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A generalized preimage for the digital analytical hyperplane recognition

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ABSTRACT

A new digital hyperplane recognition method is presented. This algorithm allows the recognition of digital analytical hyperplanes, such as Naive, Standard and Supercover ones. The principle is to incrementally compute in a dual space the generalized preimage of the ball set corresponding to a given hypervoxel set according to the chosen digitization model. Each point in this preimage corresponds to a Euclidean hyperplane the digitization of which contains all given hypervoxels. An advantage of the generalized preimage is that it does not depend on the hypervoxel locations. Moreover, the proposed recognition algorithm does not require the hypervoxels to be connected or ordered in any way.

1. Introduction

In digital geometry, objects are usually considered as digital point or hypervoxel (*pixels* in 2D and *voxels* in 3D) sets. Indeed, this is the structural decomposition mostly used to store digital information. A drawback of this kind of representation is that it does not provide any information on the shape or topology of digital objects. Another way of obtaining the description of digital objects is hyperplane decomposition. This process, called *digital hyperplane recognition*, consists of determining if a digital point set forms a hyperplane segment, that is a hyperplane bounded region.

The recognition problem has so far mainly been studied in dimensions 2 and 3 (see [1,2] for an overview on 2D recognition algorithms and digital planarity), with various approaches such as linear programming techniques [3,4], computational geometry methods [5–7] or *preimage* computation based algorithms [8,9]. Very few papers handle the problem in arbitrary dimensions [10–12]. Computational and efficiency aspects of digital hyperplane recognition problems are investigated in [13].

The present paper is an extension of [9] in which we propose a generalized approach for the recognition of digital analytical hyperplanes such as Naive, Standard and Supercover hyperplanes using generalized preimages. Informally, the preimage [14] of a hypervoxel set consists of all Euclidean hyperplanes the digitization of which contains the given hypervoxels. More precisely, the preimage of a hypervoxel set is computed in a dual space where each point is mapped onto a Euclidean hyperplane. Preimage computation algorithms depending on the hypervoxel locations have been proposed in dimensions 2 and 3 [8,15].

In this work, we perform the recognition of digital analytical hyperplanes by computing the set of Euclidean hyperplanes which intersect the ball set associated to a given hypervoxel set according to the chosen digitization model. In order to do that, we incrementally compute the *generalized preimage* of the balls corresponding to the hypervoxels. This preimage is defined in any dimension and is independent of the hypervoxel connectivity and location. More precisely, it is computed from the dual of the ball corresponding to each hypervoxel. Indeed, each point in this dual object corresponds to a Euclidean hyperplane which cuts the ball corresponding to the hypervoxel. Hence, a major part of this paper is devoted to determining the formulas describing the dual of a polytope in order to compute the one corresponding to the balls associated to an

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analytical digitization model. First, a positive and a negative extrusion are defined. Then, we show that the dual of a polytope can be computed from the extrusions of the dual of its vertices. Finally, the intersection of all ball duals forms the generalized preimage. The recognition process consists therefore simply in computing the generalized preimage of a ball set corresponding to a hypervoxel set (i.e. computing the dual of a ball set corresponding to a hypervoxel set). More precisely, we start with the dual of a ball corresponding to a hypervoxel and add the duals of the balls corresponding to the other hypervoxels as long as the generalized preimage is not empty.

In Section 2, we introduce some notations and definitions as well as the Naive, Standard and Supercover analytical hyperplane descriptions. In Section 3, we determine the dual of a polytope and introduce the notion of generalized preimage of a polytope set. Then, we explain in Section 4 how our digital analytical hyperplane recognition algorithm works. We especially focus on the Naive, Standard and Supercover hyperplane cases. Conclusion and future works are proposed in Section 5.

2. Preliminaries

In this section, we first propose some notations and give the definitions of a hypervoxel and a ball. Then, we present four digitization analytical models considered in this work: the Naive and closed Naive models, the Standard model and the Supercover model.

2.1. Notations and definitions

Let $n \in \mathbb{Z}, n > 0$. In the following, we will denote by $\mathcal{E}_n$ the classical n-dimensional Euclidean space, and by $[\![1, k]\!]$ the subset of integer values $\{1, \dots, k\} \subset \mathbb{Z}$. Moreover, a point with integer-valued coordinates $p \in \mathbb{Z}^n$ will be called a *digital point*.

We define an α*-hypercube*, $\alpha \in \mathbb{R}$, as follows:

Definition 1 (*Hypervoxel*). The hypervoxel (or n-dimensional cube) centered on the digital point $(c_1, \dots, c_n) \in \mathbb{Z}^n$, is the set of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ verifying

$$\forall i \in [\![1, n]\!], \quad c_i - \frac{1}{2} \leq x_i \leq c_i + \frac{1}{2}.$$

Hypervoxels in dimensions 2 and 3 are respectively called *pixels* and *voxels*.

Definition 2 (*Ball*). Let d be a distance in $\mathbb{R}^n$. Then, the ball $B_d(c, r)$ with center $c \in \mathbb{R}^n$ and radius $r \in \mathbb{R}$ is defined by

$$B_d(c, r) = \{x \in \mathbb{R}^n | d(c, x) \leq r\}.$$

2.2. Discrete analytical models

In this work, we study four digital analytical models: the *Naive model* [16,17], the *closed Naive model* [18], the *Standard model* [19] and the *Supercover model* [20,21]. These models are defined in any dimension and provide a digitization of Euclidean objects. Moreover, a distance and a ball is associated to each model.

In this section, we give for each model the definition of the digital hyperplane (or n-dimensional planes) and describe precisely the digitization of a Euclidean hyperplane according to the distance and the ball associated to the model.

2.2.1. The Naive models [16,18]

Naive and closed Naive hyperplanes are defined analytically as follows (see Fig. 1):

Definition 3 (*Naive Hyperplane [16]*). The Naive hyperplane with parameters $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ is the set of points $(x_1, \dots, x_n) \in \mathbb{Z}^n$ verifying

$$-\frac{\max\limits_{1 \leq i \leq n} |c_i|}{2} \leq c_0 + \sum_{i=1}^{n} c_i x_i < \frac{\max\limits_{1 \leq i \leq n} |c_i|}{2}$$

where $c_1 \geq 0$, or $c_1 = 0$ and $c_2 \geq 0$, or ..., or $c_1 = c_2 = \cdots = c_{n-1} = 0$ and $c_n \geq 0$.

Definition 4 (*Closed Naive Hyperplane [18]*). The closed Naive hyperplane with parameters $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ is the set of points $(x_1, \dots, x_n) \in \mathbb{Z}^n$ verifying

$$-\frac{\max\limits_{1 \leq i \leq n} |c_i|}{2} \leq c_0 + \sum_{i=1}^{n} c_i x_i \leq \frac{\max\limits_{1 \leq i \leq n} |c_i|}{2}.$$

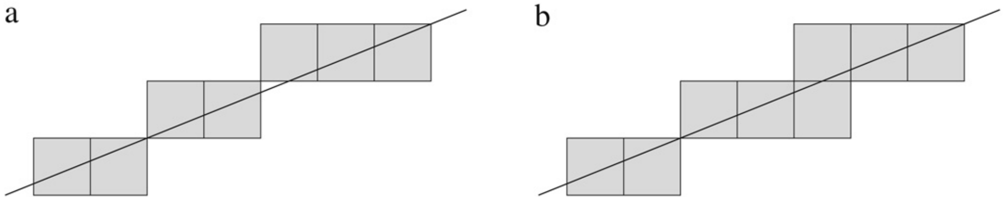

Fig. 1. Examples of Naive and closed Naive hyperplanes in dimension 2: (a) Naive line, (b) Closed Naive line.

