\varphi' \rangle = \int_{\mathbb{T}} f h_0 \, d\mu, \quad f \in L^\infty(\mathbb{T}).$$

Next we prove that $R_0 h_0 = h_0$. Take an arbitrary $f \in L^\infty(\mathbb{T})$,

$$\int_{\mathbb{T}} f h_0 \, d\mu = \langle \varphi | \pi(f) S\varphi' \rangle = \langle U\varphi | U\pi(f) S\varphi' \rangle$$

$$= \langle \pi(m_0)\varphi | \pi(f(z^N)) SU'\varphi' \rangle$$

$$= \langle \pi(m_0)\varphi | S\pi'(f(z^N)) \pi'(m_0')\varphi' \rangle$$

$$= \langle \pi(m_0)\varphi | \pi(f(z^N) m_0'(z)) S\varphi' \rangle$$

$$= \langle \varphi | \pi(f(z^N) m_0'(z) \overline{m_0(z)}) S\varphi' \rangle$$

$$= \int_{\mathbb{T}} f(z^N) m_0'(z) \overline{m_0(z)} h_0(z) \, d\mu = \int_{\mathbb{T}} f(z) R_0 h_0 \, d\mu.$$

As f is arbitrary in $L^\infty(\mathbb{T})$ this implies that $R_0 h_0 = h_0$.

Uniqueness of h_0 is clear and we concentrate on last inequality stated in the theorem. For all $f, g \in L^\infty(\mathbb{T})$ we have

$$\left| \int_{\mathbb{T}} \overline{f} g h_0 \, d\mu \right| = |\langle \pi(f)\varphi | \pi(g) S\varphi' \rangle| \leqslant \|\pi(f)\varphi\|_{\mathcal{H}} \|S\| \|\pi'(g)\varphi'\|_{\mathcal{H}}$$

$$\leqslant \left(\int_{\mathbb{T}} |f|^2 h \, d\mu \right)^{1/2} \|S\| \left(\int_{\mathbb{T}} |g|^2 h' \, d\mu \right)^{1/2}.$$

Since h_0, h, h' are in $L^1(\mathbb{T})$, almost every point is a Lebesgue density point for all of them. Take z a Lebesgue point and $f = g = \chi_I$ where I is a small segment centered at z. The inequality above implies

$$\left| \int_I h_0 \, d\mu \right| \leqslant \|S\| \left(\int_I h \, d\mu \right)^{1/2} \left(\int_I h' \, d\mu \right)^{1/2},$$

and, dividing by $\mu(I)$,

$$\left| \frac{1}{\mu(I)} \int_I h_0 \, d\mu \right| \leqslant \|S\| \left(\frac{1}{\mu(I)} \int_I h \, d\mu \right)^{1/2} \left(\frac{1}{\mu(I)} \int_I h' \, d\mu \right)^{1/2}.$$

Then let I shrink to $\{z\}$,

$$|h_0(z)| \leqslant \|S\| h^{1/2}(z) h'^{1/2}(z),$$

and the proof is complete. $\square$

3. Applications to wavelets

We have wavelet representations of $\mathfrak{A}_N$: $(U, \pi, L^2(\mathbb{R}), \varphi, m_0)$. Then, by Theorem 2.2, it is the cyclic representation associated to some positive h with $R_{m_0, m_0} h = h$. Let us see what the corresponding h is. We must have

$$\langle \varphi \mid \pi(f)\varphi \rangle = \int_{\mathbb{T}} f h \, d\mu \quad (f \in L^\infty(\mathbb{T})),$$

and by the unitarity of the Fourier transform

$$\frac{1}{2\pi} \int_{\mathbb{R}} \bar{\hat{\varphi}} f \hat{\varphi} \, d\mu = \int_{\mathbb{T}} f h \, d\mu,$$

and after periodization

$$\int_{\mathbb{T}} f \operatorname{Per}(|\hat{\varphi}|^2) \, d\mu = \int_{\mathbb{T}} f h \, d\mu,$$

which implies $\operatorname{Per}(|\hat{\varphi}|^2) = h$. Here, for $f \in L^1(\mathbb{R})$,

$$\operatorname{Per}(f)(\omega) = \sum_{k \in \mathbb{Z}} f(\omega + 2k\pi), \quad \omega \in [0, 2\pi].$$

Hence the wavelet representation is the cyclic representation corresponding to $\operatorname{Per}(|\hat{\varphi}|^2)$.

Next, we try to give a necessary and sufficient condition for the biorthogonality of wavelets. In the case of biorthogonal wavelets we have the following identity for the filters m_0, m_0':

$$\frac{1}{N} \sum_{w^N = z} \overline{m_0(w)} m_0'(w) = 1, \quad z \in \mathbb{T},$$

which can be rewritten as

$$R_{m_0, m_0'} \mathbb{1} = \mathbb{1}.$$

If φ, φ' are scaling functions corresponding to m_0, m_0', respectively, then we see that $\operatorname{Per}(\bar{\hat{\varphi}} \hat{\varphi}')$ is also an eigenvector for $R_{m_0, m_0'}$.

