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NOTE

Note on a Combinatorial Application of Alexander Duality

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The Möbius number of a finite partially ordered set equals (up to sign) the difference between the number of even and odd edge covers of its incomparability graph. We use Alexander duality and the nerve lemma of algebraic topology to obtain a stronger result. It relates the homology of a finite simplicial complex Δ that is not a simplex to the cohomology of the complex Γ of nonempty sets of minimal non-faces that do not cover the vertex set of Δ. © 1997 Academic Press

1. THE MÖBIUS NUMBER RESULT

Recall that if $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ is a finite poset, the vertices of its incomparability graph G are the elements of P and the edges of G are the 2-element antichains in P. The Möbius number $\mu_{\hat{P}}(\hat{0}, \hat{1})$ is, by Philip Hall's theorem, the reduced Euler characteristic $\tilde{\chi}(\Delta)$ of the order complex

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$\Delta = \Delta(P)$, whose simplices are the nonempty chains in P. We refer the reader to [5] for all unexplained terminology.

PROPOSITION 1 [3]. *For a finite poset P,*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = (-1)^{|P|-1} \sum_{k \geqslant 0} (-1)^k N_k,$$

where N_k is the number of k-subsets of edges of the incomparability graph that cover P.

Proof. If $P \neq \emptyset$ is a chain, then the Möbius number is 0 and $N_k = 0$ for all k. If $P = \emptyset$, then the Möbius number is -1 and the only nonzero N_k is $N_0 = 1$.

If P is not a chain, let $\hat{B} = 2^P$ be the poset of subsets of P, and let Q be the subposet of nonempty chains in P. The atoms in the lattice $\hat{B} - Q$ are the 2-element antichains in P, so by Rota's crosscut theorem [5, dual of Corollary 3.9.4]

$$\mu_{\hat{B}-Q}(\hat{0}, \hat{1}) = \sum_{k \geqslant 0} (-1)^k N_k.$$

By Stanley's combinatorial Alexander duality theorem [5, Proposition 3.14.5],

$$\mu_{\hat{Q}}(\hat{0}, \hat{1}) = (-1)^{|P|-1} \mu_{\hat{B}-Q}(\hat{0}, \hat{1}).$$

The order complex of Q is the first barycentric subdivision sd Δ of the order complex of P, so Philip Hall's theorem implies $\mu_{\hat{P}}(\hat{0}, \hat{1}) = \mu_{\hat{Q}}(\hat{0}, \hat{1})$. ▮

This result will be seen in the next section to be a consequence of a much stronger homological fact. The elementary combinatorial proof given above, unlike the proof in [3], points in the direction of the topological generalization.

2. THE GENERAL RESULT

Let Δ be a simplicial complex on a finite vertex set V. Let C denote the set of minimal nonfaces of Δ, i.e., the inclusionwise minimal nonempty subsets of V that are not in Δ. Define a simplicial complex Γ on the vertex set C as those nonempty subsets of C whose union is not all of V.

THEOREM 2. *If $V \notin \Delta$, then $\tilde{H}_i(\Delta) \cong \tilde{H}^{|V|-3-i}(\Gamma)$.*

Proof. Let Q be the poset of simplices in Δ ordered by inclusion and let $\hat{B} = B \cup \{\hat{0}, \hat{1}\} = 2^V$, so that the order complex $\Delta(Q)$ is sdΔ. Apply

[1, Theorem 10.8], which is proved using Borsuk's nerve lemma, to the lattice $\hat{B} - Q$ with crosscut C; the crosscut complex is Γ so

$$\tilde{H}^{|V|-i-3}(\Delta(B-Q)) \cong \tilde{H}^{|V|-i-3}(\Gamma).$$

Now apply Alexander duality [4, Theorem 71.1] to the subspace $|\Delta(Q)|$ of the $(|V|-2)$-sphere $|\Delta(B)|$; the subspace $|\Delta(B-Q)|$ is a strong deformation retract of $|\Delta(B)| - |\Delta(Q)|$, so

$$\tilde{H}_i(\Delta(Q)) \cong \tilde{H}^{|V|-i-3}(\Delta(B-Q)). \tag{1}$$

Since $\Delta(Q) = \text{sd}\,\Delta$, the result follows. ∎

Equation (1), which is mentioned in [5], has been used by several other authors (see [2] for some references) in the form:

$$\tilde{H}_i(\Delta) \cong \tilde{H}^{|V|-i-3}(\Delta^*), \tag{2}$$

where $\Delta^* = \{A \subseteq V \mid V - A \notin \Delta\}$. Equation (2) follows from Eq. (1) since $\Delta(Q) = \text{sd}\,\Delta$ and $\Delta(B-Q) \cong \text{sd}\,\Delta^*$.

COROLLARY 3. *Let $G = (V, E)$ be a simple graph. If $|V| = n$ and $|E| \neq 0$, then*

$$\tilde{H}_i(\Delta_G) \cong \tilde{H}^{n-3-i}(\Gamma_G),$$

where Δ_G is the independent set complex *of nonempty subsets of V that contain no edges, and Γ_G is the* covering complex *of nonempty subsets of E that do not cover V.*

To deduce Proposition 1, let $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ be a finite poset with incomparability graph G, so that Δ_G is the order complex $\Delta = \Delta(P)$. Use the Euler–Poincaré formula to see that Corollary 3 implies $\tilde{\chi}(\Delta(P)) = (-1)^{|P|-1}\,\tilde{\chi}(\Gamma_G)$, when P is not a chain. In this case $|E| \neq 0$, so $\tilde{\chi}(\Gamma_G)$ equals $\sum(-1)^k N_k$, where N_k is the number of k-subsets of E that cover P.

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