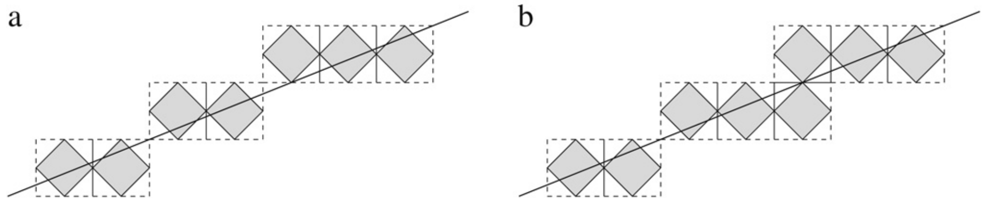

Fig. 2. Illustration of the balls associated to the Naive models: (a) Balls associated to a Naive line, (b) Balls associated to a closed Naive line.

One practical way of defining some discrete analytical models is to define them with a distance. Let $p = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $p' = (x'_1, \dots, x'_n) \in \mathbb{R}^n$. The distance associated to the closed Naive model is the distance d_1 defined by $d_1(p, p') = \sum_{i=1}^n |x_i - x'_i|$ and the corresponding ball is $B_{d_1}(c, \frac{1}{2})$, $c \in \mathbb{Z}^n$. For instance in dimension 2, the ball $B_{d_1}(c, \frac{1}{2})$ is a regular rhombus. The closed Naive model $N(E)$ of a Euclidean object E can be defined by

$$N(E) = \left(B_{d_1}\left(0, \frac{1}{2}\right) \otimes E\right) \cap \mathbb{Z}^n.$$

The Minkowski sum of two sets A and B is defined by $A \otimes B = \{a+b | a \in A, b \in B\}$. Definition 4 and the one based on distance d_1 are equivalent for the hyperplane. The definition based on a distance is more general but the Definition 4 is analytical and thus more practical. The definition of the Naive hyperplane as given in Definition 3 is only analytical as there is no easy way of defining the Naive Hyperplane with a distance. As we can see however, there is only a minor difference between the two models. The main interest of the distance definition in the case of the hyperplane is that it provides a geometrical way for understanding the Closed Naive Hyperplane and the Naive hyperplane. The closed Naive digitization of a Euclidean hyperplane consists of the centers of all balls which are intersected by the hyperplane (see Fig. 2(b)), whereas the Naive one consists of the centers of all balls cut by the hyperplane except when a ball vertex is intersected (see Fig. 2(a)). In this case, several hypervoxels adjacent to the corresponding hypervoxel do not belong to the Naive digitization. This is due to the fact that one inequality in Definition 4 is strict.

Proposition 5. *Let B be a ball $B_{d_1}(c', \frac{1}{2})$, $c' \in \mathbb{Z}^n$, and let H be a Euclidean hyperplane with equation $c_0 + \sum_{i=1}^n c_i x_i = 0$ that passes through a vertex $v = (v_1, \dots, v_n)$ of B. Moreover, we assume that the first $c_i \neq 0$ verifies $c_i > 0$.*

Let $j \in [\![1, n]\!]$ such that $|c_j| = \max_{i=1}^n |c_i|$. Then, if $c_j > 0$ (resp. $c_j < 0$), the digital point $(v_1, \dots, v_{j-1}, v_j - \frac{1}{2}, v_{j+1}, \dots, v_n)$ (resp. $(v_1, \dots, v_{j-1}, v_j + \frac{1}{2}, v_{j+1}, \dots, v_n)$) belongs to the Naive digitization of H.

Proof. By definition, a digital point $p = (x_1, \dots, x_n)$ belonging to a Naive hyperplane verifies the following inequalities:

$$-\frac{\max\limits_{1\le i\le n} |c_i|}{2} \le c_0 + \sum_{i=1}^n c_i x_i < \frac{\max\limits_{1\le i\le n} |c_i|}{2}.$$

Since $|c_j| = \max_{i=1}^n |c_i|$, we want to determine $k \in \{-1, 1\}$ such that

$$-\frac{|c_j|}{2} = c_0 + \sum_{i=1, i\neq j}^n c_i v_i + c_j\left(v_j + \frac{1}{2}k\right).$$

Then, since $c_0 + \sum_{i=1}^n c_i v_i = 0$, we have

$$-\frac{|c_j|}{2} = \frac{1}{2}kc_j, k \in \{-1, 1\}.$$

Hence, if $c_j > 0$, we have

$$-\frac{c_j}{2} = \frac{1}{2}kc_j, k \in \{-1, 1\}$$

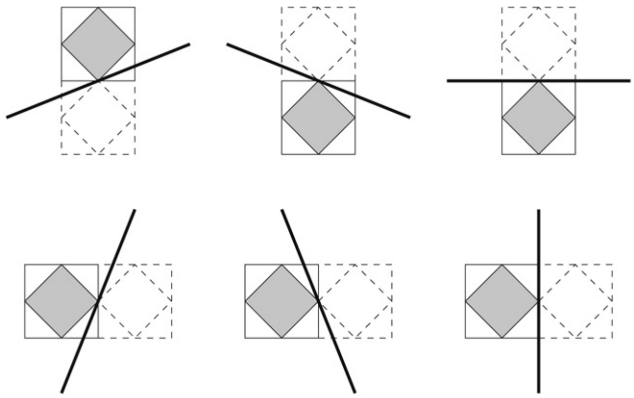

Fig. 3. Digital points belonging to the Naive digitization of a Euclidean line according to the slope of the line (in dark grey).

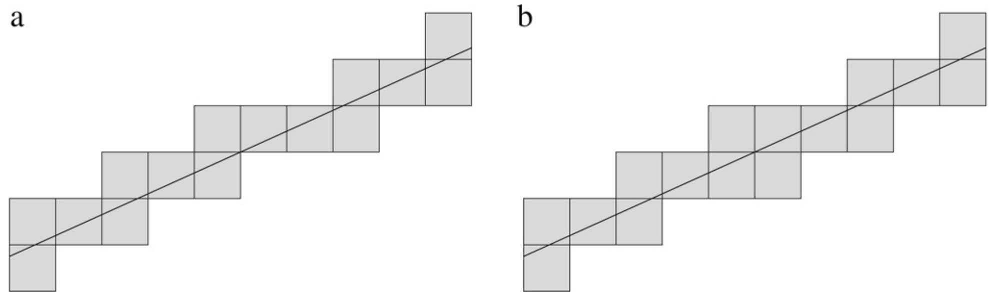

Fig. 4. Examples of Standard and Supercover hyperplanes in dimension 2: (a) Standard line, (b) Supercover line.

and so we deduce that $k = -1$. Else, if $c_j > 0$, we have

$$\frac{c_j}{2} = \frac{1}{2} k c_j,\ k \in \{-1, 1\}$$

and then we deduce that $k = 1$. $\square$

Proposition 5 is illustrated in Fig. 3.

