Singular perturbation to shock wave solutions of Burger’s K-dV equation

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1. Introduction

It is well known that both Burger’s equation and K-dV equation present interesting and complex nonlinear phenomena [1]. In this paper, we consider a singular perturbation problem in a mix-type partial differential equation—the Burger’s K-dV equations (BKDVE)

\[ U_t + U U_x - \gamma U_{xx} + \beta U_{xxx} = 0 \]  

with constants \( \gamma, \beta > 0 \). We study its shock wave solutions via traveling wave solutions, and show that under certain conditions, the shock wave solutions are the exact solutions of the BKDVE. Our study of the singular perturbation to BKDVE supports an expansion of the monotone shock wave solutions in terms of small parameter \( \beta > 0 \) given by Jeffrey in a communication article [2], which provided no analytical details. We answered the following two questions raised with his expansion. (1) Whether the expansion addresses the singular perturbation when \( \beta \) tends to 0? (2) Whether the expansion represents the true solution of the nonlinear partial differential equation BKDVE? We give some preliminary results based on conclusions in [3]. We also study the global structure of the phase-plane derived from Eq. (1), and provide results of the shock wave solutions of the BKDVE.

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2. Bounded traveling wave solutions of BKDVE

Let \( \xi = x - ct \), a transformation with constant \( c > 0 \). Consider the solutions of a special form of Eq. (1):

\[
U(x, t) = U(x - ct).
\]

These solutions represent waves traveling with a constant speed \( c \). With the transformation, Eq. (1) becomes an ordinary differential equation

\[
-cU_\xi + UU_\xi - \gamma U_\xi U_\xi + \beta U_\xi \xi = 0
\]

then

\[
-cU + \frac{1}{2} U^2 - \gamma U_\xi + \beta U_\xi \xi = c_1
\]

where \( c_1 \) is an arbitrary constant. It is then equivalent to the following system

\[
\begin{align*}
\frac{dU}{d\xi} &= v, \\
\frac{dv}{d\xi} &= \frac{1}{\beta}(c_1 + cU - \frac{1}{2}U^2 + \gamma v).
\end{align*}
\]

We have the following results for the solutions.

**Lemma 1.** System (4) has two singular points: \( A(a, 0) \) and \( B(b, 0) \). They are independent of the parameter \( \beta \). \( A \) is a saddle point and \( B \) a nodal point if \( \gamma^2 \geq 4\beta\sqrt{c^2 + 2c_1} \) or a spiral point if \( \gamma^2 < 4\beta\sqrt{c^2 + 2c_1} \).

**Proof.** Setting the right-hand side of Eq. (4) to 0, one readily obtains the only two singular points, \( A(a, 0) \) and \( B(b, 0) \), with

\[
a = c - \sqrt{c^2 + 2c_1}, \quad b = c + \sqrt{c^2 + 2c_1}.
\]

Obviously, they are independent of \( \beta \). The eigenvalues and corresponding eigenvectors determined by the linearization of system (4) are

\[
\lambda_{1,2}(A) = \frac{\gamma \pm \sqrt{\gamma^2 + 4\beta\sqrt{c^2 + 2c_1}}}{2\beta}
\]

and

\[
V_{1,2}(A) = (1, \lambda_{1,2}(A))
\]

at \( A \), and

\[
\lambda_{1,2}(B) = \frac{\gamma \pm \sqrt{\gamma^2 - 4\beta\sqrt{c^2 + 2c_1}}}{2\beta},
\]

and

\[
V_{1,2}(B) = (1, \lambda_{1,2}(B))
\]
at \( B \). \( A \) is a saddle point with \( \lambda_1(A) > 0 \) and \( \lambda_2(A) < 0 \), while \( B \) a nodal point with \( \lambda_1(B) > 0 \) and \( \lambda_2(B) > 0 \) for \( \gamma^2 \geq 4\beta \sqrt{c^2 + 2c_1} \), or a spiral point with complex eigenvalues \( \lambda_1(B) \) and \( \lambda_2(B) \) for \( \gamma^2 < 4\beta \sqrt{c^2 + 2c_1} \). Furthermore, one notices the following limits when \( \beta \to 0 \)