Indeed, one has the corresponding scaling equations

$$U\varphi = \pi(m_0)\varphi, \qquad U\varphi' = \pi(m_0')\varphi',$$

or the Fourier transform versions

$$\hat{\varphi}(\omega) = \frac{m_0(\omega/N)}{\sqrt{N}} \hat{\varphi}\left(\frac{\omega}{N}\right), \qquad \hat{\varphi}'(\omega) = \frac{m_0'(\omega/N)}{\sqrt{N}} \hat{\varphi}'\left(\frac{\omega}{N}\right).$$

Then

$$\begin{aligned}
\text{Per}\big(\bar{\hat{\varphi}}\hat{\varphi}'\big)(\omega) &= \sum_{k\in\mathbb{Z}} \bar{\hat{\varphi}}\hat{\varphi}'(\omega + 2k\pi) = \sum_{l=0}^{N-1} \sum_{k\in\mathbb{Z}} \bar{\hat{\varphi}}\hat{\varphi}'(\omega + 2kN\pi + 2l\pi) \\
&= \sum_{l=0}^{N-1} \sum_{k\in\mathbb{Z}} \frac{\overline{m_0}\big(\frac{\omega+2l\pi}{N}\big)}{\sqrt{N}} \bar{\hat{\varphi}}\left(\frac{\omega + 2l\pi}{N} + 2k\pi\right) \frac{m_0'\big(\frac{\omega+2l\pi}{N}\big)}{\sqrt{N}} \\
&\quad \times \hat{\varphi}'\left(\frac{\omega + 2l\pi}{N} + 2k\pi\right) \\
&= \frac{1}{N} \sum_{l=0}^{N-1} \overline{m_0}\left(\frac{\omega + 2l\pi}{N}\right) m_0'\left(\frac{\omega + 2l\pi}{N}\right) \\
&\quad \times \sum_{k\in\mathbb{Z}} \bar{\hat{\varphi}}\hat{\varphi}'\left(\frac{\omega + 2l\pi}{N} + 2k\pi\right) \\
&= \frac{1}{N} \sum_{l=0}^{N-1} \overline{m_0}\left(\frac{\omega + 2l\pi}{N}\right) m_0'\left(\frac{\omega + 2l\pi}{N}\right) \\
&\quad \times \text{Per}\big(\bar{\hat{\varphi}}\hat{\varphi}'\big)\left(\frac{\omega + 2l\pi}{N}\right) \\
&= R_{m_0,m_0'}\big(\text{Per}\big(\bar{\hat{\varphi}}\hat{\varphi}'\big)\big)(\omega).
\end{aligned}$$

Thus, if we know that $R_{m_0,m_0'}$ has only one eigenvector (up to a multiplicative constant) in some subspace containing $\mathbb{1}$ and $\text{Per}(\bar{\hat{\varphi}}\hat{\varphi}')$, then we get that

$$\text{Per}\big(\bar{\hat{\varphi}}\hat{\varphi}\big) = \mathbb{1}$$

which is the Fourier transform version of the biorthogonality of φ and φ'.

We shall see that, under some mild regularity assumptions on φ and φ', the converse also holds so the biorthogonality implies that $R_{m_0,m_0'}$ has a 1-dimensional eigenspace corresponding to the eigenvalue 1.

We set up the framework for this converse. Suppose we have cyclic vectors φ, φ' for the wavelet representation $\tilde{\pi}$ of $\mathfrak{A}_N$ on $L^2(\mathbb{R})$, satisfying the scaling equations

$$U\varphi = \pi(m_0)\varphi, \qquad U\varphi' = \pi(m_0')\varphi'$$

with m_0, m_0' non-singular in $L^\infty(\mathbb{T})$.

The wavelet representation $(U, \pi, L^2(\mathbb{R}), \varphi, m_0)$ is the cyclic representation corresponding to $h = \text{Per}(|\hat{\varphi}|^2)$. Similarly, for φ', the wavelet representation corresponds to $h' = \text{Per}(|\hat{\varphi}'|^2)$.

Theorem 3.1. *Let $m_0, m_0', \varphi, \varphi'$ as above. Suppose the following conditions are satisfied*:

(i) $\hat{\varphi}(0) \neq 0 \neq \hat{\varphi}'(0)$.

(ii) *The integer translates of φ and φ' form Riesz bases for their corresponding linear spans.*

(iii) *$\hat{\varphi}$ and $\hat{\varphi}'$ are continuous at 0 and*

$$\sum_{k \neq 0} |\hat{\varphi}|^2(x + 2k\pi) \to 0 \quad as \ x \to 0,$$

$$\sum_{k \neq 0} |\hat{\varphi}'|^2(x + 2k\pi) \to 0 \quad as \ x \to 0.$$

(iv) *φ and φ' are biorthogonal or equivalently*

$$\sum_{k \in \mathbb{Z}} \bar{\hat{\varphi}}(x + 2k\pi)\hat{\varphi}'(x + 2k\pi) = 1, \quad a.e. \ on \ \mathbb{R}.$$

Then there exist exactly one (*up to a constant multiple*) *solution for*

$$R_{m_0, m_0'} h_0 = h_0, \quad h_0 \in L^\infty(\mathbb{T}),$$

which is continuous at $z = 1$. (*The solution is $h_0 = \mathbb{1}$.*)

Proof. Employing Schwarz' inequality we have

$$1 = \left|\mathrm{Per}\big(\bar{\hat{\varphi}}\hat{\varphi}'\big)\right|^2(\omega) = \left|\sum_{k \in \mathbb{Z}} \bar{\hat{\varphi}}\hat{\varphi}'(\omega + 2k\pi)\right|^2$$

$$\leqslant \left(\sum_{k \in \mathbb{Z}} |\hat{\varphi}|^2(\omega + 2k\pi)\right)\left(\sum_{k \in \mathbb{Z}} |\hat{\varphi}'|^2(\omega + 2k\pi)\right)$$

$$= \big(\mathrm{Per}\,|\hat{\varphi}|^2\big)\big(\mathrm{Per}\,|\hat{\varphi}'|^2\big)(\omega) = h(\omega)h'(\omega).$$

