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Is Weibull distribution the correct model for predicting probability of failure initiated by non-interacting flaws?

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ABSTRACT

The utility of the Weibull distribution has been traditionally justified with the belief that it is the mathematical expression of the weakest-link concept in the case of flaws locally initiating failure in a stressed volume. This paper challenges the Weibull distribution as a mathematical formulation of the weakest-link concept and its suitability for predicting probability of failure locally initiated by flaws. The paper shows that the Weibull distribution predicts correctly the probability of failure locally initiated by flaws if and only if the probability that a flaw will be critical is a power law or can be approximated by a power law of the applied stress.

Contrary to the common belief, on the basis of a theoretical analysis and Monte Carlo simulations we show that in general, for non-interacting flaws randomly located in a stressed volume, the distribution of the minimum failure stress is not necessarily a Weibull distribution. For the simple cases of a single group of identical flaws or two flaw size groups each of which contains identical flaws, for example, the Weibull distribution fails to predict correctly the probability of failure. Furthermore, if in a particular load range, no new critical flaws are created by increasing the applied stress, the Weibull distribution also fails to predict correctly the probability of failure of the component. In all these cases however, the probability of failure is correctly predicted by the suggested alternative equation. This equation is the correct mathematical formulation of the weakest-link concept related to random flaws in a stressed volume. The equation does not require any assumption concerning the physical nature of the flaws and the physical mechanism of failure and can be applied in cases of locally initiated failure by non-interacting entities.

1. Introduction

The most important aspect of the load-strength interaction is the interaction of the upper tail of the load distribution and the lower tail of the strength distribution. The values from the lower tail of the strength distribution control reliability, not the high or central values. Consequently, an adequate model of the strength distribution should faithfully represent its lower tail.

In a number of cases, the *Weibull distribution* (Weibull, 1951) has been a suitable model for the variation of the strength of materials whose failure is locally initiated by flaws (e.g. ceramics, glasses, low-carbon steels at low temperatures, and other brittle materials). For the probability of failure of a chain consisting of n links, Weibull (1951) proposed the following equation:

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$$p_f(\sigma) = 1 - \exp[-n\varphi(\sigma)] \quad (1)$$

where $\varphi(\sigma)$ is positive, non-decreasing and vanishing at some value σ_l. Weibull approximated $\varphi(\sigma)$ with the function $\varphi(\sigma) \approx [(\sigma - \sigma_l)/\sigma_a]^\beta$ satisfying these conditions and obtained the distribution:

$$p_f(\sigma) = 1 - \exp\left[-\left(\frac{\sigma - \sigma_l}{\sigma_a}\right)^\beta\right], \quad \beta > 0 \quad (2)$$

As a result, the probability of failure $p_f(\sigma)$ at a loading stress σ is given by the Weibull distribution (2) where σ_l, σ_a and β are the location, scale and shape parameters, respectively. Often, $\sigma_l = 0$ is assumed which ensures conservatism in the calculations. Material with non-interacting flaws locally initiating failure has been compared to a chain with many links each of which corresponds to a flaw. The material fails when any of the flaws initiates failure during loading.

As a result, for a long time, the Weibull model

$$p_f(\sigma) = 1 - \exp\left(-V\left(\frac{\sigma - \sigma_l}{\sigma_0}\right)^m\right), \quad m > 0 \quad (3)$$

has been used to model the probability of failure $p_f(\sigma)$ locally initiated by flaws of a loaded component in uniaxial homogeneous tensile stress state. In Eq. (3), σ is the loading tensile stress, V is the stressed volume, σ_0 and m are constants. Eq. (3) is a three-parameter Weibull distribution, where σ_l is a location parameter or a threshold stress below which the probability of failure is zero.

The utility of the Weibull distribution has been traditionally justified with its capability to fit well a wide range of failure data. The theoretical justification of the Weibull distribution is the extreme value theory (Gumbel, 1958). According to the extreme value theory, the Weibull model is the asymptotic distribution for the minimum of a large number of bounded on the left, identically distributed random variables.

Freudenthal (1968) and Trustrum and Jayatilaka (1983), for example, used arguments based on the extreme value theory and concluded that the distribution of the fracture stress is insensitive to the flaw size distribution and that distributions of different types lead to a Weibull distribution.

In most publications related to the Weibull distribution, the utility of the Weibull distribution has also been justified with the belief that it is the mathematical formulation of the *weakest-link concept*. In other words, if a number of random flaws are present in a stressed volume it is believed (e.g. Freudenthal, 1968) that the Weibull distribution is the model for the distribution of the minimum failure stress characterizing these flaws.

There exists mounting experimental evidence however, showing that in some cases the Weibull distribution fails to fit data related to failure locally initiated by flaws.

According to Danzer et al. (2007), the Weibull distribution is not an appropriate model for brittle materials containing bi- or multi-modal flaw size distributions or materials having a high defect density. Furthermore, according to Danzer et al. (2007), published data claimed to be Weibull distributed which are based on small samples may not necessarily come from a Weibull population. This is because it is very difficult to decide on the basis of a small sample whether the data follow a Weibull distribution or not. Danzer (2006) noted that on the basis of a small sample size (e.g. containing fewer than 30 specimens) it is not possible to differentiate between a Weibull, a Gaussian or other similar distribution functions. Because of the flexibility of the three-parameter Weibull distribution, a strength distribution built on the basis of a small sample appears to be a Weibull distribution in almost any case.

In Zhang and Knott (2000), the value of the conventional fitting of fracture toughness to Weibull distribution has been questioned. Good estimates for the lower-tail fracture toughness values were reported for a single-phase homogeneous bainite or martensite and for a fine-mixed bainite/martensite microstructure. For a coarse-grained bainite/martensite mixed microstructures however, the Weibull fits resulted in ultra-pessimistic estimates for the lower-tail fracture toughness values. These were below the fracture toughness values of the phase characterized by a smaller fracture toughness (bainite). As a result, no physically reasonable lower-bound fracture toughness could be obtained from a Weibull fit of coarse mixed microstructures.

In cases where measured strength does not follow the Weibull distribution, fitting a Weibull distribution to the data sets and extrapolating towards low strength values may result in wrong estimates for the lower tail of the strength which is of significant importance to estimating the risk of structural failure. It seems that in some cases, the Weibull distribution is a good model for fracture locally initiated by flaws, while in other cases it is clearly not an appropriate model.

Furthermore, experiments on notched specimens reported by Milella and Bonora (2000), showed that the Weibull modulus m depends on the specimen geometry (the notch radius). These experimental findings were confirmed by experiments involving failure of notched ceramic specimens conducted by Gerguri et al. (2004). They reported that the calculated Weibull modulus depends on whether the specimen has a notch or not. For notched graphite bars, a value $m = 29$ was obtained, which was almost three times higher than the value $m = 10$ obtained for bars without notches. Similar results were obtained for silicon nitride bars. In other words, without altering the flaw population and the material of the specimens, different notch radii yield different Weibull moduli m. A major implication from these experimental results is that the Weibull modulus m is probably not a material constant.

A recent theoretical study (Todinov, 2008) showed that existing statistical theories of fracture (e.g. Batdorf and Crose, 1974; Evans, 1978; Lamon, 1988; Weibull, 1951), can be reduced to $p_f(\sigma) = 1 - \exp[-n_{cr}(\sigma)V]$ where $p_f(\sigma)$ is the probability of failure at a loading stress σ, $n_{cr}(\sigma)$ is the number density of the flaws causing failure at a loading stress σ and V is the stressed volume. These theories assume a power-law stress dependence of the number density of the critical flaws (Batdorf and Heinisch, 1978; Lamon and Evans, 1983; Evans and Jones, 1978). In the same work (Todinov, 2008), for material containing flaws, simulation counter-examples were developed that demonstrate cases where this assumption is violated.