\[
\begin{align*}
\lambda_1(A) & \to +\infty, & V_1(A) & \to V_1^0(A) = (1, +\infty), \\
\lambda_1(B) & \to +\infty, & V_1(B) & \to V_1^0(B) = (1, +\infty), \\
\lambda_2(A) & \to \frac{\sqrt{c^2 + 2c_1}}{\gamma}, & V_2(A) & \to V_2^0(A) = (1, \lambda_2^0(A)), \\
\lambda_2(B) & \to \frac{\sqrt{c^2 + 2c_1}}{\gamma}, & V_2(B) & \to V_2^0(B) = (1, \lambda_2^0(B)).
\end{align*}
\]

Lemma 2. For \( c^2 + 2c_1 > 0 \), Eq. (3) has a unique bounded solution subject to a translation, which corresponds to the heteroclinic trajectory in the phase plane between the only two singular points of system (4). This unique solution satisfies \( U_\xi < 0 \) if \( \gamma^2 \geq 4\beta \sqrt{c^2 + 2c_1} \).

The proof is given in [3] and partly in [1].

Since we are interested in the singular perturbation of Eq. (1), we assume that \( \beta \) is sufficiently small. Therefore, only the case \( \gamma^2 \geq 4\beta \sqrt{c^2 + 2c_1} \) holds. Thus \( B \) is a nodal point. For the remaining of this paper, we assume \( \gamma^2 \geq 4\beta \sqrt{c^2 + 2c_1} \).

With Lemmas 1 and 2, the global structure of the solutions of system (4) is clear as shown in Fig. 1. The asymptotic behavior of the bounded solutions of (3) at singular points \( A \) and \( B \) are as follows.

Lemma 3.

\[
U(\xi) \sim a \{1 + \exp(\lambda_2(A)\xi)\} \{1 + o(1)\} \quad \text{as } \xi \to +\infty \text{ at } A; \\
U(\xi) \sim b \{1 - \exp(\lambda(B)\xi)\} \{1 + o(1)\} \quad \text{as } \xi \to -\infty \text{ at } B;
\]

where \( \lambda(B) = \lambda_1(B) \) or \( \lambda_2(B) \) and will be determined later.

The proof is straight-forward by observing the eigenvalues of system (4) at \( A \) and \( B \).

Lemma 4. \( U_\xi \) and \( U_{\xi\xi} \in L^2(\mathbb{R}) \).

This can be easily concluded from Lemma 3 for the asymptotic behavior of the solution close to the singular points \( A \) and \( B \).

Theorem 1. Let \( U(\xi) \) be the unique bounded solution of (3). \( U(\xi), U_\xi(\xi), U_{\xi\xi}(\xi) \) and \( U_{\xi\xi\xi}(\xi) \) are uniformly bounded in \( \beta \).

Proof. By Lemma 2, \( U_\xi(\xi) < 0 \), then \( a = U(+\infty) < U(\xi) < U(-\infty) = b \), the independence of \( a \) and \( b \) of \( \beta \) implies the uniform boundedness of \( U(\xi) \) in \( \beta \).
Multiply (3) by $U_\xi$ and integrate on both sides
\[
\int_{-\infty}^{\xi} \left( -cU + \frac{1}{2}U^2 - \gamma U_\xi + \beta U_{\xi\xi} \right) U_\xi \, d\xi = \int_{-\infty}^{\xi} c_1 U_\xi \, d\xi.
\]
then
\[
U_\xi^2 = \frac{2\gamma}{\beta} \int_{-\infty}^{\xi} U_\xi^2 \, d\xi = \frac{2}{\beta} \left[ c_1(U - b) + \frac{c}{2}(U^2 - b^2) - \frac{1}{6}(U^3 - b^3) \right].
\]