2.2.2. The standard [19] and supercover [20,21] models

Standard and Supercover hyperplanes are defined analytically as follows (see Fig. 4):

Definition 6 (*Standard Hyperplane [19]*). The Standard hyperplane with parameters $(c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ is the set of points $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ verifying

$$-\frac{\sum_{i=1}^{n} |c_i|}{2} \leq c_0 + \sum_{i=1}^{n} c_i x_i < \frac{\sum_{i=1}^{n} |c_i|}{2}$$

where $c_1 \geq 0$, or $c_1 = 0$ and $c_2 \geq 0$, or ..., or $c_1 = c_2 = \cdots = c_{n-1} = 0$ and $c_n \geq 0$.

Definition 7 (*Supercover Hyperplane [21]*). The Supercover hyperplane with parameters $(c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ is the set of points $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ verifying

$$-\frac{\sum_{i=1}^{n} |c_i|}{2} \leq c_0 + \sum_{i=1}^{n} c_i x_i \leq \frac{\sum_{i=1}^{n} |c_i|}{2}.$$

If we replace the distance d_1 in the close Naive model definition based on the Minkowski sum, we obtain the Supercover model. Let $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n$. The distance associated to the Supercover models is the distance d_∞ defined by $d_\infty(p, p') = \sup_{i \in [\![1,n]\!]} |x_i - x'_i|$ and the corresponding ball is $B_{d_\infty}(c, 1)$, $c \in \mathbb{Z}^n$. In dimension n, the ball $B_{d_\infty}(c, 1)$ is a hypervoxel. Just as for the closed Naive and the Naive models, we have a geometrical interpretation based

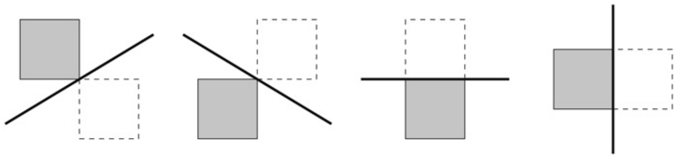

Fig. 5. Digital points belonging to the Naive digitization of a Euclidean line according to the slope of the line (in dark grey).

on the distance definition of the Supercover model. The Supercover digitization of a Euclidean hyperplane also consists of the centers of all hypervoxels which are intersected by the hyperplane (see Fig. 4(b)), whereas the Standard one consists of the centers of all hypervoxels cut by the hyperplane except when a hypervoxel vertex is intersected (see Fig. 4(a)). In this case, several hypervoxels adjacent to this vertex do not belong to the Standard digitization. This is due to the fact that one inequality in Definition 6 is strict.

Proposition 8. *Let B be a ball $B_{d_\infty}(c', 1)$, $c' \in \mathbb{Z}^n$, and let H be a Euclidean hyperplane with equation $c_0 + \sum_{i=1}^n c_i x_i = 0$ that passes through a vertex $v = (v_1, \ldots, v_n)$ of B. Moreover, we assume that the first $c_i \neq 0$ verifies $c_i > 0$.*

Then, each digital point $(x_1, \ldots, x_n)$ belonging to the standard digitization of H verifies for all $i \in [\![1, n]\!]$:

- $x_i = v_i + \frac{1}{2}$ *if* $c_i < 0$,
- $x_i = v_i - \frac{1}{2}$ *if* $c_i > 0$,
- $x_i = v_i + \frac{1}{2}$ *or* $x_i = v_i - \frac{1}{2}$ *if* $c_i = 0$.

Proof. By definition, a digital point $p = (x_1, \ldots, x_n)$ belonging to a Standard hyperplane verifies the following inequalities:

$$-\frac{\sum_{i=1}^n |c_i|}{2} \leq c_0 + \sum_{i=1}^n c_i x_i < \frac{\sum_{i=1}^n |c_i|}{2}.$$

We want to determine $k_i \in \{-1, 1\}$, $i \in [\![1, n]\!]$, such that

$$-\frac{\sum_{i=1}^n |c_i|}{2} = c_0 + \sum_{i=1}^n c_i \left(v_i + \frac{1}{2} k_i \right)$$

that is, since $c_0 + \sum_{i=1}^n c_i v_i = 0$,

$$-\frac{\sum_{i=1}^n |c_i|}{2} = \sum_{i=1}^n \frac{1}{2} k_i c_i.$$

Hence, we have

$$\sum_{i=1}^n (|c_i| - k_i c_i) = 0$$

and then

$$\sum_{i=1}^n (k'_i c_i - k_i c_i) = \sum_{i=1}^n (k'_i - k_i) c_i = 0$$

with $k'_i \in -1, 1$ and $k'_i c_i \geq 0$

However, since $\forall i \in [\![1, n]\!]$, $(k'_i - k_i) c_i \geq 0$ we deduce that

$$i \in [\![1, n]\!], k_i = k'_i$$

that is

- if $c_i > 0$ then $k_i = -1$,
- if $c_i < 0$ then $k_i = 1$,
- if $c_i = 0$ then $k_i = -1$ or $k_i = 1$. $\square$

Proposition 8 is illustrated in Fig. 5.

3. Dual of a polytope

In order to define the dual of a polytope, we use a dual transformation similar to the well known Hough transform which is an efficient tool usually used in image processing to recognize parametric shapes in an image. A review on existing variations of this method is presented in [22].

In the following two sections, we first define the parameter space in which our dual transformation is performed as well as the positive and negative extrusions of a point. Then, we describe the dual of a polytope and define the notion of generalized preimage, which is the basis of the recognition algorithm presented in Section 4.

3.1. Definitions and properties

In this work, we use the n-dimensional parameter space $\mathcal{P}_n \subset \mathbb{R}^n$, and define the two functions $\mathcal{D}_{\mathcal{E}} : \mathcal{E}_n \to \mathcal{P}_n$ and $\mathcal{D}_{\mathcal{P}} : \mathcal{P}_n \to \mathcal{E}_n$ by:

$$\mathcal{D}_{\mathcal{E}}(x_1, \ldots, x_n) = \left\{ (y_1, \ldots, y_n) \in \mathcal{P}_n | y_n = -\sum_{i=1}^{n-1} x_i y_i + x_n \right\}$$

$$\mathcal{D}_{\mathcal{P}}(y_1, \ldots, y_n) = \left\{ (x_1, \ldots, x_n) \in \mathcal{E}_n | x_n = \sum_{i=1}^{n-1} y_i x_i + y_n \right\}.$$

Informally, each point in $\mathcal{E}_n$ (resp. $\mathcal{P}_n$) is transformed by $\mathcal{D}_{\mathcal{E}}$ (resp. $\mathcal{D}_{\mathcal{P}}$) into a hyperplane in $\mathcal{P}_n$ (resp. $\mathcal{E}_n$). In the rest of this paper, we will generically write *Dual* for $\mathcal{D}_{\mathcal{E}}$ or $\mathcal{D}_{\mathcal{P}}$.

Definition 9 (*Dual Object*). Let O be a subset of $\mathbb{R}^n$. Then,

$$Dual(O) = \bigcup_{p \in O} Dual(p)$$

is called the *dual* of O.

Proposition 10. *Let O_1 and O_2 be two subsets of $\mathbb{R}^n$ such that $O_1 \subseteq O_2$. Then*

$$Dual(O_1) \subseteq Dual(O_2).$$

Proof. Since $O_1 \subseteq O_2$, we deduce that $Dual(O_2) = \bigcup_{p \in O_2} Dual(p) = \left[\bigcup_{p \in O_1} Dual(p)\right] \cup \left[\bigcup_{p \in O_2 \setminus O_1} Dual(p)\right]$. Then, $Dual(O_1) \subseteq Dual(O_2)$. $\square$

Moreover, the following properties can be deduced from our definition of the duality.