Despite this analysis, no fundamental reason has been given as to why despite the violation of the power law stress dependence, the Weibull distribution fits well such a large amount of data. Furthermore, no analysis has been conducted related to the correctness of the Weibull distribution as a mathematical formulation of the weakest-link concept in the case of failure locally initiated by non-interacting flaws.

This paper aims to fill this gap by: (i) deriving a necessary and sufficient condition for the validity of the Weibull distribution, (ii) testing the widely held belief that the Weibull distribution is the mathematical formulation of the weakest-link concept in the case of failure initiated by random non-interacting flaws, (iii) suggesting the correct mathematical formulation of the weakest-link concept in the case of failure locally initiated by non-interacting random flaws and (iv) generating insight into why the Weibull distribution fits well such a vast range of failure data.

2. Analysis of the Weibull distribution and counter-examples

Consider a bar containing random flaws, loaded in tension (Fig. 1) where the loading stress σ *is below* the minimum fracture stress σ_M of the homogeneous matrix. Consequently, in this case, failure can only be initiated by a flaw residing in the stressed volume. A flaw that will initiate failure with certainty, if it is present in the volume of the loaded bar will be referred to as *critical flaw* (Batdorf and Crose, 1974). A critical flaw, for example, can be a flaw whose size exceeds a particular limit that depends on the loading stress. Assume a population of fracture initiating flaws with finite number density λ, whose locations in the stressed volume of the bar follow a homogeneous Poisson process. The critical flaws whose number density at a loading stress σ will be denoted by $\lambda_{cr}(\sigma)$ will also follow a homogeneous Poisson process in the volume of the loaded bar (the filled circles in Fig. 1).

The probability that no critical flaws will be present in the stressed volume V at a loading stress σ is $\exp[-V\lambda_{cr}(\sigma)]$. Failure initiated by flaws will occur if and only if at least one critical flaw resides in the stressed volume V. Consequently, the probability of failure at a loading stress σ (the probability that at least one critical flaw will be present in the volume V) is

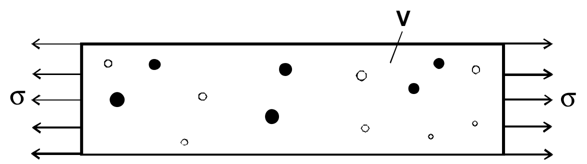

Fig. 1. Stressed bar with volume V containing flaws with finite number density λ.

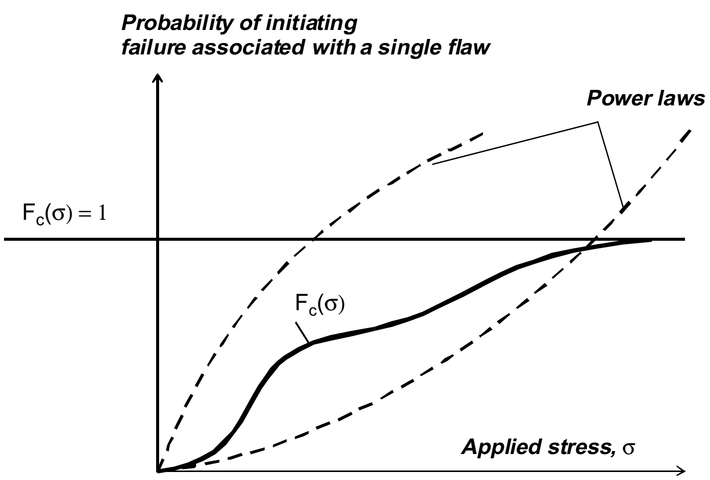

Fig. 2. If no new critical flaws are created by the applied stress, with increasing the magnitude of the applied stress, the probability that a flaw will be critical approaches unity and cannot be approximated by a power-law stress dependence in this region.

$$p_f(\sigma) = 1 - \exp[-V\lambda_{cr}(\sigma)] \tag{4}$$

Now assume that the Weibull Eq. (3) holds. Since Eqs. (4) and (3) have the same functional form, from the comparison, the dependence

$$\lambda_{cr}(\sigma) = \left(\frac{\sigma - \sigma_l}{\sigma_0}\right)^m$$

in the Weibull distribution (3), must necessarily give the number density of the critical flaws at a loading stress σ. In other words, the Weibull model (3) requires the number density of the critical flaws $\lambda_{cr}(\sigma)$ to be a power law dependence of the applied stress σ. Let us now make use of the concept conditional individual probability of initiating failure $F_c(\sigma)$ at a stress level σ, given that a flaw resides with certainty in the stressed volume V (Todinov, 2005). The probability $F_c(\sigma)$ can also be interpreted as the probability that a flaw residing in the stressed volume will be critical.

The expected number of critical flaws is then equal to the product $\lambda V F_c(\sigma)$ of the expected number λV of flaws residing in the volume V and the probability $F_c(\sigma)$ that a flaw will be critical. As a result, the number density $\lambda_{cr}(\sigma)$ of the critical flaws at a stress level σ is linked with the number density λ of all flaws with the relationship $\lambda_{cr}(\sigma) = \lambda F_c(\sigma)$. This means that the Weibull distribution holds *if and only if*, the probability $F_c(\sigma)$ that a flaw will be critical is a power law dependence (or can be approximated well by a power law dependence) of the applied stress.

$$F_c(\sigma) = \left(\frac{\sigma - \sigma_l}{\sigma_m}\right)^m, \quad m > 0 \tag{5}$$

where $\sigma_m = \sigma_0 \lambda^{-1/m}$.

This is a necessary and sufficient condition for the validity of the Weibull distribution in the case of non-interacting flaws whose locations follow a homogeneous Poisson process.

In the stress range corresponding to low values of the applied stress, the power law dependence often provides a good approximation of the probability $F_c(\sigma)$ that a flaw will be critical. According to the weakest-link concept, in a stressed volume containing a large number of flaws, failure will be initiated by the flaw characterized by the smallest failure stress. Consequently, for a relatively large number of flaws in a tested specimen, the recorded failure stress is likely to remain within the lower tail of the dependence $F_c(\sigma)$ – the region that can often be approximated well by a power law stress dependence. This argument goes towards explaining the wide range of data that are fitted well by the Weibull distribution. In the general case however, a good approximation by a power law of all regions of the $F_c(\sigma)$ curve is not possible.

Indeed, the probability $F_c(\sigma)$ is bounded by $F_c(\sigma) = 1$ (Fig. 2). It is therefore impossible to approximate $F_c(\sigma)$ by a power law stress dependence, beyond a stress level σ^* for which $F_c(\sigma^*) = 1$. A further increase of the applied stress σ beyond σ^* does not result in an increase of the probability $F_c(\sigma) = 1$ that a flaw will be critical.

Consider dependence (5). It is strictly increasing for $\sigma > 0$, because $\frac{d}{d\sigma}\left([(\sigma - \sigma_l)/\sigma_m]^m\right) > 0$. As a result, according to the Weibull model, the probability that a flaw will be critical must increase with increasing the loading stress. Consequently, a power law approximation of the type given by Eq. (5), requires that the probability $F_c(\sigma)$ increases, with increasing the loading stress σ beyond σ^* for which $F_c(\sigma^*) = 1$. In fact, the probability $F_c(\sigma)$ for $\sigma > \sigma^*$ should remain equal to one!

In short, the probability that a flaw will be critical must have an upper bound. The power law dependence however, of the conditional probability of failure associated with a single flaw does not have an upper bound (Fig. 2).

This point will further be illustrated by a counterexample.