Denote $f(U) = c_1(U - b) + c(U^2 - b^2)/2 - (U^3 - b^3)/6$, then
\[
\frac{d}{d\xi} \left[ e^{-2\gamma \xi / \beta} \int_{-\infty}^{\xi} U_\xi^2 \, d\xi \right] = \frac{2}{\beta} f(U) e^{-2\gamma \xi / \beta}.
\]
Since $f(U)$ is bounded, the right-hand side is integrable on $[\xi, +\infty)$
\[
\int_{-\infty}^{\xi} U_\xi^2 \, d\xi = -\frac{2}{\beta} \int_{\xi}^{+\infty} e^{-2\gamma \xi / \beta} f(U) \, d\xi.
\]
Taking derivative both sides implies
\[
U_\xi^2 = -\frac{2}{\beta} e^{2\gamma \xi / \beta} \int_\xi^{+\infty} e^{-2\gamma \xi / \beta} f'(U) U_\xi \, d\xi.
\]

Let \( \phi(\xi) = U_\xi \exp(-\gamma \xi / \beta) \), then \( \phi < 0, \phi \to 0 \) as \( \xi \to +\infty \). Hence
\[
\phi^2 = -\frac{2}{\beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} f'(U) \phi \, d\xi.
\]
Taking derivative both sides implies \( \phi' \) and then \( \phi \), hence
\[
U_\xi = -\frac{1}{\beta} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} f'(U) \, d\xi. \tag{5}
\]
Since \( |f'(U)| \leq c^* \) uniformly in \( \beta \) for some constant \( c^* > 0 \), it follows that
\[
|U_\xi| \leq \frac{c^*}{\beta} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} \, d\xi = \frac{c^*}{\gamma}.
\]
Taking derivative of (5) implies
\[
U_{\xi\xi} = -\frac{1}{\beta} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} f''(U) U_\xi \, d\xi. \tag{6}
\]
Thus
\[
|U_{\xi\xi}| \leq \frac{c^*}{\beta \gamma} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} |f''(U)| \, d\xi.
\]
Since \( |f''(U)| \leq \tilde{c} \) uniformly in \( \beta \) for some constant \( \tilde{c} > 0 \)
\[
|U_{\xi\xi}| \leq \frac{c^* \tilde{c}}{\beta \gamma} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-2\gamma \xi / \beta} \, d\xi = \frac{c^* \tilde{c}}{\gamma^2}
\]
uniformly in \( \beta \). Taking derivative of (6) implies
\[
U_{\xi\xi\xi} = -\frac{\gamma}{\beta^2} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} f'''(U) U_\xi \, d\xi + \frac{1}{\beta} f''(U) U_\xi
\]
\[
= -\frac{1}{\beta} e^{\gamma \xi / \beta} \int_\xi^{+\infty} e^{-\gamma \xi / \beta} \left[ f'''(U) U_\xi^2 + f''(U) U_{\xi\xi} \right] \, d\xi.
\]
Since $U, U_\xi, U_{\xi\xi}$ and thus $f''(U), f'''(U)$ are uniformly bounded in $\beta$, so is

$$|f'''(U)U^2_\xi + f''(U)U_{\xi\xi}| < c^{**}$$

for some constant $c^{**}$. It can be easily derived in the similar way as above that

$$|U_{\xi\xi\xi}| < c^{**}$$

i.e., $U_{\xi\xi\xi}$ is also uniformly bounded in $\beta$. $\square$

3. Traveling wave solutions of Burger’s equation

When $\beta = 0$, Eq. (2) becomes Burger’s equation

$$-cU_\xi + UU_\xi - \gamma U_{\xi\xi} = 0.$$ 

Integrate both sides with the same constant $c_1$,

$$-cU + \frac{1}{2}U^2 - \gamma U_\xi = c_1.$$ 

(7)

If $c^2 + 2c_1 > 0$, (7) has unique bounded solution as follows

$$U(\xi) = \frac{b - ac_2 e^{\frac{b - a}{\gamma} \xi}}{1 - c_2 e^{\frac{b - a}{\gamma} \xi}},$$

where $c_2 > 0$ is a constant depending on the initial condition $U(0)$. $a = U(+\infty) < u < U(-\infty) = b$, $a$ and $b$ are defined as before.

$$U - a \sim -\frac{b - a}{c_2} \exp\left[-\frac{\sqrt{c^2 + 2c_1}}{\gamma} \xi\right], \quad \xi \to +\infty,$$

$$U - b \sim (b - a)c_2 \exp\left[\frac{\sqrt{c^2 + 2c_1}}{\gamma} \xi\right], \quad \xi \to -\infty,$$

also $U_\xi(\xi) < 0$.