Proposition 11. *Let O_1 and O_2 be two subsets of $\mathbb{R}^n$. Then,*

$$Dual(O_1 \cup O_2) = Dual(O_1) \cup Dual(O_2).$$

Proof. $Dual(O_1 \cup O_2) = \bigcup_{p \in O_1 \cup O_2} Dual(p) = \left[\bigcup_{p \in O_1} Dual(p)\right] \cup \left[\bigcup_{p \in O_2} Dual(p)\right] = Dual(O_1) \cup Dual(O_2)$. $\square$

Proposition 12. *Let O_1 and O_2 be two subsets of $\mathbb{R}^n$. Then,*

$$Dual(O_1 \cap O_2) \subseteq Dual(O_1) \cap Dual(O_2).$$

Proof. Since $O_1 \cap O_2 \subseteq O_1$ and $O_1 \cap O_2 \subseteq O_2$, we deduce that $Dual(O_1 \cap O_2) \subseteq Dual(O_1)$ and $Dual(O_1 \cap O_2) \subseteq Dual(O_2)$. Thus, $Dual(O_1 \cap O_2) \subseteq Dual(O_1) \cap Dual(O_2)$. $\square$

Remark 13. Let $p \in \mathbb{R}^n$ be a point. The dual of each point which lies in $Dual(p)$ is a hyperplane which passes through p.

Moreover, in order to describe the dual of a polytope, we need to define the positive and negative extrusions of a point as follows:

Definition 14 (*Positive and Negative Extrusions*). Let $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a point. The positive extrusion of p is defined by:

$$p^+ = \{p' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n | \forall i \in [\![1, n-1]\!], x_i = x'_i \text{ and } x_n \le x'_n\}.$$

In the same way, the negative extrusion of p is defined by:

$$p^- = \{p' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n | \forall i \in [\![1, n-1]\!], x_i = x'_i \text{ and } x_n \ge x'_n\}.$$

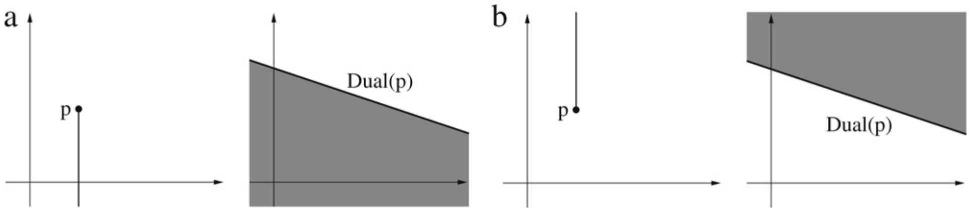

Fig. 6. Positive and negative extrusions of a point p (half-lines) and their dual object: a half-space, (a) Positive extrusion of p, (b) Negative extrusion.

Let O_1 and O_2 be two subsets of $\mathbb{R}^n$ such that $O_1 \subseteq O_2$. Then, $O_1^+ \subseteq O_2^+$ and $O_1^- \subseteq O_2^-$. Moreover, the following properties can be deduced from Definition 14.

Proposition 15. *Let O_1 and O_2 be two subsets of $\mathbb{R}^n$. Then,*

$$(O_1 \cup O_2)^+ = O_1^+ \cup O_2^+.$$

In the same way, $(O_1 \cup O_2)^- = O_1^- \cup O_2^-$.

Proof. $(O_1 \cup O_2)^+ = \bigcup_{p \in O_1 \cup O_2} p^+ = \left[\bigcup_{p \in O_1} p^+\right] \cup \left[\bigcup_{p \in O_2} p^+\right] = O_1^+ \cup O_2^+$. The proof of $(O_1 \cup O_2)^- = O_1^- \cup O_2^-$ is obtained in the same way. □

Proposition 16. *Let $p \in \mathbb{R}^n$ be a point. Then,*

$$Dual(p)^+ = Dual(p^+).$$

In the same way, $Dual(p)^- = Dual(p^-)$.

Proof. Let us consider $p = (x_1, \ldots, x_n) \in \mathcal{E}_n$. Then,

$$\begin{aligned} Dual(p^+) &= \mathcal{D}_{\mathcal{E}}(p^+) = \bigcup_{p' \in p^+} Dual(p') = \bigcup_{p' = (x'_1, \ldots, x'_n) \in p^+} \left\{ (y_1, \ldots, y_n) \in \mathcal{P}_n | y_n = -\sum_{i=1}^{n-1} x'_i y_i + x'_n \right\} \\ &= \left\{ (y_1, \ldots, y_n) \in \mathcal{P}_n | y_n \geq -\sum_{i=1}^{n-1} x_i y_i + x_n \right\} = \bigcup_{p' \in \mathcal{D}_{\mathcal{E}}(p)} p'^+ = \mathcal{D}_{\mathcal{E}}(p)^+ = Dual(p)^+. \end{aligned}$$

The proof of $Dual(p)^- = Dual(p^-)$ can be obtained in the same way. □

Proposition 16 is illustrated in Fig. 6.

3.2. Polytope dual representation

In this work, we need to define the dual of a polytope. An *n-polytope*, $n \in \mathbb{Z}$, is defined as follows:

Definition 17 (*n-polytope*). Let P be a polytope in dimension n, or n-polytope. Then, there exists a finite set of k half-spaces $\overline{\mathcal{H}} = \{\overline{H}_1, \ldots, \overline{H}_k\}$ such that $P = \bigcap_{i=1}^{k} \overline{H}_i$, and such that if H_i is the hyperplane forming the boundary of the half-space $\overline{H}_i$ (or *boundary hyperplane* of $\overline{H}_i$), then $\forall i \in [\![1, k]\!]$, $H_i \cap P \neq \emptyset$.

Notations. Let P be an n-polytope, and let $\overline{\mathcal{H}}$ be the corresponding half-space set. We define three subsets of $\overline{\mathcal{H}}$, denoted by $\overline{\mathcal{H}}_0$, $\overline{\mathcal{H}}_+$ and $\overline{\mathcal{H}}_-$, as follows:

- $\overline{\mathcal{H}}_0$ is the half-space set in $\overline{\mathcal{H}}$ defined by an inequality of the form $c_n + \sum_{i=1}^{n-1} c_i X_i \geq 0$ or similar to $c_n + \sum_{i=1}^{n-1} c_i X_i \leq 0$, with $(c_1, \ldots, c_n) \in \mathcal{E}^n$.
- $\overline{\mathcal{H}}_+$ is the half-space set in $\overline{\mathcal{H}}$ defined by an inequality of the form $X_n \geq c_n + \sum_{i=1}^{n-1} c_i X_i$, $(c_1, \ldots, c_n) \in \mathcal{E}^n$.
- $\overline{\mathcal{H}}_-$ is the half-space set in $\overline{\mathcal{H}}$ defined by an inequality of the form $X_n \leq c_n + \sum_{i=1}^{n-1} c_i X_i$, $(c_1, \ldots, c_n) \in \mathcal{E}^n$.

Moreover, we denote by $\mathcal{H}_0$, $\mathcal{H}_+$ and $\mathcal{H}_-$ the three boundary hyperplane sets corresponding respectively to the half-space sets $\overline{\mathcal{H}}_0$, $\overline{\mathcal{H}}_+$ and $\overline{\mathcal{H}}_-$.