A piece of wire with length L and unit cross sectional area $S = 1$, contains only a single type of identical flaws (e.g. random tool marks) with number density λ. The wire is subjected to tensile loading in the range $(\sigma_{\min}, \sigma_{\max})$ which is below the minimum fracture stress σ_M of the homogeneous wire (with no flaws on it) (Fig. 3). Suppose that the stress level $\sigma_{\min}$ is such that

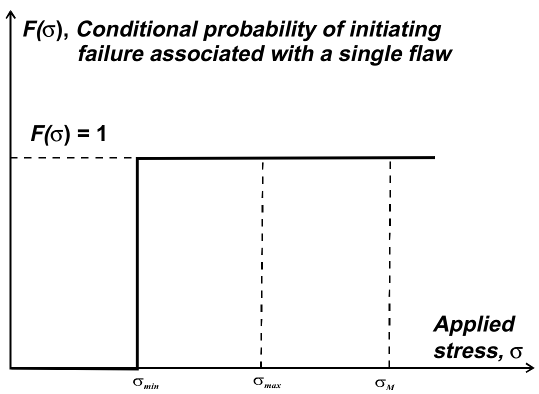

Fig. 3. Variation of the conditional probability of initiating failure associated with a single tool mark.

any flaw (tool mark), will cause failure if present on the stressed piece of wire. In other words, beyond the stress level $\sigma_{\min}$, all flaws are critical ($F_c(\sigma) = 1$, $\sigma > \sigma_{\min}$), Fig. 3.

Clearly, the probability of failure in the stress region ($\sigma_{\min}, \sigma_{\max}$) is equal to the probability of existence of a flaw on the stressed piece of wire. This probability, which is given by

$$p(\sigma) = 1 - \exp(-\lambda L) \tag{6}$$

is constant in the stress range ($\sigma_{\min}, \sigma_{\max}$). However, the Weibull distribution gives

$$p(\sigma) = 1 - \exp\left(-L[(\sigma - \sigma_l)/\sigma_0]^m\right), \quad m > 0 \tag{7}$$

for the probability of failure. According to the Weibull distribution, within the stress range ($\sigma_{\min}, \sigma_{\max}$) as σ varies from $\sigma_{\min}$ towards $\sigma_{\max}$, the probability of failure *will increase* ($p(\sigma_{\max}) > p(\sigma_{\min})$). In fact, the probability of failure will remain the same because the expected number of flaws (tool marks) in the stressed volume has not been altered!

This counterexample and the previous analysis show that *in the case where the probability that a flaw will be critical remains the same during increasing the loading stress, the Weibull distribution is incapable of predicting correctly the probability of failure initiated by flaws*.

Furthermore, with increasing the loading stress, the probability that a flaw will be critical may increase, without necessarily following a power law dependence.

Now let us introduce another counterexample based on flaws of the same type but with different size: a larger size a_1 with a number density λ_1 and a smaller size a_2 ($a_2 < a_1$) with a number density λ_2. The size distribution of the flaws is therefore given by the discrete distribution:

$$\begin{array}{lcc} \text{Flaw size}: & a_1 & a_2 \\ \text{Probability}: & \lambda_1/(\lambda_1+\lambda_2) & \lambda_2/(\lambda_1+\lambda_2) \end{array} \tag{8}$$

where $\lambda_1/(\lambda_1 + \lambda_2)$ and $\lambda_2/(\lambda_1 + \lambda_2)$ are the probabilities that a random flaw will be of size a_1 or a_2, respectively. The first (larger) size of flaws contains identical flaws with number density λ_1 and the probability that a flaw will be critical is

$$F_{c1}(\sigma) = \begin{cases} 0, & \sigma \leqslant \sigma_{10} \\ 1, & \sigma > \sigma_{10} \end{cases} \tag{9}$$

where σ_{10} is the stress threshold beyond which all flaws with size a_1 become critical (initiate failure). Compared with the first size, the second flaw size group contains identical flaws with number density $\lambda_2 = k \times \lambda_1$ where $k \geqslant 0$. Since the flaws from the second group are of smaller size compared to the flaws from the first group, they will initiate failure at a higher stress threshold $\sigma_{20} > \sigma_{10}$. The probability that a flaw from the second group will be critical is then

$$F_{c2}(\sigma) = \begin{cases} 0, & \sigma \leqslant \sigma_{20} \\ 1, & \sigma > \sigma_{20} \end{cases} \tag{10}$$

The probability that a flaw will be critical irrespective of the group to which it belongs is then given by

$$F_c(\sigma) = \frac{1}{1+k} \times F_{c1}(\sigma) + \frac{k}{1+k} F_{c2}(\sigma) \tag{11}$$

where $\frac{1}{1+k} = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{k}{1+k} = \frac{\lambda_2}{\lambda_1+\lambda_2}$ are the probabilities that the flaw will belong to the first or the second size group, correspondingly. This dependence, which has been graphically illustrated in Fig. 4a, is not a power law.

The dependence related to the probability of failure associated with the whole stressed volume is given in Fig. 4b. In the stress range $0 \leqslant \sigma \leqslant \sigma_{10}$, the probability of failure initiated by flaws is zero, because no flaw can initiate failure below the stress threshold σ_{10}. In the stress range $\sigma_{10} < \sigma \leqslant \sigma_{20}$, the probability of failure of the stressed volume is equal to $1 - \exp(-\lambda_1 V)$ – the probability of existence of at least a single flaw of size a_1, because only flaws of the larger size a_1 can initiate failure below the stress threshold σ_{20}. In the stress range $\sigma_{20} < \sigma < \sigma_M$, located below the fracture stress σ_M of the matrix, the probability of failure of the stressed volume is equal to $1 - \exp[-(\lambda_1 + \lambda_2)V]$. This is the probability of existence of at least a single flaw from any size, because flaws from both sizes can initiate failure beyond the stress threshold σ_{20}.

As can be seen, the dependence from Fig. 4b *cannot be approximated* by a Weibull distribution. This counterexample shows that *there exist simple flaw size distributions for which the distribution of the minimum failure stress is not a Weibull distribution*.

In short, for non-interacting flaws with number density λ, characterized by a strength distribution $F_c(\sigma)$, the distribution of the minimum failure stress is not necessarily the Weibull distribution.

3. Distribution of the minimum failure stress and a mathematical formulation of the weakest-link concept

The case where the minimum failure stress $\sigma_{\min,f}$ characterizing the flaws in the stressed volume is greater than the loading stress σ, is equivalent to the case where no critical flaws are present in the stressed volume V. Indeed, suppose that for the minimum failure stress $\sigma_{\min,f}$ characterizing the flaws in the stressed volume, $\sigma_{\min,f} > \sigma$ is fulfilled. This means that there will be no failure initiated by a flaw in the stressed volume, therefore no critical flaws are present in the stressed volume. On

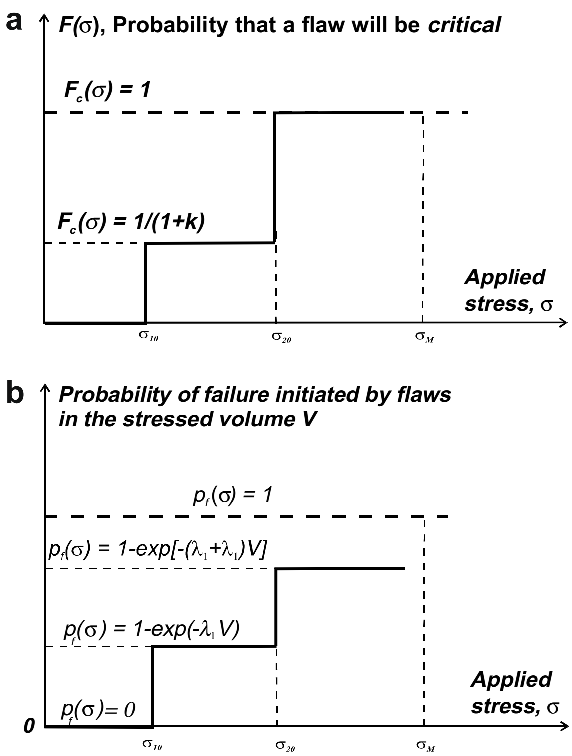

Fig. 4. (a) Variation of the probability that a flaw will be critical for flaws from by two size groups; (b) variation of the probability of failure of a bar containing flaws of the same type, from two different size groups.

the other hand, if no critical flaws reside in the stressed volume, the minimum failure stress characterizing the flaws in the stressed volume will certainly be greater than the loading stress σ.