4. Limiting solution of BKDVE

Denote the bounded solution of (3) by $U(\xi, \beta)$, and that of (7) by $\overline{U}(\xi)$. We have the following results.

**Theorem 2.** Assume $U(0, \beta) = \overline{U}(0)$, then $\forall \xi \in \mathbb{R}^1$,

$$\lim_{\beta \to 0^+} U(\xi, \beta) = \overline{U}(\xi),$$

$$\lim_{\beta \to 0^+} U_\xi(\xi, \beta) = \overline{U}_\xi(\xi).$$
\[
\lim_{\beta \to 0^+} U_{\xi \xi} (\xi, \beta) = \overline{U}_{\xi \xi} (\xi),
\]
i.e., the limiting solution of (3) is the solution of (7) as \( \beta \to 0^+ \).

**Proof.** Let \( W = U(\xi) - \overline{U}(\xi) \), then \( W(0) = 0 \) and
\[
-cW + \frac{1}{2} W(U + \overline{U}) - \gamma W_{\xi} + \beta U_{\xi \xi} = 0,
\]
then
\[
\frac{d}{d\xi} \left( W \exp \left[ - \int_0^\xi \frac{U + \overline{U} - 2c}{2\gamma} d\xi' \right] \right) = \frac{\beta}{\gamma} U_{\xi \xi} \exp \left[ - \int_0^\xi \frac{U + \overline{U} - 2c}{2\gamma} d\xi' \right].
\]
Integrating both sides implies
\[
W = \frac{\beta}{\gamma} \exp \left[ \int_0^\xi \frac{U + \overline{U} - 2c}{2\gamma} d\xi' \right] \int_0^\xi U_{\xi \xi} \exp \left[ - \int_0^\xi \frac{U + \overline{U} - 2c}{2\gamma} d\xi' \right] d\xi.
\]
Since \( \overline{U} \), \( U \) and \( U_{\xi \xi} \) are uniformly bounded in \( \beta \), so is the following for fixed \( \xi \):
\[
e^{\int_0^\xi \frac{U + \overline{U} - 2c}{2\gamma} d\xi'} \int_0^\xi U_{\xi \xi} e^{-\int_0^\xi \frac{U + \overline{U} - 2c}{2\gamma} d\xi'} d\xi.
\]
then
\[
\lim_{\beta \to 0^+} |W| = 0.
\]
i.e.,
\[
\lim_{\beta \to 0^+} U(\xi, \beta) = \overline{U}(\xi) \quad \forall \xi \in \mathbb{R}^1.
\]
From (8)
\[
W_{\xi} = \frac{1}{\gamma} \left[ \beta U_{\xi \xi} - cW + W(U + \overline{U})/2 \right].
\]
By the uniform boundedness of \( U \), \( U_{\xi \xi} \) and \( \overline{U} \)
\[
\lim_{\beta \to 0^+} W_{\xi} = 0.
\]
Taking derivative on both sides of (8) implies
\[
W_{\xi \xi} = \frac{1}{\gamma} \left[ -cW_{\xi} + \frac{1}{2} W_{\xi} (U + \overline{U}) + \frac{1}{2} W (\overline{U}_{\xi} + U_{\xi}) + \beta U_{\xi \xi} \right]
\]
then
\[
\lim_{\beta \to 0^+} W_{\xi \xi} = 0.
\]
i.e.,

\[
\lim_{\beta \to 0^+} U_{\xi\xi} = \overline{U}_{\xi\xi}.
\]

\[\Box\]

**Corollary 1.** For any \( n \geq 0 \), the \( n \)th derivative \( U^{(n)}_{\xi} \) of the bounded solution of (3) is uniformly bounded in \( \beta \). And

\[
\lim_{\beta \to 0^+} U^{(n)}_{\xi} = \overline{U}^{(n)}_{\xi}.
\]

The proof is straightforward by mathematical induction on \( n \).

**Corollary 2.** The integral curve \( l_{AB} \) connecting \( A \) and \( B \) in the phase plane of the bounded solution of system (3) is tangent to the vector \( V_2(B) \) rather than \( V_1(B) \) at singular point \( B \).

