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q-Coverings, Codes, and Line Graphs

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In this paper we consider the relationship between q-coverings of a regular graph and perfect 1-codes in line graphs. An infinite class of perfect 1-codes in the line graphs $L(O_k)$ is constructed.

1. INTRODUCTION

Let e be a positive integer. A *perfect e-code* in a graph Γ is a non-empty subset C of the vertex set of Γ with the property that every vertex of the graph is at distance at most e from a unique vertex in C. In [2] Biggs proves that if a regular graph Γ contains a perfect 1-code, then -1 is an eigenvalue of its adjacency matrix. In Section 2 we generalize this result to a particular class of q-coverings and we also establish a connection between the existence of these q-coverings of Γ and perfect 1-codes in the line graph of Γ. In Section 3 we construct an infinite family of perfect 1-codes in the regular line graphs $L(O_k)$ $(k > 2)$.

2. q-COVERINGS AND PERFECT 1-CODES

Let q denote a mapping of the non-negative integers into the rationals with the property that $q(i) = 0$ implies $q(j) = 0$ for all $j > i$. Let σ be the largest integer for which q is non-zero. Γ denotes a connected, finite graph with distance function ∂, diameter d, and vertex set $V\Gamma$. Let ∂_L be the distance function in the line graph, $L(\Gamma)$, of Γ. The above notation will remain fixed throughout the paper.

DEFINITION. If q and Γ are given, such that $\sigma < d$, then a subset C of $V\Gamma$ will be called a *q-covering of Γ* when the numbers

$$\alpha(v) = \sum_{c \in C} q(\partial(c, v)) \qquad (v \in V\Gamma)$$

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are all equal to some constant α. If $\alpha = q(0)$, then the q-covering is said to be *sparse*.

If $\partial(v, C)$ is at most e' for each $v \in V\Gamma$, then we say that C has *external distance* e' in Γ.

In [4] Biggs states that if $q(i) = 1$ $(0 \leqslant i \leqslant e)$ and $q(i) = 0$ $(i > e)$, then a sparse q-covering of Γ is a perfect e-code. A necessary condition for the existence of a perfect 1-code in a regular graph is that -1 is an eigenvalue of its adjacency matrix [2, p. 2]. We generalize this result to sparse q-coverings with q satisfying

$$q(1) = 1 \qquad \text{and} \qquad q(i) = 0 \qquad (i > 1). \tag{*}$$

LEMMA 1. *If the regular graph Γ contains a sparse q-covering, with the particular q defined in* (*), *then $-q(0)$ is an eigenvalue of the adjacency matrix of Γ.*

Proof. For each $v \in V\Gamma$

$$q(0) = \alpha(v) = \sum_{c \in C} q(\partial(c, v)) = q(0)\,|\theta_0(v)| + |\theta_1(v)|,$$

where $\theta_i(v) = \{c \in C \mid \partial(v, c) = i\}$. If $\partial(v, C) = 1$, then $|\theta_0(v)| = 0$ and so $|\theta_1(v)| = q(0)$. Let Γ have adjacency matrix A and valency k. If $\mathbf{c}$ is defined by

$$(\mathbf{c})_u = \begin{cases} 1, & u \in C \\ 0, & \text{otherwise,} \end{cases}$$

then

$$A\mathbf{c} = q(0)(\mathbf{u} - \mathbf{c}),$$

where $\mathbf{u} = [1,\ldots, 1]'$. Let $\mathbf{x} = \mathbf{u} - (1 + k/q(0))\,\mathbf{c}$. Then

$$\begin{aligned} A\mathbf{x} &= A\mathbf{u} - (1 + k/q(0))\,A\mathbf{c} \\ &= k\mathbf{u} - (1 + k/q(0))\,q(0)(\mathbf{u} - \mathbf{c}) \\ &= -\,q(0)\,\mathbf{x}. \end{aligned}$$

Thus $-q(0)$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. ∎

From [4, p. 117] we know that a sparse q-covering defined by (*) has minimum distance at least 2 and it it is not difficult to see it also has external distance 1. If $q(0) = 1$, then such a q-covering is a perfect 1-code with minimum distance 3; and if $q(0) > 1$, then, in the terminology of [6], it is locally regular code with minimum distance 2, external distance 1, and parameter $p_{11}(C) = q(0)$. As we shall now see, the case where $q(0) = k - 1$ is particularly interesting.