Since the expected number of critical flaws in the stressed volume V is $\lambda V F_c(\sigma)$, for the probability that the minimum failure stress will be greater than σ, $P(\sigma_{\min f} > \sigma) = \exp[-\lambda V F_c(\sigma)]$ holds. The probability distribution function of the minimum failure stress characterizing the flaws in the stressed volume is therefore given by:

$$P(\sigma_{\min f} \leqslant \sigma) = 1 - \exp[-\lambda V F_c(\sigma)] \tag{12}$$

The dependence $F_c(\sigma)$ is not necessarily a power law. *Consequently, the distribution of the minimum strength of the flaws is not necessarily described by the Weibull distribution*. Eq. (12) does not require any assumption concerning the physical nature of the flaws and the physical nature of failure. The flaws in real materials are rarely simple cracks that satisfy the equations of the fracture mechanics! The equation can therefore be applied in any case of a locally initiated failure by non-interacting entities, where a random entity is characterized by a probability $F_c(\sigma)$ of initiating failure given that it is present with certainty.

Eq. (12) is a special case of the equation

$$p_f = 1 - \exp(-\lambda V F_c) \tag{13}$$

part of a methodology proposed in earlier work (Todinov, 2007, 2008) for determining the probability of failure of components with complex shape initiated by flaws, where F_c is the conditional individual probability of initiating failure characterizing a single flaw given that it resides with certainty in the component/structure.

The distribution of the minimum failure stress can also be derived in the case where the locations of the flaws do not necessarily follow a homogeneous Poisson process.

Indeed, suppose that the locations of the flaws in the stressed volume follow a non-homogeneous Poisson process with density $\lambda(x,y,z)$ and the distribution of the strength of the flaws is given by $F_c(\sigma)$. The probability that the failure stress will be greater than σ is a sum of the probability of the following mutually exclusive and exhaustive events: the probability that there will be no flaws in the stressed volume V which is given by $\exp(-\int_V \lambda(x,y,z)dv)$, the probability that there will be a single flaw in the volume V and its strength will be greater than σ which is given by $\exp(-\int_V \lambda(x,y,z)dv) \times (\int_V \lambda(x,y,z)d) \times [1 - F_c(\sigma)], \ldots$, the probability that there will be k flaws in the volume V and the strength of each flaw will be greater than σ, which is given by $\exp(-\int_V \lambda(x,y,z)dv) \times (\int_V \lambda(x,y,z)d)^k \times \frac{[1-F_c(\sigma)]^k}{k!}$ and so on...

Adding these probabilities results in,

$$P(\sigma_{\min f} > \sigma) = \exp\left(-\int_V \lambda(x,y,z)d\right) \times \left(1 + \left(\int_V \lambda(x,y,z)\right)^1 \times \frac{[1-F_c(\sigma)]^1}{1!} + \left(\int_V \lambda(x,y,z)\right)^2 \times \frac{[1-F_c(\sigma)]^2}{2!} + \ldots\right) \quad (14)$$

After some algebraic manipulation, for the distribution of the minimum failure stress

$$P(\sigma_{\min f} < \sigma) = 1 - \exp\left(-F_c(\sigma)\int_V \lambda(x,y,z)d\right) \quad (15)$$

is obtained. Denoting, $\bar{\lambda} = \frac{1}{V}\int_V \lambda(x,y,z)d$ as an average number density of flaws, Eq. (15) can also be presented as

$$P(\sigma_{\min f} < \sigma) = 1 - \exp\left(-\bar{\lambda}VF_c(\sigma)\right) \quad (16)$$

If $\lambda(x,y,z)$ is constant, Eq. (16) transforms into Eq. (12).

The distribution of the minimum failure stress in the stress range $0 \leqslant \sigma \leqslant \sigma_M$ is given by Eq. (16). Eq. (16), not the Weibull distribution, *is the correct mathematical expression of the weakest-link concept in the case of flaws whose locations follow a Poisson process in a stressed volume V*.

As can be verified, Eq. (12) gives a correct result for the probability of failure of the wire containing flaws. Indeed, beyond the stress level $\sigma_{\min}$, all flaws are critical and the conditional probability of initiating failure associated with a single flaw is $F_c(\sigma) = 1$ in Eq. (12). The probability of failure of the stressed length L is then given by Eq. (6). This probability is constant in the stress interval $(\sigma_{\min}, \sigma_{\max})$ as it should be. As can be verified, Eq. (12) yields also correct results in all stress ranges for the second counterexample, involving flaws from the same type and two different sizes, and avoids a major drawback discussed in relation with the Weibull model.

Eq. (12) regarding the distribution of the minimum failure stress can be generalized naturally to model the distribution of the minimum failure stress associated with multiple type of flaws (M type of flaws) present in the material.

$$P(\sigma_{\min f} \leqslant \sigma) = 1 - \exp\left(-V\sum_{i=1}^{M}\bar{\lambda}_i F_{ci}(\sigma)\right) \quad (17)$$

where $\bar{\lambda}_i$ is the average number density and F_{ci} is the probability that a flaw from the ith type will be critical. Denoting by $\bar{\lambda} = \sum_{i=1}^{M}\bar{\lambda}_i$ the total average number density of the flaws, Eq. (17) can be presented as

$$P(\sigma_{\min f} \leqslant \sigma) = 1 - \exp\left(-\bar{\lambda}V \times F_c(\sigma)\right) \quad (18)$$

where the expression

$$F_c(\sigma) = \sum_{i=1}^{M}\frac{\bar{\lambda}_i}{\bar{\lambda}_1 + \ldots + \bar{\lambda}_M} \times F_{ci}(\sigma) \quad (19)$$

is the probability that a flaw will be critical. This probability is formed from the sum of the probabilities that a flaw will be critical given that it belongs to the kth type of flaws, where $k = 1, 2, \ldots, M$. The probability that failure will be initiated by the kth type of flaws is equal to the product $\frac{\bar{\lambda}_k}{\bar{\lambda}_1+\ldots+\bar{\lambda}_M} \times F_{ck}(\sigma)$ of the probability $\frac{\bar{\lambda}_k}{\bar{\lambda}_1+\ldots+\bar{\lambda}_M}$ that the flaw will belong to the kth type of flaws and the conditional probability $F_{ck}(\sigma)$ that given that the flaw belongs to the kth type, it will initiate failure at a stress level σ.

In general, for multiple type of flaws, $F_c(\sigma)$ is not necessarily a power law even if all $F_{ci}(\sigma)$ are given by power laws. This can be demonstrated for two types of flaws only. In this case, the probability that a flaw will be critical is

$$F_c(\sigma) = a_1\sigma^{m1} + a_2\sigma^{m2} \quad (20)$$

where $a_1 > 0$, $a_2 > 0$, $\sigma \geqslant 0$, $m1 > 1$ and $0 \leqslant m2 < 1$ are assumed. The second derivative of $F_c(\sigma)$ in Eq. (20) is

$$\frac{\partial^2}{\partial\sigma}(F_c(\sigma)) = a_1 \times m1 \times (m1-1)\sigma^{m1-2} + a_2 \times m2 \times (m2-1)\sigma^{m2-2} \quad (21)$$

Since $m2 - 1 < 0$ there will exist a positive value $\sigma > 0$ where the second derivative will be zero, therefore, in this case, $F_c(\sigma)$ cannot be described by a power law.

The requirement for non-interacting flaws is essential for the validity of the weakest-link concept. Clustering of two or more flaws within a critical distance is associated with an interaction of their stress fields and the combined impact on the probability of failure is stronger compared to the case where no such interaction is present.

In many cases, clustering of flaws within a critical distance is strongly correlated with the probability of failure, particularly for thin fibers and wires. Indeed, clustering of two or more flaws within a small critical distance often decreases dangerously the load-bearing cross section and increases the stress concentration which further decreases the load-bearing capacity.

