An Operator Splitting-Radial Basis Function Method for the Solution of Transient Nonlinear Poisson Problems

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Abstract—This paper presents an operator splitting-radial basis function (OS-RBF) method as a generic solution procedure for transient nonlinear Poisson problems by combining the concepts of operator splitting, radial basis function interpolation, particular solutions, and the method of fundamental solutions. The application of the operator splitting permits the isolation of the nonlinear part of the equation that is solved by explicit Adams-Bashforth time marching for half the time step. This leaves a nonhomogeneous, modified Helmholtz type of differential equation for the elliptic part of the operator to be solved at each time step. The resulting equation is solved by an approximate particular solution and by using the method of fundamental solution for the fitting of the boundary conditions. Radial basis functions are used to construct approximate particular solutions, and a grid-free, dimension-independent method with high computational efficiency is obtained. This method is demonstrated for some prototypical nonlinear Poisson problems in heat and mass transfer and for a problem of transient convection with diffusion. The results obtained by the OS-RBF method compare very well with those obtained by other traditional techniques that are computationally more expensive. The new OS-RBF method is useful for both general (irregular) two- and three-dimensional geometry and provides a mesh-free technique with many mathematical flexibilities, and can be used in a variety of engineering applications. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Method of fundamental solutions, Operator splitting, Nonlinear Poisson problem, Particular solution method, Radial basis functions, Convection-diffusion-reaction equation, Helmholtz equation.

1. INTRODUCTION

Transient nonlinear Poisson problems are widely encountered in the modeling of physical phenomena. For example, transient heat conduction or mass diffusion with source terms arises in model equations in many different areas of computational physics and engineering. Representative prototype problems include transient diffusion with chemical reaction in a catalyst pellet, microwave heating process, spontaneous combustion, and thermal explosion problems and transient convection. Efficient solution of nonlinear Poisson problems is quintessential to numerical

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simulation of many other complex problems, e.g., time-dependent multidimensional viscous and viscoelastic flows. The numerical solution procedure usually depends on finite-difference, finite-element, or spectral methods. These methods involve a grid generation that is a time consuming process, especially for three-dimensional problems where elaborate bookkeeping and interelement assembly process are required.

Recently, there is considerable interest in developing grid-free methods for such problems. Such methods are especially attractive for three-dimensional problems in any general nonregular geometry. The boundary element method (BEM) is one such method suitable for linear problems, and a large number of papers and books (e.g., [1-3]) have addressed the solution of the steady-state problems with this method. For transient problems, the BEM can be used in conjunction with finite differencing in time (e.g., [4-6]). The resulting formulation is a steady-state type of Poisson equation that can be solved by dual reciprocity methods [1,6]. Thus, the advantages of the ‘boundary only’ discretization are retained, and the internal points are needed only for the interpolation of the nonhomogeneous terms. Various papers (e.g., [4-6]) have shown the application of this solution procedure to transient problems. Higher-order time integration schemes [7] or cubic Hermitian algorithms [8] can also be used for temporal discretization. For linear transient problems, alternative boundary based methods are solutions in the Laplace domain followed by numerical inversion [9,10] or the direct use of time-dependent fundamental solutions [11,12].

The disadvantage of all the BEM-based techniques is the need for the evaluation of singular or near-singular integral which can be time consuming and the need to do surface meshing in 3-D. As an alternative, solution methods based on the method of fundamental solution (MFS) are gaining considerable attention [13-17]. These methods are based on fitting of the boundary conditions with the fundamental solutions of the Laplace equation as the basis functions. The poles or singularities of the fundamental solutions are placed outside the domain, thus avoiding the need for evaluation of the singular integrals in contrast to traditional BEM. This method has been demonstrated for various linear differential equations [13-19] and in conjunction with the method of particular solutions for nonlinear Poisson problems [20-22]. The growing field of the radial basis function (RBF) has provided a considerable impetus for this problem since approximate particular solutions are often needed and can be easily constructed with RBF interpolation.

In view of the rapid development of the MFS-RBF method in recent years, the applications to transient problems would be interesting. However, the application of the MFS-RBF method to transient problems has been limited. For linear transient problems, this method can be combined with the Laplace transform method, as demonstrated by Chen et al. [23]. This procedure is suitable only when $f$ is a linear function of $u$. For other cases, procedures based on finite differencing in time need to be used. The simplest method is to use an explicit Euler method for approximating the time derivatives, and a recent paper by Golberg and Chen [24] provides a detailed computational study based on this approach. Also, the forcing function $f$ was approximated at the previous time step in their study. The explicit scheme presented in their study is first-order accurate and has stability restrictions. Using the implicit Euler method can circumvent this stability restriction. The accuracy is again of $O(\Delta t)$ (the time step size), but the method is unconditionally stable. The disadvantage of the implicit scheme is the necessity to solve a set of algebraic equations at each time step. In view of the above limitations of the Euler method, there is considerable motivation to examine other methods for handling the time derivative terms and for approximating the forcing function. In this respect, a procedure based on operator splitting for handling the nonlinear source term explicitly using higher-order accurate schemes is gaining considerable attention in the context of other numerical methods. For example, a number of papers [25-33] use the concept of operator splitting for the context of finite-difference, finite-element, and/or spectral simulations of viscous and viscoelastic flows. The method (operator splitting) does not appear to have been applied in the context of MFS for transient problems, and this is the focus of this work. Here we present a combined operator splitting-radial basis function (OS-RBF) method for the solution of transient problems in con-
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The goal of this paper is to document the method and to demonstrate its accuracy for some test problems. The OS-RBF method is shown to be a versatile tool for the solution of a variety of transient nonlinear Poisson equations and can be easily applied to any complex geometry with mixed boundary conditions. This paper is organized as follows. In Section 2, we discuss the key concepts of operator splitting and show how the nonlinear Poisson problem is reduced to the solution of a diffusion-reaction problem of the steady-state type for each time step. In Section 3, the method of fundamental solution is developed to solve the nonhomogeneous steady-state Helmholtz (diffusion-reaction) equation. These two sections thus outline the complete procedure for the solution of transient problem. Section 4 extends the procedure to convection diffusion reaction problems. Section 5 provides some numerical results for a variety of nonlinear problems. Section 6 offers conclusions and future directions.