Suppose h_0 is a solution in $L^\infty(\mathbb{T})$ of $R_{m_0, m_0'} h_0 = h_0$ which is continuous at $z = 1$. Since $h_0 \in L^\infty(\mathbb{T})$, there is a $c < \infty$ such that $|h_0|^2 \leqslant c$ almost everywhere on $\mathbb{T}$ and the previous inequality implies

$$|h_0|^2 \leqslant chh'.$$

Therefore, by Theorem 2.4, h_0 induces an operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$US = SU, \quad \pi(f)S = S\pi(f) \quad \big(f \in L^\infty(\mathbb{T})\big),$$

$$\langle \varphi | S\pi(f)\varphi' \rangle = \int_{\mathbb{T}} f h_0 \, d\mu.$$

Fourier transform the last equation and then periodize to obtain for $f \in L^\infty(\mathbb{T})$

$$\int_\mathbb{T} f h_0 \, d\mu = \frac{1}{2\pi} \langle \hat{\varphi} \mid f \, \widehat{(S\varphi')} \rangle = \int_\mathbb{T} \text{Per}\big(\hat{\varphi}(S\varphi')\big) f \, d\mu.$$

So

$$h_0 = \text{Per}\big(\hat{\varphi}(S\varphi')\big). \tag{3.1}$$

Also the commuting properties of S imply that $S\varphi'$ is a solution for the scaling equation

$$U S\varphi' = \pi(m_0') S\varphi'.$$

Assume $S\varphi' \neq c\varphi'$ where c is some constant. We want to prove that $\widehat{S\varphi'}$ can not be continuous at 0. Otherwise, consider the Fourier version of the scaling equation

$$\widehat{S\varphi'}(\omega) = \frac{m_0'(\omega/N)}{\sqrt{N}} \widehat{S\varphi'}\left(\frac{\omega}{N}\right)$$

and by induction

$$\widehat{S\varphi'}(\omega) = \left[\prod_{i=1}^n \frac{m_0'(\omega/N^i)}{\sqrt{N}} \right] \widehat{S\varphi'}\left(\frac{\omega}{N^n}\right).$$

A similar equation can be constructed for $\hat{\varphi}'$. If $\widehat{S\varphi'}$ is continuous at 0 then

$$\frac{\widehat{S\varphi'}(x)}{\hat{\varphi}'(x)} = \lim_{n \to \infty} \frac{\widehat{S\varphi'}(x/N^n)}{\hat{\varphi}'(x/N^n)} = \frac{\widehat{S\varphi'}(0)}{\hat{\varphi}'(0)}.$$

So $S\varphi' = c\varphi$ with $c = \widehat{S\varphi'}(0)/\hat{\varphi}'(0)$, a contradiction.

On the other hand, from (3.1) we get

$$\bar{\hat{\varphi}}(x)\widehat{S\varphi'}(x) = h_0(x) - \sum_{k \neq 0} \bar{\hat{\varphi}}(x + 2k\pi)\widehat{S\varphi'}(x + 2k\pi). \tag{3.2}$$

$\bar{\hat{\varphi}}$ and h_0 are continuous at $x = 0$. We prove that the sum in (3.2) converges to 0 as $x \to 0$. By the Schwarz inequality

$$\left| \sum_{k \neq 0} \bar{\hat{\varphi}} \widehat{S\varphi'}(x + 2k\pi) \right| \leqslant \left(\sum_{k \neq 0} |\hat{\varphi}|^2(x + 2k\pi) \right) \left(\sum_{k \neq 0} |\widehat{S\varphi'}|^2(x + 2k\pi) \right). \tag{3.3}$$

We try to bound the second factor. Since S commutes with π and U, the same is true for S^* and S^*S. By Theorem 2.7, S^*S induces some h_0' with $R_{m_0', m_0'} h_0' = h_0'$, $|h_0'|^2 \leqslant ch'^2$ for some $c > 0$ and

$$\langle \varphi' \mid \pi(f) S^* S\varphi' \rangle = \int_\mathbb{T} f h_0' \, d\mu \quad (f \in L^\infty(\mathbb{T})).$$

Then

$$\langle S\varphi' | \pi(f) S\varphi' \rangle = \int_{\mathbb{T}} f h_0' \, d\mu.$$

Using again the Fourier transform and periodization we obtain

$$\mathrm{Per} |\widehat{S\varphi'}|^2 = h_0' \leqslant \sqrt{c} h'.$$

Since φ' generates a Riesz basis (see [6]), there is a $B < \infty$ such that

$$h' = \mathrm{Per} |\hat{\varphi}'|^2 \leqslant B.$$

Thus $\mathrm{Per} |\widehat{S\varphi'}|^2$ is bounded and, using the hypothesis (iii), in (3.3) we obtain that

$$\lim_{x \to 0} \left| \sum_{k \neq 0} \bar{\hat{\varphi}} \widehat{S\varphi'}(x + 2k\pi) \right| = 0.$$

Now apply this to (3.2) and use the fact that $\hat{\varphi}$ is continuous at 0 with $\hat{\varphi}(0) \neq 0$ to conclude that $\widehat{S\varphi'}$ is continuous 0, again a contradiction which leads to $S\varphi' = c\varphi'$. Without loss of generality, take the constant c to be 1.