In order for the weakest-link concept to be applicable, failure must be initiated locally, by a single flaw. In the case of dense flaws, failure in fact can occur due to at least two failure modes: (i) due to individual flaws triggering failure and (ii) due to clustering of flaws within a critical distance. Furthermore, these failure modes are not statistically independent. Indeed, the fact that there exists clustering of flaws within a critical distance affects the probability that there will exist flaws, some of which may locally initiate failure. Let A_1 denote the event *no failure initiated from individual flaws* and A_2

denote the event *no failure initiated by clustering of two or more flaws within a critical distance*. $A_1 \cap A_2$ is the event that *no failure will occur during loading at the stress level* σ. The probability of the intersection of events A_1 and A_2 (the probability of no failure) is given by

$$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1) \tag{22}$$

where $P(A_2|A_1)$ in Eq. (22) is the conditional probability of no failure due to clustering of flaws within a critical distance given that 'no failure has been initiated by individual flaws'. Detailed discussion related to determining these probabilities can be found in Todinov (2005).

3.1. Monte Carlo verification regarding the distribution of the minimum failure stress of a stressed volume containing random non-interacting flaws

The expression regarding the distribution of the minimum failure stress has been verified by Monte Carlo simulations. A loading stress σ^* is first specified. The strength of the individual flaws follows a particular distribution $F_c(\sigma)$ (e.g. normal, log-normal, exponential).

The algorithm of the simulation procedure in pseudo-code is as follows.

Algorithm 1

```
Define loading stress sigma_star;
failure_counter=0;
for i=1 to Number_of_trials do
{
  num_flaws=Generate_random_number_of_flaws();
  min_failure_stress=BIG_Number;
  for j=1 to num_flaws do
    {
    flaw_strength=Sample_the_flaw_strength_distribution();
    if (min_failure_stress < flaw_strength) min_failure_stress=flaw_strength;
      }
      if (min_failure_stress < sigma_star) failure_counter=failure_counter+1;
}
      probability_of_failure=failure_counter/Number_of_trials;
```

For each simulation trial, a random number of flaws (num_flaws) following a homogeneous Poisson process is generated inside the stressed volume V by the procedure **Generate_random_number_of_flaws**(), whose algorithm can be found in books on Monte Carlo simulation. The locations of the flaws follow a homogeneous Poisson process with a specified density λ.

Next, for each of the generated num_flaws a strength is generated in the variable flaw_strength by sampling the specified strength distribution $F_c(\sigma)$ of the flaws. The sampling from the strength distribution is performed by using the *inverse transformation method* – a well-documented method in books on simulation. Simultaneously, the flaw with the smallest strength among the generated flaws is determined. After exiting the inner loop, the minimum strength characterizing the generated flaws, stored in the variable min_failure_stress, is compared with the specified loading stress sigma_star. If the min_failure_stress is smaller than the loading stress sigma_star, the specimen will fail and the failure counter is incremented.

The probability of failure is obtained as a ratio of the number of simulations during which the minimum failure stress has been smaller than the loading stress σ^* and the total number (Number_of_trials) of Monte Carlo simulation trials.

The results from the simulations coincided with the results obtained from a direct calculation using Eq. (12). A stressed volume of $V = 100\ \text{cm}^3$ with a flaw number density $\lambda = 0.15\ \text{cm}^{-3}$ were assumed.

For a flaw strength given by the exponential distribution $F_c(\sigma) = 1 - \exp(-\sigma/\mu)$, for example, where $\mu = 900$ MPa, and the loading stress is $\sigma^* = 200$ MPa, the simulations yielded $p_f \approx 0.95$ for the probability of failure. Since $F_c(\sigma^*) = 1 - \exp(-200/900) = 0.199$, the substitution in Eq. (12) yields $p_f = 1 - \exp(-0.15 \times 100 \times 0.199) \approx 0.95$ for the probability of failure.

For a flaw strength described by a normal distribution with mean $\mu = 900$ MPa, standard deviation $s = 108$ MPa and the loading stress is $\sigma^* = 1000$ MPa, the simulations yielded $p_f \approx 0.38$ for the probability of failure. Since $F_c(\sigma^*) = \Pr(\sigma \leqslant 1000) \approx 0.032$ for the normal distribution, the substitution in Eq. (12) yields $p_f = 1 - \exp(-0.15 \times 100 \times 0.032) \approx 0.38$ for the probability of failure.

For a flaw strength described by a log-normal distribution with mean $\mu = 8$ MPa, standard deviation $s = 0.6$ MPa of the log-data, and the loading stress is $\sigma^* = 900$ MPa, the simulations yielded $p_f \approx 0.29$ for the probability of failure. Since $F_c(\sigma^*) = \Pr(\sigma \leqslant 900) \approx 0.023$ for the log-normal distribution, the substitution in Eq. (12) yields $p_f = 1 - \exp(-0.15 \times 100 \times 0.023) \approx 0.29$ for the probability of failure.

4. Physical meaning of the probability that a flaw will be critical

4.1. Case I: Failure controlled by the size of the flaws

Suppose that the material of the loaded specimen contains flaws with a number density λ, that become unstable if the maximum tensile stress exceeds a particular critical value, inversely proportional to the square root of the flaw size. According to the stress intensity approach (discussed in any book on Fracture Mechanics), fast fracture occurs if the stress intensity factor $K_I = Y\sigma\sqrt{\pi a}$ becomes equal to the fracture toughness K_{Ic}:

$$Y\sigma\sqrt{\pi a} = K_{Ic} \tag{23}$$

where Y is the geometry factor and a is the flaw size. The failure criterion therefore has the form

$$\sigma_c = \frac{C}{\sqrt{a}} \tag{24}$$

where C is a constant depending on the material and geometry. From this equation, for a specified loading stress σ, the critical flaw size that causes fracture becomes:

$$a_{cr} = C^2/\sigma^2 \tag{25}$$

All flaws with size $a \geqslant a_{cr}$ are also critical and will cause fracture if present in the stressed volume.

In the case of fracture controlled by the size of the flaws during uniaxial tension, the probability that a flaw will be critical $F_c(\sigma)$ is simply the probability $F_c(\sigma) = P(a > a_{cr})$ that the size of the flaw will be greater than the critical flaw size $a_{cr} = C^2/\sigma^2$ corresponding to the applied stress σ. For the probability $F_c(\sigma)$, we have

$$F_c(\sigma) = P(a > a_{cr}) = 1 - G(a_{cr}) \tag{26}$$

where $G(a)$ is the cumulative distribution of the flaw size. Substituting this in Eq. (12) gives

$$P(\sigma_{\min,f} \leqslant \sigma) = 1 - \exp\left(-\lambda V[1 - G(C^2/\sigma^2)]\right) \tag{27}$$

for the distribution of the minimum failure stress. This dependence is valid for any flaw size distribution. Eq. (27) is particularly suited for determining the probabilities of failure from the lower tail of the distribution of the fracture stress – the region corresponding to the largest flaws in the material.

For fracture controlled by the size of flaws, characterized by constant number densities $\lambda_1, \ldots, \lambda_M$, Eq. (19) becomes

$$F_c(\sigma) = \sum_{i=1}^{M} \frac{\lambda_i}{\lambda_1 + \ldots + \lambda_M}[1 - G_i(C^2/\sigma^2)] \tag{28}$$

where $G_i(\bullet)$ is the cumulative size distribution of the flaws from the ith type.

Now, assume a cumulative distribution of the flaw size, given by the exponential distribution:

$$G(a) = 1 - \exp[-a/a_m] \tag{29}$$

where a_m is the mean flaw size in µm. Since $1 - G(C^2/\sigma^2) = \exp[-k/\sigma^2]$ where $k = C^2/a_m$ is a constant, Eq. (27) results in

$$P(\sigma_{f,\min} \leqslant \sigma) = 1 - \exp\left(-\lambda V \times \exp[-k/\sigma^2]\right) \tag{30}$$

for the probability of failure of the component at a loading stress σ.