**Proof.** By Theorem 2 and Corollary 1

\[
\lim_{\beta \to 0^+} \frac{dU_{\xi}}{dU} = \lim_{\beta \to 0^+} \frac{U_{\xi\xi}}{U_{\xi}} = \frac{d\overline{U}_{\xi}}{dU}
\]

and also at \( B \)

\[
\frac{d\overline{U}_{\xi}}{dU} \sim \lambda_2^0(B) + o(1)(\xi \to -\infty),
\]

\[
\frac{dU_{\xi}}{dU} \sim \lambda(B) + o(1)(\xi \to -\infty),
\]

where \( \lambda(B) = \lambda_1(B) \) or \( \lambda_2(B) \). Since \( \lambda_2(B) \to \lambda_2^0(B) \), \( \lambda_1(B) \to +\infty \) as \( \beta \to 0^+ \), \( \lambda(B) = \lambda_2(B) \) for sufficiently small \( \beta > 0 \). It can then be concluded that \( \lambda(B) = \lambda_2(B) \) for all \( \beta > 0 \) by the dependency of solutions of (4) on positive \( \beta \). \[\Box\]

5. **Exact shock wave solutions of BKDVE**

It is artefact that the BKDVE takes the traveling wave form in its solutions

\[ U(x,t) = U(x-ct) \]

Are these the true solutions of (1)? We show in the following that under certain conditions, the solutions of the BKDVE take the traveling wave form.

**Theorem 3.** Suppose \( U(x,t) \) is the solution of BKDVE (1) with arbitrary parameters \( \beta > 0, a < b \) and \( a + b > 0 \). If \( U(x,t) \) satisfies the following conditions

\[
\forall t \geq 0,
\begin{cases}
\lim_{x \to -\infty} U(x,t) = b, & \lim_{x \to +\infty} U(x,t) = a; \\
\lim_{x \to -\infty} U_x(x,t) = \lim_{x \to +\infty} U_{xx}(x,t) = 0; \\
U(x,t) - a \in L^2(0, +\infty), & U(x,t) - b \in L^2(-\infty, 0); \\
\sup_{x \in [0,T]} |U_x(x,t)| \leq C(T) < +\infty,
\end{cases}
\]

then for any initial value \( U(x,0) = U_0(x) \) satisfying condition (10), the solution of BKDVE (1) exists and is unique.
Proof. The existence is given in Lemma 1. Now we prove the uniqueness.

Suppose there exists two solutions $U_1(x,t)$ and $U_2(x,t)$ satisfying condition (10). Let $W(x,t) = U_1(x,t) - U_2(x,t)$, then $W(x,0) = 0$. For fixed $t$, $W \to 0$ as $|x| \to \infty$, $W \in L^2(R^1)$, and

$$W_t + (U_1 U_{1x} - U_2 U_{2x}) = \gamma W_{xx} - \beta W_{xxx}.$$ 

Since

$$U_1 U_{1x} - U_2 U_{2x} = W U_{1x} + U_2 W_x,$$

then

$$W_t + W U_{1x} + U_2 W_x = \gamma W_{xx} - \beta W_{xxx}.$$ (11)

Multiply by $W$ and integrate both sides

$$\int_0^T \int_{-\infty}^{+\infty} \left[ \frac{1}{2} W^2 \right]_x dx dt + \int_0^T \int_{-\infty}^{+\infty} \left[ W^2 U_{1x} + U_2 W_x W \right] dx dt$$

$$= \int_0^T \int_{-\infty}^{+\infty} \left[ \gamma W_{xx} W - \beta W_{xxx} W \right] dx dt.$$ (12)

One can easily derive the following by straight-forward calculation.