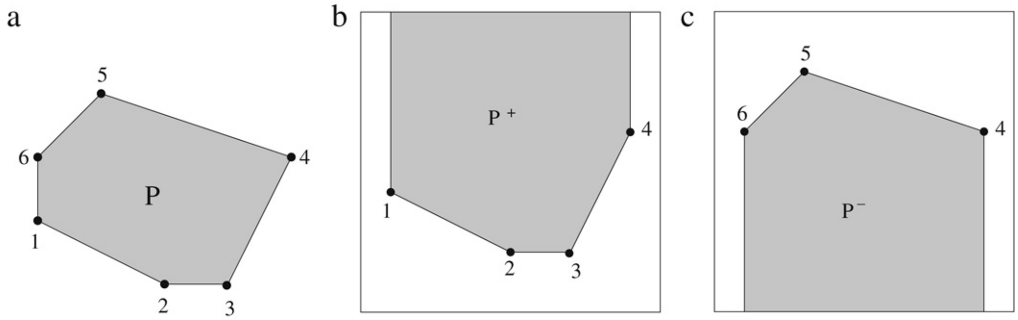

Fig. 7. Positive and negative extrusions of a polytope in dimension 2: (a) A 2-polytope P, (b) Positive extrusion of P, (c) Negative extrusion of P.

Proposition 18. *Let P be an n-polytope. Then,*

$$P = P^+ \cap P^-$$

with

$$P^+ = \bigcap_{\overline{H} \in (\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_+})} \overline{H}$$

and

$$P^- = \bigcap_{\overline{H} \in (\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_-})} \overline{H}.$$

Proof. Let us prove $P_c^+ = \bigcap_{\overline{H} \in (\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_+})} \overline{H}$. The proof of $P_c^- = \bigcap_{\overline{H} \in (\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_-})} \overline{H}$ can be obtained in the same way.

Let $p = (p_1, \ldots, p_n) \in P_c^+$. Then, there exists $p' = (p'_1, \ldots, p'_n) \in P$ such that for all $i \in [\![1, n-1]\!]$,

$$c_i = c'_i \quad \text{and} \quad c_n = c'_n.$$

Hence, for all $\overline{H} \in \overline{\mathcal{H}_0}$ and for all $\overline{H} \in \overline{\mathcal{H}_+}$, $p \in \overline{H}$. We deduce that $p \in \bigcap_{\overline{H} \in (\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_-})} \overline{H}$.

Now, let $p = (p_1, \ldots, p_n) \in \bigcap_{\overline{H} \in (\overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_-})} H$. Let us proceed by contradiction and assume that $p \notin P_c^+$. Then, for all $p' = (p'_1, \ldots, p'_n) \in P_c$, there exists $i \in [\![1, n-1]\!]$ such that $c_i \neq c'_i$ *or* $c_n \neq c'_n$. Then, there exists $\overline{H} \in \overline{\mathcal{H}_0}$ or $\overline{H} \in \overline{\mathcal{H}_+}$ such that $p \notin \overline{H}$. We deduce that $p \notin \bigcap_{\overline{H} \in \overline{\mathcal{H}_0} \cup \overline{\mathcal{H}_-}} \overline{H}$. □

Proposition 18 is illustrated in Fig. 7 in the case of dimension 2.

Let us now describe the dual of an n-polytope P from its vertices.

Let $\mathcal{V}$ be the set of vertices of P. We define two subsets of $\mathcal{V}$, denoted by $\mathcal{V}_+$ and $\mathcal{V}_-$, as follows:

$$\mathcal{V}_+ = \{v \in \mathcal{V} | \exists H \in \mathcal{H}_+, v \in H \cap P\}$$
$$\mathcal{V}_- = \{v \in \mathcal{V} | \exists H \in \mathcal{H}_-, v \in H \cap P\}.$$

We can see in Fig. 7 that the vertices numbered 1, 2, 3 and 4 belong to the vertex set $\mathcal{V}_+$ of P. In the same way, vertices numbered 4, 5 and 6 belong to the vertex set $\mathcal{V}_-$.

The dual of an n-polytope can then be defined by:

Theorem 19 (*Dual of a Polytope*). *Let P be an n-polytope, $\mathcal{V}_+$ and $\mathcal{V}_-$ the two vertex sets defined previously. Then:*

$$Dual(P) = \left[\bigcup_{v \in \mathcal{V}_+} Dual(v)^+\right] \cap \left[\bigcup_{v \in \mathcal{V}_-} Dual(v)^-\right].$$

Proof. Let us first prove the following lemma:

Lemma 20. *Let P be an n-polytope. Then,*

$$Dual(P) = Dual(P)^+ \cap Dual(P)^-.$$

Proof. In the following, we assume that $H \in \mathcal{E}_n$.

Since $Dual(P) \subseteq Dual(P)^+$ and $Dual(P) \subseteq Dual(P)^-$, we deduce that $Dual(P) \subseteq Dual(P)^+ \cap Dual(P)^-$.

We now prove that $Dual(P)^+ \cap Dual(P)^- \subseteq Dual(P)$. Consider a point $p = (x_1, \ldots, x_n) \in Dual(P)^+ \cap Dual(P)^-$. Then,

$$\exists p' = (x'_1, \ldots, x'_n) \in Dual(P) \mid p \in p'^+$$

and

$$\exists p'' = (x''_1, \ldots, x''_n) \in Dual(P) \mid p \in p''^-.$$

We deduce that $\forall i \in [\![1, n-1]\!], x'_i = x_i = x''_i$ and $x'_n \le x_n \le x''_n$.

Next we prove that $Dual(p) \cap H \neq \emptyset$, which would imply $p \in Dual(P)$. Since $p' \in Dual(P)$ and $p'' \in Dual(P)$, we have $Dual(p') \cap P \neq \emptyset$ and $Dual(p'') \cap P \neq \emptyset$. Let $q' = (q'_1, \ldots, q'_n) \in Dual(p') \cap P$ and $q'' = (q''_1, \ldots, q''_n) \in Dual(p'') \cap P$. Then, we have

$$q'_n = \sum_{i=1}^{n-1} x_i q'_i + x'_n \quad \text{and} \quad q''_n = \sum_{i=1}^{n-1} x_i q''_i + x''_n.$$

Since $x'_n \le x_n \le x''_n$, we deduce that

$$q'_n \le \sum_{i=1}^{n-1} x_i q'_i + x_n \quad \text{and} \quad q''_n \ge \sum_{i=1}^{n-1} x_i q''_i + x_n.$$

Thus, $Dual(p) \cap [q', q''] \neq \emptyset$. Finally, since P is convex we know that $[q', q''] \subset P$. We then deduce that $Dual(p) \cap P \neq \emptyset$. $\square$

Let us now define two object sets $\mathcal{F}_+$ and $\mathcal{F}_-$ by

$$\mathcal{F}_+ = \{H \cap P, H \in \mathcal{H}_+\}$$

and

$$\mathcal{F}_- = \{H \cap P, H \in \mathcal{H}_-\}.$$

Let $\mathcal{S}$ be a set. In the following, we will denote by $|\mathcal{S}|$ the cardinal of the set $\mathcal{S}$. Especially, we remark that $|\mathcal{F}_+|$ (resp. $|\mathcal{F}_-|$) is equal to $|\mathcal{H}_+|$ (resp. $|\mathcal{H}_-|$).

For instance, in dimension 2, the set $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) corresponds to the segments which belong to the boundary of P such that there two endpoints are vertices in $\mathcal{V}_+$ (resp. $\mathcal{V}_-$). In Fig. 7, $\mathcal{F}_+$ is composed of the segments [1, 2], [2, 3] and [3, 4]. In the same way, $\mathcal{F}_-$ is composed of the segments [4, 5] and [5, 6]. In dimension 3, these two sets are composed of faces of P.