$$\int_{\mathbb{T}} f h_0 \, d\mu = \langle \varphi | \pi(f) S\varphi' \rangle = \langle \varphi | \pi(f) \varphi' \rangle = \int_{\mathbb{T}} f \, d\mu,$$

the last equality follows from (iv) using the usual Fourier-periodization technique. The equality holds for all $f \in L^\infty(\mathbb{T})$ so $h_0 = \mathbb{1}$. $\square$

Corollary 3.2. *If φ and φ' are compactly supported, biorthogonal,*

$$m_0(0) = \sqrt{N} = m_0'(0),$$
$$m_0\left(\frac{2k\pi}{N}\right) = 0 = m_0'\left(\frac{2k\pi}{N}\right), \quad k \in \{1, \dots, N-1\},$$

then $\mathbb{1}$ is the only solution of $R_{m_0, m_0'} h_0 = h_0$ which is continuous at $z = 1$ (up to a multiplicative constant).

Proof. We have to check the conditions of Theorem 3.1. The Fourier coefficients of $\mathrm{Per} |\hat{\varphi}|^2$ are given by

$$\int_{\mathbb{T}} e_k \, \mathrm{Per} |\hat{\varphi}|^2 \, d\mu = \langle \varphi | \pi(e_k) \varphi \rangle = \langle \varphi | \varphi(\omega - k) \rangle,$$

where $e_k = e^{-ik\theta}$. Therefore the coefficients are zero except for a finite number of them, so $\mathrm{Per} |\hat{\varphi}|^2$ is a trigonometric polynomial. The same is true for $\mathrm{Per} |\hat{\varphi}'|^2$. Also the fact that φ and φ' are compactly supported implies that $\hat{\varphi}, \hat{\varphi}'$ are continuous. From the scaling equation we obtain that $\hat{\varphi}(0) = 1 = \hat{\varphi}'(0)$.

For $k \in \mathbb{Z}$, $k \neq 0$, we can write $k = N^p l$ with $l = Nq + r$, $r \in \{1, \ldots, N-1\}$. Then

$$
\begin{aligned}
\hat{\varphi}(2k\pi) &= \hat{\varphi}(2N^p l\pi) \\
&= \frac{m_0(2N^{p-1}l\pi)}{\sqrt{N}} \cdots \frac{m_0(2Nl\pi)}{\sqrt{N}} \frac{m_0(2l\pi)}{\sqrt{N}} \frac{m_0(2l\pi/N)}{\sqrt{N}} \hat{\varphi}\left(\frac{2l\pi}{N}\right) \\
&= 0
\end{aligned}
$$

because

$$
m_0\left(\frac{2l\pi}{N}\right) = m_0\left(\frac{2(Nq+r)\pi}{N}\right) = m_0\left(\frac{2r\pi}{N}\right) = 0.
$$

Thus $\hat{\varphi}(2k\pi) = 0$ for $k \neq 0$. This shows that $\mathrm{Per}\,|\hat{\varphi}|^2(0) = 1$. Then

$$
\sum_{k \neq 0} |\hat{\varphi}|^2(x + 2k\pi) = \mathrm{Per}\,|\hat{\varphi}|^2(x) - \hat{\varphi}(x) \to \mathrm{Per}\,|\hat{\varphi}|^2(0) - \hat{\varphi}(0) = 0
$$

as $x \to 0$.

The same argument applies to $\hat{\varphi}'$.

Condition (ii) is obtained as follows: look at the first inequality in the proof of Theorem 3.1. We have

$$
1 \leqslant \left(\mathrm{Per}\,|\hat{\varphi}|^2\right)\left(\mathrm{Per}\,|\hat{\varphi}'|^2\right)(\omega).
$$

As both factors are trigonometric polynomials they are bounded by a common constant $0 < A < \infty$. Hence

$$
1/A \leqslant \left(\mathrm{Per}\,|\hat{\varphi}|^2\right) \leqslant A
$$

which implies that the translates of φ form a Riesz basis for their linear span (see [6]). Similarly for φ'. $\square$

4. Some examples

We know that in the case of a quadrature mirror filter m_0 for which the transfer operator R_{m_0,m_0} has 1 as a simple eigenvalue in $C(\mathbb{T})$, the cyclic representation that corresponds to the constant function $\mathbb{1}$ is in fact the wavelet representation on $L^2(\mathbb{R})$. Then the commutant of this representation is in one-to-one correspondence with $L^\infty(\mathbb{T})$-solutions for $R_{m_0,m_0} h = h$. We will describe this commutant and give the form of all corresponding $L^\infty(\mathbb{T})$-solutions.

We recall that the wavelet representation of $\mathfrak{A}_N$ is generated by

$$
U : \psi \mapsto \frac{1}{\sqrt{N}} \psi\left(\frac{x}{N}\right), \qquad V : \psi \mapsto \psi(x - 1),
$$

$V = \pi(z)$ where π is the representation of $L^\infty(\mathbb{T})$ given by

$$\left(\pi(f)\psi\right)\hat{} = f\hat{\psi} \quad \left(f \in L^\infty(\mathbb{T}), \ \psi \in L^2(\mathbb{R})\right).$$

It will be useful to consider the representation in Fourier space and in this case U has the form

$$\widehat{U}\hat{\psi}(x) = \sqrt{N}\hat{\psi}(Nx). \tag{4.1}$$

Thus we have another representation of $\mathfrak{A}_N$ on $L^2(\mathbb{R})$, which is generated by $\widehat{U}$ given in (4.1) and the representation $\hat{\pi}$ of $L^\infty(\mathbb{T})$ given by

$$\hat{\pi}(f)\hat{\psi} = f\hat{\psi} \quad \left(f \in L^\infty(\mathbb{T}), \ \hat{\psi} \in L^2(\mathbb{R})\right).$$

This representation is equivalent to the wavelet representation via the Fourier transform.