Eq. (30) has been verified by a simulation experiment. The mean flaw size a_m in Eq. (29) was assumed to be $a_m = 300$ µm. The constant C in Eq. (24) was assumed to be 2000×10^6, $k = C^2/a_m = 13333 \times 10^{12}$ is a constant, $\lambda V = 50$ is the expected number of flaws in the stressed volume. The simulation experiments followed Algorithm 1. Random flaws were generated, whose number inside the volume V of the 'specimen' follows a Poisson distribution with mean $\lambda V = 50$. Each flaw size has been obtained from sampling the tested flaw size distribution. For each flaw, the critical stress that makes it unstable was calculated. The critical stress was calculated from Eq. (24). For each simulation, the minimum failure stress associated with the volume V was determined as the minimum critical stress characterizing the generated population of flaws in the simulation trial. The minimum failure stresses from 1000 simulation trials were finally analysed by a double-logarithm plot.

Taking a double logarithms from Eq. (30) results in

$$\ln[-\ln(1 - p_f(\sigma))] = \ln(\lambda V) - k/\sigma^2 \tag{31}$$

If $z_i = \ln[-\ln(1 - \hat{F}_i)]$ are plotted versus $1/\sigma_f^2$, where σ_f is the simulated minimum failure stress, a plot which conforms to a straight line will be obtained if the stimulated minimum failure stress complies with Eq. (30). $\hat{F}_i \approx i/(n+1)$ are rank approximations for the probability of failure, x_i, $i = 1, 2, \ldots, n$ are the ordered simulated minimum failure stresses and n is their number ($n = 1000$).

The inverse of the square of the simulated minimum failure stress plotted versus z_i produced points falling closely along a straight line (Fig. 5). This shows that the simulated minimum failure stress complies with Eq. (30).

In earlier work (Todinov, 2008), we showed that flaws whose size follows a normal distribution, result in a dependence for the conditional probability of failure with an inflection point. Here we show that any unimodal flaw size distribution must have an inflection point.

Indeed, assume that the size distribution of the flaws in the material is unimodal (Fig. 6a). With increasing the loading stress σ, the probability $F_c(\sigma)$ will increase or stay the same because more and more flaw sizes will become critical. The largest increase of the probability $F_c(\sigma)$ will occur when the loading stress makes critical the flaws corresponding to the mode of the flaw size distribution (Fig. 6a).

In the vicinity of the stress $\sigma = \frac{C}{\sqrt{a_m}}$, an elementary increase of the stress $\Delta\sigma$ will correspond to a maximum elementary increase $\Delta n = n(a_m) \times \Delta a$ of the number density of critical flaws. Therefore, the inflection point $\sigma_{\inf}$, marking the fastest increase of the probability of initiating failure associated with a single flaw, is linked with the mode a_m of the flaw size number density by $\sigma_{\inf} = \frac{C}{\sqrt{a_m}}$. The probability $F_c(\sigma)$ therefore, cannot be approximated by a three-parameter power law. Despite this, the three-parameter Weibull distribution often gives good fits even for the type of stress dependence in Fig. 6b.

Again, the reason is that if a large number of flaws are present in the stressed volume, there is increased likelihood that relatively large flaw sizes will be present, associated with failure stress from the lower tail of $F_e(\sigma)$. If the lower tail of $F_c(\sigma)$ can be closely approximated by a power law dependence, the Weibull distribution will yield a good fit.

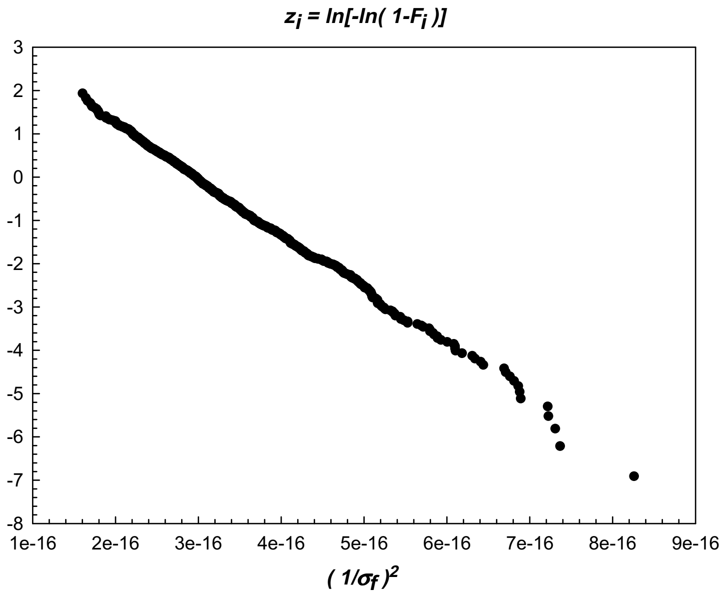

Fig. 5. A probability plot of the minimum simulated failure stress. The plot confirms the validity of Eq. (30).

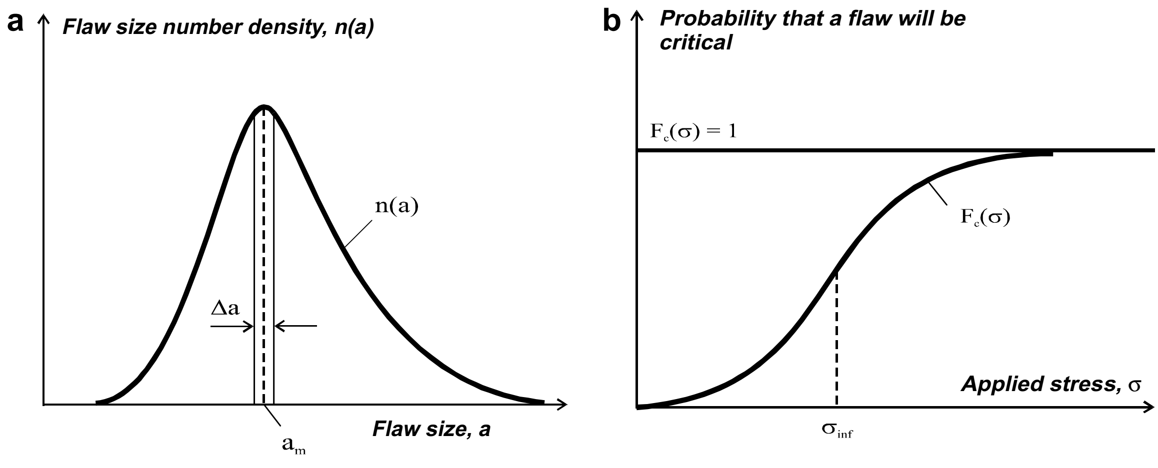

Fig. 6. For a unimodal distribution of the flaw size number density, the probability that a flaw will be critical is characterized by an inflection point that is linked with the mode of the flaw size distribution.

4.2. Case II: Failure controlled by the orientation of the flaws

Suppose that in the bar from Fig. 7a, subjected to a uniaxial tension, a number of flaws exist, shaped as thin discs of the same size but with different orientation. The condition expressing the instability of a flaw is dependent solely on its orientation. Suppose for simplicity, that if the normal stress σ_n to such a disc-shaped flaw exceeds a particular critical value σ_{cr}, the flaw will initiate fracture.

Clearly, if the loading stress is smaller than the critical value $\sigma_{cr}(\sigma < \sigma_{cr})$, irrespective of the flaw orientation, there will be no locally initiated failure. In this case, the probability that a flaw will be critical is zero ($F_c(\sigma) = 0$).