The following relation is then verified:

Lemma 21. *Let P be an n-polytope. Then,*

$$P^+ = \bigcup_{F \in \mathcal{F}_+} F^+.$$

In the same way, $P^- = \bigcup_{F \in \mathcal{F}_-} F^-$.

Proof. Let us prove that

$$P^+ = \bigcup_{F \in \mathcal{F}_+} F^+ = \bigcup_{i \in [\![1, |\mathcal{F}_+|]\!], H_i \in \mathcal{H}_+} (H_i \cap P)^+ = \left[\bigcup_{i \in [\![1, |\mathcal{C}_+|]\!], H_i \in \mathcal{H}_+} H_i \cap P \right]^+.$$

First, we have

$$\bigcup_{i \in [\![1, |\mathcal{C}_+|]\!], H_i \in \mathcal{H}_+} H_i \cap P \subseteq P.$$

Hence,

$$\left[\bigcup_{i \in [\![1, |\mathcal{C}_+|]\!], H_i \in \mathcal{H}_+} H_i \cap P \right]^+ \subseteq P^+.$$

Now let $p \in P^+$. We know that $P^+ = \bigcap_{\overline{H} \in \overline{\mathcal{H}}_0 \cup \overline{\mathcal{H}}_+} \overline{H}$, which is equivalent to $P^+ = \bigcap_{H \in \mathcal{H}_0 \cup \mathcal{H}_+} H^+$. Hence, we deduce that for all $H_i \in \mathcal{H}_+$, $i \in [\![1, |\mathcal{C}_+|]\!]$, there exists $p_i = (p_{i_1}, \ldots, p_{i_n}) \in H_i$ such that $p \in p_i^+$. Let $p' = (p_{i_1}, \ldots, p_{i_{n-1}}, p'_n)$ be the point which verifies $\forall i \in [\![1, |\mathcal{C}_+|]\!], p'_n \ge p_{i_n}$. Then, since P is a polytope, we have $p' \in P$.

The second equality can be obtained in the same way. $\square$

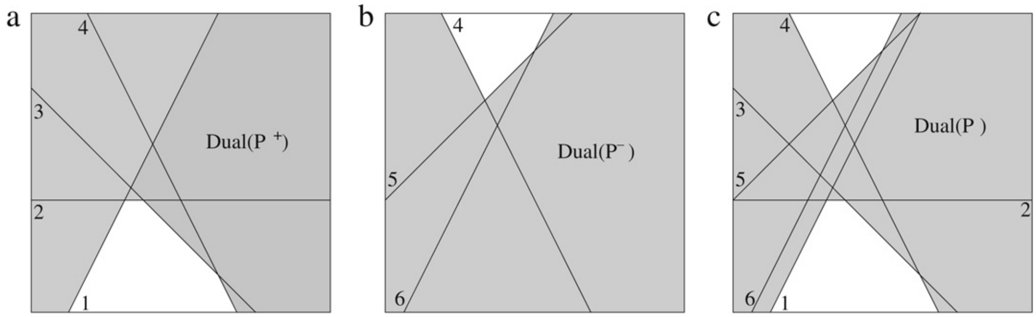

Fig. 8. Dual of a 2-polytope P: (a) Dual of the positive extrusion of P, (b) Dual of the negative extrusion of P, (c) Dual of P.

Lemma 22. *Let P be an n-polytope. Then,*

$$Dual(P^+) = \bigcup_{v \in \mathcal{V}_+} Dual(v)^+.$$

In the same way, $Dual(P^-) = \bigcup_{v \in \mathcal{V}_-} Dual(v)^-$.

Proof. Let us prove that $Dual(P^+) = \bigcup_{v \in \mathcal{V}_+} Dual(v)^+$.

By definition, for each vertex v in $\mathcal{V}_+$, there exists $F \in \mathcal{F}_+$ such that $v \in F$. Hence, $\bigcup_{v \in \mathcal{V}_+} v \subseteq \bigcup_{F \in \mathcal{F}_+} F$. Moreover, $(\bigcup_{v \in \mathcal{V}_+} v)^+ \subseteq (\bigcup_{F \in \mathcal{F}_+} F)^+$. Then, $\bigcup_{v \in V_+} v^+ \subseteq \bigcup_{F \in \mathcal{F}_+} F^+$. However, according to Lemma 21, we have $\bigcup_{F \in \mathcal{F}_+} F^+ = P^+$. We deduce that $Dual(\bigcup_{v \in V_+} v^+) \subseteq Dual(P^+)$, and then $\bigcup_{v \in V_+} Dual(v^+) \subseteq Dual(P^+)$.

Let us prove the second inclusion. Let $p \in Dual(P^+) = Dual(\bigcup_{F \in \mathcal{F}_+} F^+)$. Then, there exists $F \in \mathcal{F}_+$ such that $Dual(p) \cap F^+ \neq \emptyset$. Let us prove that there exists one vertex v in V such that $Dual(p) \cap v^+ \neq \emptyset$.

Let us proceed by contradiction and assume that for all $v \in F$, $Dual(p) \cap v^+ = \emptyset$. We know that there exists $H \in \mathcal{H}_+$ such that $F = H \cap P = H \cap \left[\bigcap_{i=1}^k \overline{H_i}\right] = \bigcap_{i=1}^k (H \cap \overline{H_i})$. Hence, if we considered the hyperplane H as space, we deduce that F is an $n-1$-polytope, since for all i, $\overline{H_i} \cap H$ is a half-space in H, and then F is equal to the intersection of several half-spaces. Since F is the convex hull of its vertices, we deduce that if $\forall v \in F, Dual(p) \cap v^+ = \emptyset$, then, $\forall p' \in F, Dual(p) \cap p' = \emptyset$. Moreover, $F^+ = \bigcup_{p \in F} p^+ = \bigcup_{p \in F, k \in \mathbb{R}_+} p + k\overrightarrow{OX_n} = \bigcup_{k \in \mathbb{R}_+} C + k\overrightarrow{OX_n}$. Since F is a polytope, $F + k\overrightarrow{OX_n}$ is also a polytope and the same method can be applied to prove that $\forall p' \in F + k\overrightarrow{OX_n}, Dual(p) \cap p' = \emptyset$.

A similar proof can be used to show that $Dual(P^-) = \bigcup_{v \in \mathcal{V}_+} Dual(v)^-$. □

The proof of Theorem 19 is obtained from Lemmas 22 and 20. □

Theorem 19 allows us to describe the dual of a polytope from the dual of its vertices. More precisely, the dual of a polytope is defined by the intersection of two objects, each one being a union of several half-spaces (see Fig. 8). Each half-space is the positive or negative extrusion of the hyperplane dual of one vertex of the polytope. In Fig. 8(c), we can see the representation of the dual of the polytope in Fig. 7(a).