Theorem 4.1. *The commutant of* $(\hat{\pi}, \widehat{U}, L^2(\mathbb{T}))$ *is*

$$\left\{M_f \mid f \in L^\infty(\mathbb{R}), \ f(Nx) = f(x) \ a.e. \ on \ \mathbb{R}\right\},$$

where $M_f(\psi) = f\psi$ *for all* $\psi \in L^2(\mathbb{R})$, $f \in L^\infty(\mathbb{R})$.

Proof. (The theorem is valid even in a more general case, see [9]). Let A be an operator that commutes with $\widehat{U}$ and $\hat{\pi}(f)$ for all $f \in L^\infty(\mathbb{T})$. We prove first that A commutes with M_g, where $g \in L^\infty(\mathbb{R})$ is periodic of period $2N\pi$.

Indeed, let $f(x) = g(Nx)$ for $x \in \mathbb{R}$. Then f is 2π-periodic and bounded, so $\hat{\pi}(f)$ commutes with A. Then A commutes also with $\widehat{U}^{-1}\hat{\pi}(f)\widehat{U}$. But

$$\left(\widehat{U}^{-1}\hat{\pi}(f)\widehat{U}\psi\right)(x) = \widehat{U}^{-1}\left(f(x)\sqrt{N}\psi(Nx)\right)$$

$$= f\left(\frac{x}{N}\right)\psi(x) = g(x)\psi(x).$$

So $\widehat{U}^{-1}\hat{\pi}(f)\widehat{U} = M_g$. It follows by induction that A commutes with any multiplication by a $2\pi N^k$-periodic function in $L^\infty(\mathbb{R})$. Now, consider $f \in L^\infty(\mathbb{R})$. We claim that A commutes with M_f. Define $f_n(x) = f(x)$ on $[-\pi N^n, \pi N^n]$ and extended it on $\mathbb{R}$ by $2\pi N^n$-periodicity. First we prove that M_{f_n} converges to M_f in the strong operator topology. For this, take $\psi \in L^2(\mathbb{R})$. Then

$$\|M_{f_n}\psi - M_f\psi\|^2 = \int_{\mathbb{R}} |f_n - f|^2|\psi|^2 \, dx = \int_{|x|\geqslant \pi N^n} |f_n - f|^2|\psi|^2 \, dx$$

$$\leqslant \left(2\|f\|_\infty\right)^2 \int_{\mathbb{R}} \chi_{\{|x|\geqslant \pi N^n\}}|\psi|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Thus M_f is the limit of M_{f_n} in the strong operator topology and, consequently, A will commute with M_f also. Using then Theorem IX.6.6 in [5] we

obtain that $A = M_f$ for some $f \in L^\infty(\mathbb{R})$. Then, the fact that A and $\widehat{U}$ commute implies

$$f(Nx) = f(x) \quad \text{a.e. on } \mathbb{R}.$$

This proves one inclusion, the other one is a straightforward verification. $\square$

Using this theorem we can find all solutions to $R_{m_0, m_0'} h = h$ as follows:

Corollary 4.2. *Suppose we have the wavelet representations* $(U, \pi, L^2(\mathbb{R}), \varphi, m_0)$ *and* $(U, \pi, L^2(\mathbb{R}), \varphi', m_0')$. *Let* $h = \text{Per}(|\hat{\varphi}|^2)$ *and* $h' = \text{Per}(|\hat{\varphi}'|^2)$. *Then each solution* $h_0 \in L^1(\mathbb{R})$ *for* $R_{m_0, m_0'} h_0 = h_0$ *with* $|h_0|^2 \leqslant chh'$ *for some positive constant* c *has the form*

$$h_0 = \text{Per}(f \bar{\hat{\varphi}} \hat{\varphi}')$$

for some $f \in L^\infty(\mathbb{R})$ *with* $f(Nx) = f(x)$ *a.e.*
    Conversely, any such h_0 *is an* $L^1(\mathbb{R})$-*solution for* $R_{m_0, m_0'} h_0 = h_0$ *and* $|h_0|^2 \leqslant chh'$ *for some* $c > 0$.

Proof. The cyclic representations corresponding to h and h' are the wavelet representations given in the hypothesis. Hence the intertwining operators are in fact the ones in the commutant of the wavelet representation. We will transfer everything into the Fourier space by applying the Fourier transform and then use Theorem 4.1 and the results in Section 2.