In the case where the loading stress is greater than the critical stress σ_{cr}, the condition for instability is $\sigma_n = \sigma\cos^2\theta \geqslant \sigma_{cr}$ or $\theta \leqslant \theta^* = \arccos(\sigma_{cr}/\sigma)^{0.5}$. If the orientation of the normal is random, the conditional probability $F_c(\sigma) = P(\sigma_n \geqslant \sigma_{cr})$ that $\sigma_n = \sigma\cos^2\theta \geqslant \sigma_{cr}$ will be fulfilled is equal to the probability that the normal to the flaw will subtend with the direction of the loading stress an angle smaller than the critical angle θ^* $F_c(\sigma) = P\,(\sigma_n \geqslant \sigma_{cr}) = P(\theta \leqslant \theta^*)$, Fig. 7.

The probability $P(\theta \leqslant \theta^*)$ can be determined from the ratio of twice the curved area of the spherical cap defined by the critical angle θ^* and the surface area of a sphere with radius R. The area of a spherical cap with radius R, defined by angle θ^* is $2\pi R^2(1 - \cos\theta^*)$. The total area of the sphere is $4\pi R^2$. The probability $P(\sigma_n = \sigma\cos^2\theta \geqslant \sigma_{cr})$ is therefore given by

$$F_c(\sigma) = P(\sigma_n \geqslant \sigma_{cr}) = \frac{4\pi R^2(1 - \cos\theta^*)}{4\pi R^2} = 1 - \cos\theta^* = 1 - (\sigma_{cr}/\sigma)^{0.5} \qquad (32)$$

Finally, for the probability that a flaw will be critical, the dependence

$$F_c(\sigma) = \begin{cases} 0, & \sigma \leqslant \sigma_{cr} \\ 1 - (\sigma_{cr}/\sigma)^{0.5}, & \sigma > \sigma_{cr} \end{cases} \qquad (33)$$

is obtained.

According to Eq. (12), for random flaws whose locations follow a homogeneous Poisson process, the distribution of the minimum failure stress is given by

$$P(\sigma_{f,\min} \leqslant \sigma) = \begin{cases} 0, & \sigma \leqslant \sigma_{cr} \\ 1 - \exp[-\lambda V(1 - (\sigma_{cr}/\sigma)^{0.5})], & \sigma > \sigma_{cr} \end{cases} \qquad (34)$$

Eq. (34) has also been verified by a simulation experiment, where the critical stress σ_{cr} has been taken to be σ_{cr} = 950 MPa. The algorithm in pseudo-code is presented in what follows.

Algorithm 2

```
Define a critical stress sigma_cr;
Define a reference stress sigma_ref;
failure_counter=0;
for i=1 to Number_of_trials do
{
  num_flaws=Generate_random_number_of_flaws();
  min_failure_stress=BIG_Number;
for j=1 to num_flaws do
  {
  cos_fi=Generate_random_orientation_with_respect_to_acting_stress();
  failure_stress=sigma_cr/ (cos_fi)^2;
  if (min_failure_stress < failure_stress) min_failure_stress=failure_stress;
  }
  if (min_failure_stress < sigma_ref) failure_counter=failure_counter+1;
}
  probability_of_failure=failure_counter/Number_of_trials;
```

The difference from Algorithm 1 is that for each flaw, the cosine of a random orientation angle is generated with respect to the direction of the acting stress. The cosine of a random angle is generated from

$$\cos(\varphi_i) = 1 - 2 \times u_i \qquad (35)$$

where u_i are uniformly distributed numbers in the interval (0,1) (Sobol, 1994).

The failure stress $\sigma_{f,i}$ characterizing a randomly oriented flaw is generated from $\sigma_{f,i} = \sigma_{cr}/(\cos(\varphi_i))^2$ because failure occurs if the component of the loading stress $\sigma \times (\cos(\varphi_i))^2$ is equal to or greaterthan the critical stress σ_{cr}.

The results from the simulation were confirmed by results obtained directly from Eq. (34). Thus, for a loading stress of 950 MPa, λ = 0.15 cm^{-3} and V = 100 cm^3, both the simulations and Eq. (34) yielded zero for the probability of failure. For a loading stress of 970 MPa, both the simulation and Eq. (34) yielded probability of failure 0.144. For a loading stress of 1000 MPa, both the simulation and Eq. (34) yielded 0.316 for the probability of failure. As can be verified from Fig. 8, the

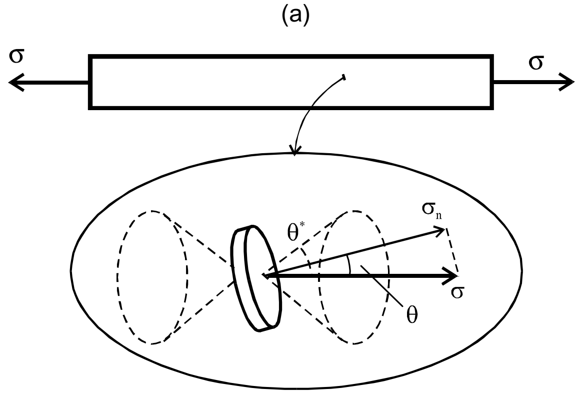

Fig. 7. A bar subjected to a uniaxial tension, containing disc-type flaws of equal size and random orientation.

probability of failure controlled by the flaw orientation quickly increases with increasing the loading stress after which the rate decreases. For a flaw number density $\lambda = 0.3\ \text{cm}^{-3}$, in the vicinity of the critical stress $\sigma_{cr} = 950$ MPa, a variation of the loading stress of 2% only causes an increase of the probability of failure by 27%.

This result is confirmed by taking the differential of expression (34) in the vicinity of $\sigma = 950$ and substituting the numbers.

$$\Delta p_f(\sigma) \approx \frac{\lambda V}{2\sigma_{cr}} \Delta\sigma \tag{36}$$

The increase of the magnitude of the probability of failure $\Delta p_f(\sigma)$ is directly proportional to the flaw number density λ.

Fig. 9 shows simulation results regarding the variation of the probability of failure of a specimen with volume $V = 100\ \text{cm}^3$ containing two types of flaws. The first type of flaws is characterized by a number density $\lambda_1 = 0.015\ \text{cm}^3$ and a critical normal stress of triggering fracture $\sigma_{cr,1} = 450$ MPa; the second type of flaws is characterized by a flaw number density $\lambda_2 = 0.085\ \text{cm}^3$ and a critical normal stress of triggering fracture $\sigma_{cr,2} = 950$ MPa.

Clearly, the distribution from Fig. 9 cannot be approximated by a Weibull distribution. Eq. (18) however, yields the correct probability of failure.

Indeed, the conditional probability for initiating failure characterizing the two types of flaws is $F_{c1}(\sigma) = 1 - (\sigma_{cr,1}/\sigma)^{0.5}$ and $F_{c2}(\sigma) = 1 - (\sigma_{cr,2}/\sigma)^{0.5}$, respectively. According to Eq. (19), the probability that a flaw will be critical is

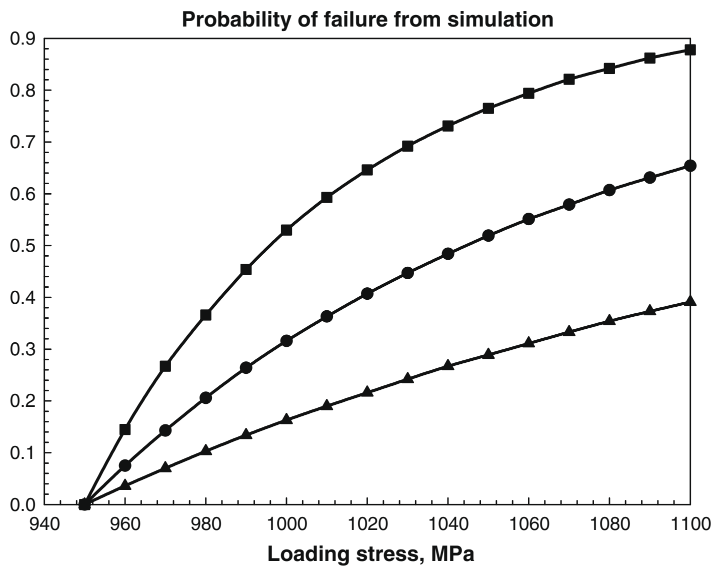

Fig. 8. Variation of the probability of failure with the loading stress in the case of failure controlled by the orientation of the flaws (a single type of flaws).