3.3. The notion of generalized preimage

In this section, we define the generalized preimage of a set of polytopes. This preimage is a geometrical object computed in the parameter space from the duals of the polytopes. Each point in the preimage is associated to a hyperplane which cuts all polytopes. The generalized preimage of a polytope set is then defined as follows:

Definition 23 (*Generalized Preimage*). Let $\mathcal{P} = (P_1, \ldots, P_k)$ be a set of k polytopes, and let $Dual(P_i)$, $i \in [\![1, k]\!]$, be the dual of P_i in the parameter space. The generalized preimage $\mathbb{G}_P$ of $\mathcal{P}$ is defined by:

$$\mathbb{G}_P(\mathcal{P}) = \bigcap_{i=1}^{k} Dual(P_i).$$

4. Digital hyperplane recognition

In this section, we present our digital hyperplane recognition algorithm. Moreover, we assume this hyperplane is analytically defined with a distance and a ball such as the digital hyperplanes defined in Section 2.2 (such as the closed Naive hyperplane or the Supercover hyperplane) or that the definition is closely related to such a definition (such as Naive hyperplane and Standard hyperplane). The aim of our algorithm is to determine if a hypervoxel set belongs to a digital hyperplane. More precisely, we want to determine all Euclidean hyperplanes the digitization of which contains the given hypervoxel set. We call these hyperplanes the *solution hyperplanes*.

In order to do that, the idea is to compute the set of Euclidean hyperplanes (if it exists) which cross all balls corresponding to the given hypervoxels by computing the generalized preimage of the balls. Then we can deduce if the hypervoxel set belongs or not to a digital hyperplane depending on whether the preimage is a nonempty set or not.

However, according to the digitization model used, some points located on the border of the dual of the ball are not associated to solution hyperplanes (because these hyperplanes cross ball vertices), and thus some points on the border of the generalized preimage are not associated to solution hyperplanes. This is for instance the case for the Standard and Naive models since one inequality in the analytical digital hyperplane definitions (see Definitions 3 and 6) is strict.

In the following, we first detail our recognition algorithm. Then, we apply our algorithm to the Naive and Standard digitization models. The algorithms for the closed Naive and the Supercover models are only marginally different.

4.1. Recognition algorithm

Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a set of k hypervoxels. The digital hyperplane recognition (see Algorithm 1) is simply performed by computing the generalized preimage $\mathbb{G}_P$ of the balls $\{B_1, \ldots, B_k\}$ associated to $\mathcal{H}$. First, $\mathbb{G}_P(B_1)$, i.e. the dual of B_1, is computed according to the polytope dual definition given by Theorem 19. Then, $\mathbb{G}_P(\{B_1, B_2\})$ is computed from the intersection of $\mathbb{G}_P(B_1)$ and $Dual(B_2)$. And so on until $\mathbb{G}_P(\{B_1, \ldots, B_k\})$ is computed or $\mathbb{G}_P$ becomes empty. Note that the balls can be considered in any order, and the corresponding hypervoxels do not need to be connected.

Algorithm 1: Standard and Supercover hyperplane recognition algorithm

```
Data: A set H of k hypervoxels H_1, ..., H_k and their associated balls B_1, ..., B_k.
begin
    GP ⟵ Dual(B_1);
    i ⟵ 2;
    while GP ≠ ∅ and i ≤ n do
        GP ⟵ GP ∩ Dual(B_i);
        i ⟵ i + 1;
    if GP ≠ ∅ then
        H belongs to a digital hyperplane.
    else
        H does not belong to a digital hyperplane.
end
```

4.2. Example: Application to Naive and Standard hyperplane recognition

For a given ball associated to a given digitization model, some parts in the generalized preimage do not correspond to solution hyperplanes. It is the case when one or several inequalities in the hyperplane digitization definition are strict, for instance for the Standard and Naive models. In the case of the Supercover and closed Naive digitization models, all points in the generalized preimage are solutions. The algorithms for the Supercover and closed Naive models are therefore the same minus the discussions about the ball vertices that we do not need to consider.

In the following, we study the case of the Naive and Standard models and describe which part of the dual of the balls corresponds to solution hyperplanes.

4.2.1. Naive hyperplanes

We want to determine which points on the boundary of the dual of a ball $B_{d_1}(c, \frac{1}{2})$ are associated to solution hyperplanes. We know that each point $(c_0, \ldots, c_{n-1})$ is associated to a hyperplane with equation $c_0 - x_n + \sum_{i=1}^{n-1} c_i x_i = 0$. Moreover, we know that this hyperplane contains a vertex of the ball.

We deduce from Proposition 5 the following property:

Proposition 24. *Let B be a ball $B_{d_1}(c', \frac{1}{2})$, $c' \in \mathbb{Z}^n$, and let H be a Euclidean hyperplane with equation $c_0 + \sum_{i=1}^{n} c_i x_i = 0$ that passes through a vertex $v = (v_1, \ldots, v_n)$ of B. Moreover, we assume that the first $c_i \neq 0$ verifies $c_i > 0$.*

Hence, there exists $j \in [\![1, n]\!]$ such that $v = (c'_1, \ldots, c'_{j-1}, c'_j + \frac{1}{2}, c'_{j+1}, \ldots, c'_n)$ (resp. $v = (c'_1, \ldots, c'_{j-1}, c'_j - \frac{1}{2}, c'_{j+1}, \ldots, c'_n)$). Then, if $c_j > 0$ (resp. $c_j < 0$), c' belongs to the Naive digitization of H.

Hence, from Proposition 24, we can easily determine which points in the dual of a ball $B_{d_1}(c', \frac{1}{2})$ are associated to solution hyperplanes. We can see in Fig. 9 an example of dual ball in dimension 2.

Fig. 10 illustrates the recognition process in dimension 2 in the case of the Naive hyperplane recognition.

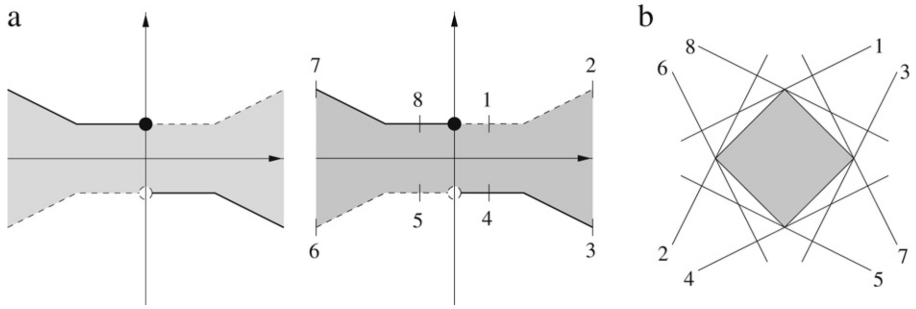

Fig. 9. Dual of a ball $B_{d_1}(c', \frac{1}{2})$: (a) Points on dashed lines are not associated to solution hyperplanes, (b) Correspondence between the ball and its dual.