If h_0 is as given then there is an operator A in the commutant of the representation such that

$$\langle \varphi | \pi(g) A \varphi' \rangle = \int_{\mathbb{T}} g h_0 \, d\mu \quad (g \in L^\infty(\mathbb{T})).$$

But after the Fourier transform, we saw that $\widehat{A}$ has the form $\widehat{A} = M_f$ where $f \in L^\infty(\mathbb{R})$ with $f(Nx) = f(x)$ a.e. Therefore

$$\frac{1}{2\pi} \langle \hat{\varphi} | g f \hat{\varphi}' \rangle = \int_{\mathbb{T}} g h_0 \, d\mu \quad (g \in L^\infty(\mathbb{T})),$$

and after periodization we obtain that

$$h_0 = \text{Per}(f \bar{\hat{\varphi}} \hat{\varphi}').$$

Conversely, it is easy to see that, when f is given,

$$\frac{1}{2\pi} \langle \hat{\varphi} | g M_f \hat{\varphi}' \rangle = \int_{\mathbb{T}} g h_0 \, d\mu \quad (g \in L^\infty(\mathbb{T})),$$

and as M_f is in the commutant, the rest follows. $\square$

Example 4.3. In the sequel we consider $N = 2$, $m_0(z) = (1/\sqrt{2})(1 + z^p)$, p being an odd integer. This example appears also in [1]. We try to find out the solutions for $R_{m_0,m_0}h = h$.

It is easy to see that

$$R_{m_0,m_0}\mathbb{1} = \mathbb{1}.$$

Also if $\varphi = (1/p)\chi_{(0,p)}$ then

$$U\varphi = \pi(m_0)\varphi.$$

Then the wavelet representation $(U, \pi, L^2(\mathbb{T}), \varphi, m_0)$ is the cyclic representation corresponding to $h_\varphi = \mathrm{Per}(|\hat\varphi|^2)$,

$$\hat\varphi(x) = e^{-ipx/2}\frac{\sin(px/2)}{px/2}.$$

Using the identity

$$\sum_{n\in\mathbb{Z}}\frac{1}{(t + 2\pi n)^2} = \frac{1}{4\sin^2(t/2)} \quad (t \in \mathbb{R}) \tag{4.2}$$

we obtain

$$h_\varphi(t) = \frac{1}{p^2}\frac{\sin^2(pt/2)}{\sin^2(t/2)} \quad (t \in \mathbb{R}).$$

We try to construct the cyclic representation corresponding to $\mathbb{1}$. Let $\rho = e^{i2\pi/p}$ and for $\eta \in \mathbb{T}$ define $\alpha_\eta(f)(z) = f(\eta z)$ for all $f \in L^\infty(\mathbb{T})$ and $z \in \mathbb{T}$. The essential observation is that we have the following identity:

$$\sum_{k=0}^{p-1}\alpha_{\rho^k}(h_\varphi) = 1 \quad \text{on } \mathbb{T}. \tag{4.3}$$

This identity follows from the following computation:

$$\sum_{k=0}^{p-1}\alpha_{\rho^k}(h_\varphi)(t) = \sum_{k=0}^{p-1}h_\varphi\left(t + \frac{2k\pi}{p}\right)$$

$$= \frac{4}{p^2}\sin^2\left(\frac{pt}{2}\right)\sum_{k=0}^{p-1}\sum_{n\in\mathbb{Z}}\frac{1}{(t + 2k\pi/p + 2n\pi)^2}$$

$$= 4\sin^2\left(\frac{pt}{2}\right)\sum_{l\in\mathbb{Z}}\frac{1}{(pt + 2\pi l)^2} \quad (l = k + pn)$$

$$= 1 \quad \left(\text{using (4.2)}\right).$$

We construct now the cyclic representation that corresponds to $\mathbb{1}$. Let

$$\mathcal{H}_1 = \underbrace{L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})}_{p \text{ times}}.$$

For $f \in L^\infty(\mathbb{T})$ define

$$
\begin{aligned}
&\pi_1(f)(\xi_0, \ldots, \xi_{p-1}) \\
&= \big(\pi\big(\alpha_{\rho^0}(f)\big)(\xi_0), \pi\big(\alpha_{\rho^1}(f)\big)(\xi_1), \ldots, \pi\big(\alpha_{\rho^{p-1}}(f)\big)(\xi_{p-1})\big), \\
&U_1(\xi_0, \ldots, \xi_{p-1}) = \big(U\xi_{\sigma(0)}, U\xi_{\sigma(1)}, \ldots, U\xi_{\sigma(p-1)}\big),
\end{aligned}
$$

where π and U come from the wavelet representation on $L^2(\mathbb{R})$ and σ is the permutation of the set $\{0, \ldots, p-1\}$ given by $\rho^{2k} = \rho^{\sigma(k)}$, so $\sigma(k) = 2k \bmod p$.

π_1 is a representation of $L^\infty(\mathbb{T})$, U_1 is unitary, and a short computation, based on the fact that $\alpha_{\rho^k}(f(z^2)) = (\alpha_{\rho^{\sigma(k)}}(f))(z^2)$, shows that

$$
U_1 \pi_1(f) = \pi_1\big(f(z^2)\big) U_1 \quad \big(f \in L^\infty(\mathbb{T})\big).
$$

Define $\varphi_1 = (\varphi, \varphi, \ldots, \varphi)$. Since $\alpha_{\rho^k}(m_0) = m_0$ for all $k \in \{0, \ldots, p-1\}$ and $U\varphi = \pi(m_0)\varphi$,

$$
U_1 \varphi_1 = \pi_1(m_0)\varphi_1.
$$

And, finally, we check that $\mathbb{1}$ is the eigenfunction that induces this representation. As $\langle \varphi | \pi(f)\varphi \rangle = \int_{\mathbb{T}} f h_\varphi \, d\mu$, we have

$$
\begin{aligned}
\big\langle \varphi_1 \big| \pi_1(f)\varphi_1 \big\rangle &= \sum_{k=0}^{p-1} \int_{\mathbb{T}} \alpha_{\rho^k}(f) h_\varphi \, d\mu = \sum_{k=0}^{p-1} \int_{\mathbb{T}} f \alpha_{\rho^{-k}}(h_\varphi) \, d\mu \\
&= \int_{\mathbb{T}} f \sum_{l=0}^{p-1} \alpha_{\rho^l}(h_\varphi) \, d\mu = \int_{\mathbb{T}} f \mathbb{1} \, d\mu \quad \text{(by (4.3))}.
\end{aligned}
$$