LEMMA 2. *A regular k-valent graph Γ contains a q-covering with* $q(0) = k - 1$, $q(1) = 1$, *and* $q(i) = 0$ $(i > 1)$ *if and only if its line graph* $L(\Gamma)$ *contains a perfect* 1*-code.*

Proof. Suppose that C_L is a perfect 1-code in $L(\Gamma)$. Let C be the set of vertices of $V\Gamma$ which are not incident with any member of C_L. Obviously C has external distance 1 and minimum distance 2. If $v \in C$, then

$$\alpha(v) = \sum_{c \in C} q(\partial(c, v)) = k - 1 + |\theta_1(v)| = k - 1$$

and if $v \notin C$, then

$$\alpha(v) = |\theta_1(v)| = k - 1,$$

since there is exactly one edge incident with v and also contained in C_L.

Conversely suppose that Γ contains a q-covering C with the relevant function q. Therefore, C has external distance 1 and for $z \in V\Gamma$ with $\partial(z, C) = 1$ there is a unique vertex u_z such that $\partial(z, u_z) = 1$ and $u_z \notin C$. Let C_L be the set of edges in Γ connecting each z, with $\partial(z, C) = 1$, and its unique u_z. Choose any edge e of Γ. If $e \notin C_L$, then e is incident with two vertices w and v of Γ such that $w \notin C$ and $v \in C$. There is exactly one edge e_1, namely, the edge between w and u_w, which is in C_L and has $\partial_L(e, e_1) = 1$. Thus C_L is a perfect 1-code in $L(\Gamma)$. ∎

Lemmas 1 and 2 imply that when $L(\Gamma)$ contains a perfect 1-code, $-(k-1)$ is an eigenvalue of A, the adjacency matrix of Γ, and -1 is an eigenvalue of A_L, the djacency matrix of $L(\Gamma)$. These two necessary conditions are satisfied simultaneously because Sachs [7] has proved that for $\lambda \neq -k$, λ is an eigenvalue of A if and only if $\lambda + k - 2$ is an eigenvalue of $L(\Gamma)$.

3. PERFECT 1-CODES AND LINE GRAPHS

A number of perfect 1-codes other than those of classical coding theory are mentioned by Biggs in [2]. New infinite classes of perfect 1-codes have been constructed by Cameron, Thas, and Payne [5], and Thas [8]. In the following result we construct an infinite class of perfect 1-codes in the family of line graphs $L(O_k)$ $(k > 2)$. The *odd graphs* O_k $(k > 1)$ have the $(k-1)$-subsets of a $(2k-1)$-set as their set of vertices and two vertices are adjacent whenever the subsets are disjoint. O_k has valency k, diameter $k-1$, and the eigenvalues of its adjacency matrix are $(-1)^{k-i}i$ $(1 \leqslant i \leqslant k)$ [1].

THEOREM. *If $k > 2$, then $L(O_k)$ contains a perfect 1-code.*

Proof. Let $X = \{2, 3, \ldots, 2k-2\}$ with $k > 2$. Let A be the set of $(k-2)$-subsets of X and let $B = \{a \cup \{1\} \mid a \in A\}$. If $b = a \cup \{1\}$ and $a \in A$, then b' denotes $X \backslash a$. Let $C_L = \{b - b' \mid b \in B\}$, where $u - v$ denotes the unique edge of Γ joining the vertices u and v. We prove that C_L is a perfect 1-code in $L(O_k)$. Choose any vertex $c - d$ of $L(O_k)$. If $c \cup d = \{1, 2, \ldots, 2k-2\}$, then $c - d \in C_L$. If $2k - 1 \in c \cup d$, then $c - d \notin C_L$. Suppose, without loss of generality, that $2k - 1 \in c$ and define c_1 to be $X \cup \{1\} \backslash d$. Then $c_1 - d \in C_L$ and $\partial_L(c - d, c_1 - d) = 1$. Thus every vertex of $L(O_k)$ is at distance at most 1 from a unique vertex of C_L. ∎

COROLLARY. *For $k > 2$, O_k contains a q-covering with $q(0) = k - 1$, $q(1) = 1$, and $q(i) = 0$ $(i > 1)$.*

Proof. This result follows immediately from the preceding theorem and Lemma 2. ∎

Most of the recent results on perfect codes (and on the more general class of q-coverings) have been in the setting of distance-regular graphs. The number of line graphs which are distance-regular is small and $L(O_k)$ is distance-regular only when $k < 4$ [3]. However, there is at least one distance-regular line graph which contains a non-trivial perfect code. This is the perfect 1-code in the line graph of the 8-cage of valency 3. Perfect 1-codes are also contained in the line graphs of Desargue's graph and the dodecahedron.

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