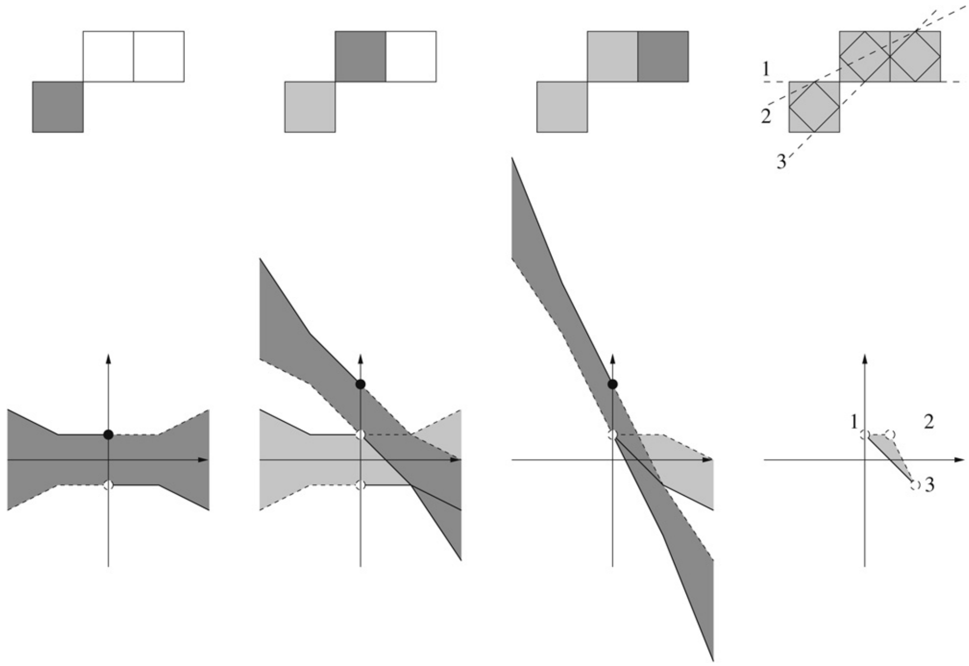

Fig. 10. Example of 2D generalized preimage computation: Naive hyperplane recognition.

4.2.2. Standard hyperplanes

We want to determine which points on the boundary of the dual of a ball $B_{d_\infty}(c, 1)$ are associated to solution hyperplanes. We know that each point $(c_0, \ldots, c_{n-1})$ is associated to a hyperplane with equation $c_0 - x_n + \sum_{i=1}^{n-1} c_i x_i = 0$. Moreover, we know that this hyperplane contains a vertex of the ball.

We deduce from Proposition 8 the following property:

Proposition 25. *Let B be a ball $B_{d_\infty}(c', 1)$, $c' \in \mathbb{Z}^n$, and let H be a Euclidean hyperplane with equation $c_0 + \sum_{i=1}^{n} c_i x_i = 0$ that passes through a vertex $v = (v_1, \ldots, v_n)$ of B. Moreover, we assume that the first $c_i \neq 0$ verifies $c_i > 0$.*

Hence, if $v_n > c'_n$ (resp. $v_n < c'_n$) and $c_n > 0$ (resp. $c_n < 0$), then c' belongs to the Standard digitization of H.

Hence, from Proposition 25, we can easily determine which points in the dual of a ball $B_{d_\infty}(c', 1)$ are associated to solution hyperplanes. We can see in Fig. 11 an example of dual ball in dimension 2.

Fig. 12 illustrates the recognition process in dimension 2 in the case of the Standard hyperplane recognition.

5. Conclusion and future works

In this article, a new digital hyperplane recognition scheme in arbitrary dimension has been presented. This algorithm determines if a given hypervoxel set belongs to a digital hyperplane by providing the set of Euclidean hyperplanes which cut all balls associated to the given hypervoxels. This set is deduced from the computation in a dual space of the generalized preimage of the balls. This preimage is defined as the intersection of the duals of the balls. The recognition algorithm does

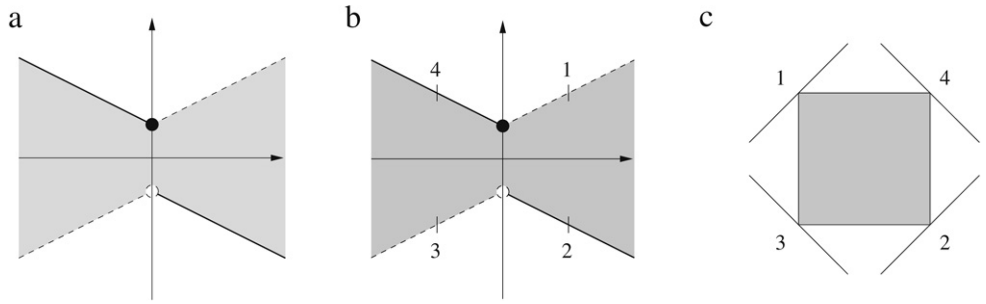

Fig. 11. Dual of a ball $B_{d_\infty}(c', 1)$: (a) Points on dashed lines are not associated to solution hyperplanes, (b) Correspondence between the ball and its dual.

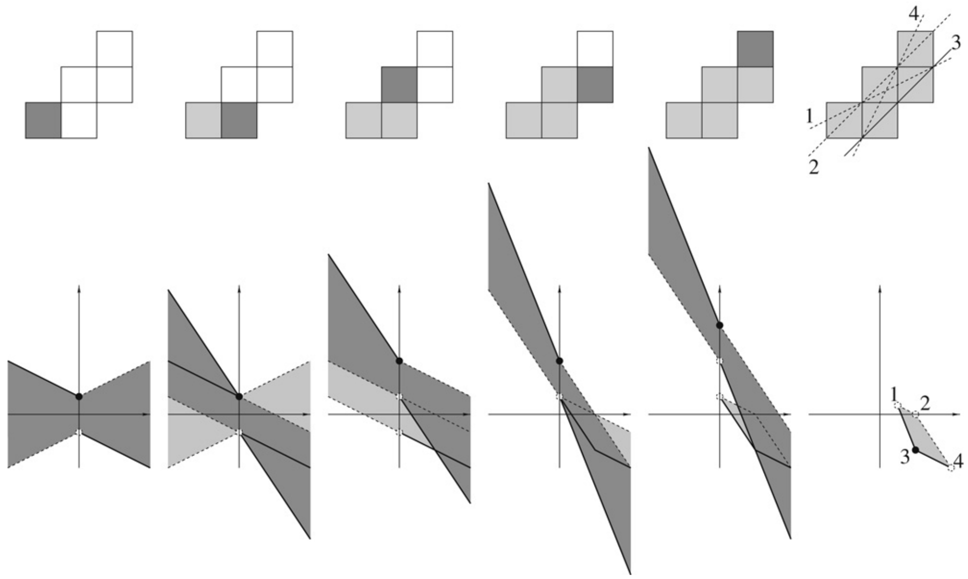

Fig. 12. Example of 2D generalized preimage computation: Standard hyperplane recognition.

not require the given hypervoxels to be connected. Moreover, during the recognition process, hypervoxels can be considered in any order.

The results proposed in this paper are very general. Indeed, since the generalized preimage is defined for any polytope set, this can easily lead to recognition algorithms in multi-scale grids or heterogeneous grids, such as for instance irregular isothetic grids [23].

One of the major tasks that needs to be addressed now is a thorough study on the complexity of the proposed method. The proposed algorithm is very general in its scope as there are no requirements on the hypervoxel connectivity or order in which they are fed to the algorithm. For instance, let us consider the Standard hyperplane recognition. To perform intersection operations, the first approach is to intersect directly the generalized preimage and each hypervoxel dual. It is not an efficient method since the dual of a hypervoxel is an open concave polytope. However, the generalized preimage of a hypervoxel set can be obtained by simply computing intersections of convex polytopes and hyperplanes, as seen in [9]. Let k be the number of given hypervoxels. These improvements lead to a complexity for our algorithm of $\mathcal{O}(k)$ in dimension 2 when applied on 4-connected curves [24]. Indeed, the generalized preimage of a pixel set is a polygon with at most four edges [14,25]. In dimension 3 [2,26] and higher, the complexity is $\mathcal{O}(k^2)$ in the worst case. In the general case however, the complexity in time and space depends on the connectivity of the surface and on the order in which the hypervoxels are added to the recognition algorithm. This is still a somewhat open question.

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