Now we Fourier transform everything and try to find the commutant of the representation. After the Fourier transform, $\hat{\pi}_1$ and $\widehat{U}_1$ have the same form as before, only now, $\hat{\pi}(f)$ is the multiplication by f ($f \in L^\infty(\mathbb{T})$), and

$$
\widehat{U}\psi(x) = \sqrt{2}\psi(2x) \quad \big(\psi \in L^2(\mathbb{R})\big).
$$

So consider $A : \mathcal{H}_1 \to \mathcal{H}_1$ that commutes with the representation, $A = (A_{ij})_{i,j=0}^{p-1}$.

Consider $g \in L^\infty(\mathbb{T})$ of period $2\pi/p$. Then $\alpha_{\rho^k}(g) = g$ for all k. Hence

$$
\hat{\pi}_1(g)(\xi_0, \ldots, \xi_{p-1}) = \big(\hat{\pi}(g)(\xi_0), \ldots, \hat{\pi}(g)(\xi_{p-1})\big).
$$

Also, since σ is permutation, there is an M such that $\sigma^M(k) = k$ for all k so

$$
\widehat{U}_1^M(\xi_0, \ldots, \xi_{p-1}) = \big(\widehat{U}^M \xi_0, \ldots, \widehat{U}^M \xi_{p-1}\big).
$$

Note that P_i, the projection on the ith component, commutes with $\hat{\pi}_1(g)$ and $\widehat{U}_1^M$. Then $P_i A P_j$ commutes with $\hat{\pi}_1(g)$ and $\widehat{U}_1^M$, and this implies that A_{ij} commutes with $\hat{\pi}(g)$ and $\widehat{U}^M$. Repeating the argument in Theorem 4.1, we obtain that $A_{ij} = M_{f_{ij}}$ for some $f_{ij} \in L^\infty(\mathbb{R})$.

Now take $f \in L^\infty(\mathbb{T})$ arbitrary. The fact that $\hat{\pi}_1(f)$ and A commute can be rewritten as

$$\sum_{j=0}^{p-1} f_{ij} \alpha_{\rho^j}(f) \xi_j = \alpha_{\rho^i}(f) \sum_{j=0}^{p-1} f_{ij} \xi_j \quad \left(i \in \{0, \dots, p-1\}, \; (\xi_j) \in \mathcal{H}_1\right).$$

Fix k and let $\xi_j = 0$ for $j \neq k$. Then

$$f_{ik} \alpha_{\rho^k}(f) \xi_k = \alpha_{\rho^i}(f) f_{ik} \xi_k,$$

and this implies that $f_{ik} = 0$ for $i \neq k$.

Now the fact that A commutes with $\widehat{U}_1$ implies

$$f_{ii}(x) = f_{\sigma(i)\sigma(i)}(2x), \quad \text{a.e. on } \mathbb{R} \quad \left(i \in \{0, \dots, p-1\}\right).$$

A simple check shows that any A of this form commutes with the representation, therefore the commutant is given by

$$\hat{\pi}_1' = \Big\{ A : \mathcal{H}_1 \to \mathcal{H}_1 \mid A(\xi_0, \dots, \xi_{p-1}) = (f_0 \xi_0, \dots, f_{p-1} \xi_{p-1}),$$
$$f_i \in L^\infty(\mathbb{R}), \; f_i(x) = f_{\sigma(i)}(2x) \text{ a.e.} \Big\}.$$

Using this and Section 2 we obtain that the $L^\infty(\mathbb{T})$-solutions for $R_{m_0,m_0} h = h$ must satisfy the identity

$$\frac{1}{2\pi} \sum_{k=0}^{p-1} \langle \hat{\varphi} | \alpha_{\rho^k}(f) f_k \hat{\varphi} \rangle = \int_{\mathbb{T}} f h \, d\mu \quad \left(f \in L^\infty(\mathbb{T})\right)$$

for some $f_k \in L^\infty(\mathbb{R})$ with $f_k(x) = f_{\sigma(k)}(2x)$ a.e. on $\mathbb{R}$. So

$$h = \sum_{k=0}^{p-1} \mathrm{Per}\left(f_k\left(x + \frac{2k\pi}{p}\right) |\hat{\varphi}|^2\left(x + \frac{2k\pi}{p}\right) \right).$$

To conclude our analysis we try to find the continuous eigenfunctions. So let us take h continuous, having the form mentioned above. We want to see what conclusions can be drawn on f_i. We prove that they are constants.