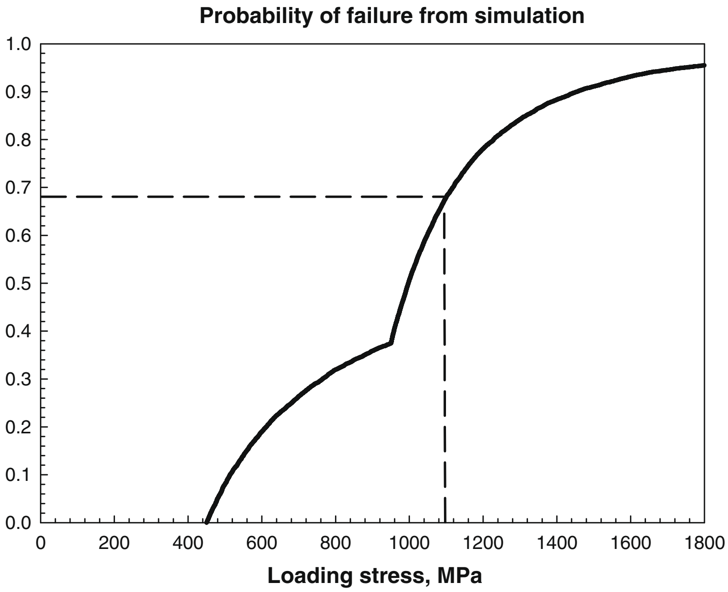

Fig. 9. Variation of the probability of failure with the loading stress in the case of failure controlled by the orientation of two types of flaws.

$$F_c(\sigma) = \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - (\sigma_{cr,1}/\sigma)^{0.5}) + \frac{\lambda_2}{\lambda_1 + \lambda_2}(1 - (\sigma_{cr,2}/\sigma)^{0.5}) \tag{37}$$

For a loading stress of σ = 1100 MPa, Eq. (37) yields $F_c(\sigma)$ = 0.114. Substituting this value in Eq. (18) gives

$$p_f(\sigma) = 1 - \exp[-0.1 \times 100 \times 0.114] \approx 0.68$$

for the probability of failure.

This result is confirmed by the simulation (see the dashed line in Fig. 9) which illustrates the validity of Eq. (18) regarding the distribution of the minimum failure stress in the case of multiple types of flaws.

5. The negative power law flaw size distribution and the Weibull distribution

According to the earlier discussion, for a finite number of flaws following a homogeneous Poisson process, if the Weibull Eq. (3) holds, the relationship:

$$\lambda_{cr}(\sigma) = \lambda \times P(a \geqslant a_{cr}) = ((\sigma - \sigma_l)/\sigma_0)^m \tag{38}$$

will also hold for the number density of the critical flaws. Assume that σ_l = 0. From Eq. (25) we get $\sigma = C/\sqrt{a_{cr}}$ and introduction of this in Eq. (38) results in

$$P(a \geqslant a_{cr}) = \left(\frac{C}{\sigma_0 \lambda^{1/m} \sqrt{a_{cr}}}\right)^m \tag{39}$$

Suppose that the smallest flaw size is a_0. For a loading stress

$$\sigma_{\max} = C/\sqrt{a_0} \tag{40}$$

the smallest flaw will become critical, therefore all existing flaws in the material will also be critical. Considering this and also the fact that for the smallest flaw size a_0, $P(a \geqslant a_0) = 1$ holds, Eq. (38) yields

$$\lambda = (\sigma_{\max}/\sigma_0)^m \tag{41}$$

for the number density of all flaws. Substituting $\sigma_{\max}$ obtained from Eq. (40) results in

$$\lambda^{1/m} = \left(\frac{C}{\sigma_0 \sqrt{a_0}}\right) \tag{42}$$

which, after the introduction in Eq. (39) gives

$$P(a \geqslant a_{cr}) = \left(\frac{a_0}{a_{cr}}\right)^{m/2} \tag{43}$$

As a result, the negative power law distribution of the size X of the flaws

$$F(x) \equiv P(X \leqslant x) = 1 - (a_0/x)^{m/2} \tag{44}$$

is compatible with the Weibull distribution in the stress range 0, $\sigma_{\max}$.

Indeed, considering Eq. (43), the number density of the critical flaws at a loading stress σ is

$$\lambda_{cr} = \lambda \times P(a \geqslant a_{cr}) = (a_0/a_{cr})^{m/2} \tag{45}$$

Since $\sqrt{a_{cr}} = C/\sigma$, substituting in Eq. (38) results in $\lambda_{cr} = (\sigma/\sigma_0)^m$, where $\sigma_0 = \frac{C}{\lambda^{1/m}\sqrt{a_0}}$, $\sigma_l = 0$. Finally, substituting $\lambda_{cr} = (\sigma/\sigma_0)^m$ in Eq. (4) yields the Weibull distribution (3).

The negative power law distribution is very common. Phase transitions in thermodynamic systems, for example, are associated with the emergence of power law distributions. Furthermore, the upper tails of various flaw size distributions can often be approximated well by the negative power law distribution. This all goes towards explaining why such a large number of fracture data sets are often fitted very well by the Weibull distribution.

Beyond the stress $\sigma_{\max}$ however, all existing flaws will become critical and increasing the stress will no longer increase the number density of critical flaws. As a result, the negative power law distribution of the flaw size is compatible with the Weibull distribution up to a stress level corresponding to the smallest flaw size a_0. Beyond this stress level, the Weibull distribution will yield a larger probability of failure initiated by flaws. The actual probability of failure will remain the same in this case, because the number density of the critical flaws will not increase.

6. Conclusions

1. The Weibull distribution is incapable of correctly predicting the probability of failure in the simple cases of:
   – identical flaws;
   – two flaw size groups, each of which contains identical flaws;
   – failure controlled by the orientation of two different types of flaws;
   – and also beyond a stress level where no new critical flaws are created by increasing the applied stress.

   In all these cases, the probability of failure is correctly predicted by the suggested alternative equation.
2. In the case of non-interacting flaws randomly distributed in a stressed volume, the Weibull distribution predicts correctly the probability of failure if and only if the stress dependence of the probability that a flaw will be critical is a power law or can be approximated well by a power law.
3. Contrary to the common belief, in the case of non-interacting flaws in a stressed volume, the Weibull distribution is not the mathematical formulation of the weakest-link concept.
4. For non-interacting flaws characterized by a strength distribution $F_c(\sigma)$, whose locations in a volume V follow a Poisson process with average number density $\bar{\lambda}$, the distribution of minimum failure stress is described by $P(\sigma_{f,\min} \leqslant \sigma) = 1 - \exp[-\bar{\lambda} V F_c(\sigma)]$. This is the mathematical expression of the weakest-link concept in the case of failure locally initiated by flaws. The equation does not require any assumptions concerning the physical nature of the flaws and the physical mechanism of failure and can be applied in any situation of a locally initiated failure by non-interacting entities.
5. For a relatively large number of flaws in the tested specimens, the recorded failure stress often remains in the stress region of the lower tail of the probability that a flaw will be critical. Often, this region can be approximated well by a power law stress dependence and the Weibull model produces a good fit of the failure data.
6. The negative power law distribution of the flaw sizes is compatible with the Weibull distribution up to the stress level corresponding to the smallest flaw size. Beyond this threshold, the Weibull distribution fails to predict correctly the probability of failure.
7. The probability of failure controlled by the orientation of the flaws is directly proportional to the flaw number density and increases quickly with increasing the loading stress after which the rate decreases.
8. The probability that a flaw will be critical has a clear physical meaning both in the case of failure controlled by the size of the flaws and in the case of failure controlled by the orientation of the flaws.

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