$$\int_0^T \int_{-\infty}^{+\infty} \left[ \frac{1}{2} W^2 \right]_x dx dt = \frac{1}{2} \int_{-\infty}^{+\infty} W^2(x,T) dx,$$

$$\int_0^T \int_{-\infty}^{+\infty} \left[ W^2 U_{1x} + U_2 W_x W \right] dx dt = \int_0^T \int_{-\infty}^{+\infty} \left[ W^2 U_{1x} - \frac{1}{2} W^2 U_{2x} \right] dx dt$$

$$= \int_0^T \int_{-\infty}^{+\infty} \frac{1}{2} W^2 U_{1x} dx dt + \frac{1}{2} \int_0^T \int_{-\infty}^{+\infty} W^2 W_x dx dt = \frac{1}{2} \int_{-\infty}^{+\infty} W^2 W_x dx dt$$

and

$$\int_0^T \int_{-\infty}^{+\infty} \left[ \gamma W_{xx} W - \beta W_{xxx} W \right] dx dt = \int_0^T \int_{-\infty}^{+\infty} \left[ \gamma W dW_x - \beta W dW_{xx} \right]$$

$$= -\int_0^T \int_{-\infty}^{+\infty} \left[ \gamma W_x^2 + \beta W_x W_x \right] dx dt = -\gamma \int_{-\infty}^{+\infty} W_x^2 dx dt.$$ (13)

Substituting these into (11) implies
\[ \int_{-\infty}^{+\infty} W_2(x,T) \, dx = - \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_2 U_{1x} \, dx \, dt - 2\gamma \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_2^2 \, dx \, dt \]

\[ \leq - \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_2 U_{1x} \, dx \, dt \leq \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_2 |U_{1x}| \, dx \, dt \leq C(T) \int_{0}^{T} \int_{-\infty}^{+\infty} W_2^2 \, dx \, dt \]

for some positive constant \( C(T) \) depending on \( T \). Let

\[ F(T) = \int_{0}^{T} \int_{-\infty}^{+\infty} W_2^2 \, dx \, dt \]

then

\[ \frac{d}{dT} F(T) \leq C(T) F(T) \]

hence \( F(T) \leq 0 \), i.e.,

\[ \int_{0}^{T} \int_{-\infty}^{+\infty} W_2^2 \, dx \, dt \leq 0. \]

It then implies \( W = 0, \forall (x, t) \in \mathbb{R} \times [0, T] \).

**Remark.** Define \( c = (a + b)/2 \) and \( c_1 = -ab/2 \) in condition (10). The substitution of \( \xi = x - ct \) into (1) and (2) implies, by Lemma 1, that there exists a unique solution subject to a translation. All above results on the bounded solution hold, especially the perturbation property. Therefore we conclude the following.

**Theorem 4.** For \( \beta > 0 \), under the boundary value condition (10) and the initial value \( U_0(x, \beta) \), such that \( U_0(x, \beta) \) satisfies (3), then Eq. (1) has a unique solution

\[ U(x, t, \beta) = U_0(x - ct, \beta), \]

and it satisfies

\[ \lim_{\beta \to 0^+} U(x, t, \beta) = U(x, t, 0), \]

where \( U(x, t, 0) \) is the solution of Burger’s equation.

Since the uniqueness of the solution holds, the artefact traveling wave solutions are regarded as the natural solution of the BKDVE (1), and then can be regarded as the shock wave solutions. The perturbation is then an action on the original equation (1) rather than on the derived equation (2).
6. Discussion

Burger’s K-dV equation (BKDVE) is important in the study of non-linear wave equations, especially in studying solitary waves and shock waves since BKDVE presents different types of wave solutions as discussed in [1].

In this paper, we considered singular perturbation of the shock wave solutions of Burger’s K-dV equation with dominant dissipation. We studied the shock wave solution of BKDVE represented by the traveling waves of the BKDVE through ordinary differential equations. We derived the asymptotic behavior of the traveling wave solution at the infinity and proved that this solution converges to the traveling wave solution of Burger’s equation, which has a simple close form. We also provide estimation of several derivatives of the shock wave solutions of the BKDVE, which supports the expansion of the shock wave solution of BKDVE by Jeffrey [2].

Furthermore, we also proved that the traveling wave solution is the unique solution of the BKDVE under certain conditions. It implies that the shock wave solution represented by the traveling wave solution is a true solution of the BKDVE, and thus the expansion of the shock wave solution in [2] provides an approximation to the true shock wave solutions of the BKDVE, not just to the traveling wave solutions of the BKDVE.

References