Fix $k \in \{0, \dots, p-1\}$.

$$h(x) = f_k\left(x + \frac{2k\pi}{p}\right) |\hat{\varphi}|^2\left(x + \frac{2k\pi}{p}\right)$$
$$+ \sum_{l \in \mathbb{Z} \setminus \{0\}} f_k\left(x + \frac{2k\pi}{p} + 2l\pi\right) |\hat{\varphi}|^2\left(x + \frac{2k\pi}{p} + 2l\pi\right)$$
$$+ \sum_{j \in \{0, \dots, p-1\} \setminus \{k\}} \sum_{l \in \mathbb{Z}} f_j\left(x + \frac{2j\pi}{p} + 2l\pi\right) |\hat{\varphi}|^2\left(x + \frac{2j\pi}{p} + 2l\pi\right).$$

Denote the first sum by $A(x)$ and the second one by $B(x)$.

$$|A(x)| \leqslant M \sum_{l \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}|^2 \left(x + \frac{2k\pi}{p} + 2l\pi \right),$$

where $M = \sup_i \sup_x |f_i(x)|$. This last sum is a continuous function, as it is the difference between $h_\varphi(x + 2k\pi/p)$ and $|\hat{\varphi}|^2(x + 2k\pi/p)$, and its value at $-2 \times k\pi/p$ is 0, since

$$|\hat{\varphi}|^2 \left(-\frac{2k\pi}{p} + \frac{2k\pi}{p} + 2l\pi \right) = 0, \quad l \neq 0.$$

Similarly,

$$|B(x)| \leqslant M \sum_{j \in \{0, \ldots, p-1\} \setminus \{k\}} \text{Per}\, |\hat{\varphi}|^2 \left(x + \frac{2j\pi}{p} \right)$$

and also this function is continuous and 0 at $x = -2k\pi/p$ (look at h_φ). Thus

$$A(x) + B(x) \to 0, \quad \text{as } x \to -\frac{2k\pi}{p}.$$

Since h is continuous, the following identity holds:

$$\lim_{x \to -2k\pi/p} f_k \left(x + \frac{2k\pi}{p} \right) |\hat{\varphi}|^2 \left(x + \frac{2k\pi}{p} \right) = h \left(-\frac{2k\pi}{p} \right).$$

But $|\hat{\varphi}|^2$ is also continuous and $|\hat{\varphi}|^2(0) = 1$ so

$$\lim_{x \to 0} f_k(x) = h \left(-\frac{2k\pi}{p} \right).$$

On the other hand, since σ is a permutation, there is a K such that $\sigma^K(i) = i$ for all $i \in \{0, \ldots, p-1\}$. Then by induction $f_k(x) = f_{\sigma^K(k)}(2^K x) = f_k(2^K x)$ a.e. on $\mathbb{R}$. This, coupled with the limit at 0, makes f_k constant a_k. Then

$$h = \sum_{k=0}^{p-1} a_k \text{Per}\, |\hat{\varphi}|^2 \left(x + \frac{2k\pi}{p} \right)$$

so

$$h = \sum_{k=0}^{p-1} a_k \alpha_{\rho^k}(h_\varphi)$$

with $a_k = a_{\sigma(k)}$ for all $k \in \{0, \ldots, p-1\}$.

We want to give a basis for the space of the continuous eigenfunctions h. For this, note that a_k is constant for k in a cycle of σ. So let $O_1, \ldots, O_{c(p)}$ be the cycles of σ. Then each continuous h will have the form

$$h = \sum_{k=1}^{c(p)} b_k \sum_{l \in O_k} \alpha_{\rho^l}(h_\varphi).$$

The functions $\sum_{l \in O_k} \alpha_{\rho^l}(h_\varphi)$ can be seen to be linearly independent if we observe that the set of zeroes are $\{\rho^{-l} \mid l \in \{0, \dots, p-1\} \setminus O_k\}$.

Also, it is easy to see that the cyclic representation associated to

$$h_{O_k} = \sum_{l \in O_k} \alpha_{\rho^l}(h_\varphi)$$

is given by $P_{O_k}\pi_1$, $P_{O_k}U_1$, where P_{O_k} is the projection on the components in O_k.

Take now, for example, $p = 9$. So $m_0(z) = (1/\sqrt{2})(1 + z^9)$, $\varphi = (1/9)\chi_{(0,9)}$,

$$h_\varphi(t) = \frac{1}{9^2} \frac{\sin^2(9t/2)}{\sin^2(t/2)}.$$

It induces the wavelet representation on $L^2(\mathbb{R})$:

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

The cycles are $O_1 = \{0\}$, $O_2 = \{1, 2, 4, 5, 7, 8\}$, $O_3 = \{3, 6\}$. $h_{O_1} = h_\varphi$ of course.

Observe that

$$h_{O_1}(x) + h_{O_3}(x) = \frac{4}{9^2} \sum_{k=0}^{2} \sum_{n \in \mathbb{Z}} \frac{\sin^2\left(\frac{9(x+2k\pi/3+2n\pi)}{2}\right)}{(x + 2k\pi/3 + 2n\pi)^2}$$

$$= \frac{4}{3^2} \sin^2\left(\frac{9x}{2}\right) \sum_{k=0}^{2} \sum_{n \in \mathbb{Z}} \frac{1}{(3x + 2\pi(k + 3n))^2}$$

$$= \frac{4}{3^2} \sin^2\left(\frac{9x}{2}\right) \sum_{l \in \mathbb{Z}} \frac{1}{(3x + 2\pi l)^2}$$

$$= \frac{1}{3^2} \frac{\sin^2(9x/2)}{\sin^2(3x/2)}.$$

Also

$$h_{O_1} + h_{O_2} + h_{O_3} = \mathbb{1}.$$

Therefore a basis for the continuous eigenfunctions is

$$\left\{ \mathbb{1}, \frac{1}{3^2} \frac{\sin^2(9x/2)}{\sin^2(3x/2)}, \frac{1}{9^2} \frac{\sin^2(9x/2)}{\sin^2(x/2)} \right\}.$$

Acknowledgments

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