2. OPERATOR SPLITTING PROCEDURE

Consider the unsteady Poisson equation,

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(u), \quad \text{on } \Omega, \quad (1)$$

with boundary conditions

$$u = u_0, \quad \text{on } \Gamma_1,$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma_2,$$

where $\Gamma = \Gamma_1 + \Gamma_2$ represents the boundary of $\Omega$, $u$ is the dependent variable, and $f$ is in general a nonlinear forcing function. This problem can be treated as a two-level problem by splitting the elliptic operator from the nonlinear forcing function, following an approach similar to the alternating direction implicit (ADI) type procedure proposed by Peaceman and Rachford [33]. In general, the time-dependent problem can be treated as a sum of $n$ operators $L_1$ through $L_n$, i.e.,

$$\frac{\partial u}{\partial t} = L_1 + L_2 + \cdots . \quad (2)$$

Here one deals with two operators $L_1$ and $L_2$, which are given by the following expressions:

$$L_1 = f(u), \quad (3)$$

$$L_2 = \nabla^2 u. \quad (4)$$

One can adopt a solution in time by discretization using finite differences using a two time level scheme. Here the nonlinear term is evaluated explicitly at each half time step, and the resulting solution is then used for an implicit solution of the Laplacian term. Thus, for the half step, if one utilizes the second-order Adams-Bashforth (AB) explicit scheme, one has

$$\frac{u^{n+1/2} - u^n}{\Delta t} = 1.5f(u^n) - 0.5f(u^{n-1}). \quad (5)$$

The solution $u^{n+1/2}$ is then utilized in the second step, which is then discretized implicitly using the second-order Adams-Moulton (AM) scheme, which gives

$$\frac{u^{n+1} - u^{n+1/2}}{\Delta t} = 0.5 \left( \nabla^2 u^{n+1} + \nabla^2 u^n \right). \quad (6)$$

As seen from equation (5), the nonlinear term is evaluated explicitly obviating the need for any iterative solution at each time step. This semisexplicit approach will, however, impose a time
step restriction similar to those encountered in the explicit numerical integration of initial value problems. If the individual steps are updated using consistent, i.e., $O(\Delta t^p)$ accurate AB and AM schemes, the accuracy of the overall methods will also be $O(\Delta t^p)$. See [28] and references therein. For nonlinear problems, the error bounds are difficult to be determined analytically and depend on the nonlinear forcing function.

In the r.h.s. of equation (6), Laplacian for the previous time step is required explicitly at each time level. Numerical evaluation of the Laplacian will result in an additional error other than that from the time discretization. In order to circumvent the numerical evaluation of the Laplacian, one resorts to the following variable transformation. The dependent variable in equation (6) is transformed to $u^*$, which is defined as follows:

$$u^* = \frac{u^n + u^{n+1}}{2}.$$  

Using this in equation (6) and eliminating $u^{n+1/2}$ using equation (5), we get

$$\nabla^2 u^* - \frac{2u^*}{\Delta t} = - (1.5f^n - 0.5f^{n-1}) - \frac{2u^n}{\Delta t}.$$  

The above equation, in the form of the modified Helmholtz (diffusion-reaction type) equation, needs to be solved for each time step. One notes that this equation has a nonhomogeneous term; however, these values are explicitly known at each time step. This equation is therefore in a form where the MFS procedure with the particular solution approach can be readily used. These details are presented in the next section. The use of MFS procedure circumvents the discretization of the Laplacian operator and the errors associated with approximating this operator. This improves the accuracy of the computations and is one of the advantages of the OS-RBF method. At each step, the values of $u^*$ are obtained by the solution of equation (8); the values of $u^{n+1}$ are then extracted from the computed values of $u^*$, and the time marching can be continued. The solution needs the function values at the current ($n$) and one previous time ($n-1$) step. Hence, the method is not self-starting, and an extrapolated explicit forward Euler method is used for simplicity for the first time step to initiate time stepping. Some differences between the current formulation and the explicit Euler formulation presented in the reference [24] are noteworthy. The function values at the current and previous time steps are now used in the time marching scheme. This is a characteristic of multistep methods. Also, the solution value at the midinterval point in time ($u^*$) is computed rather than the end point value. These features contribute to increased accuracy and numerical stability.

3. SOLUTION OF THE MODIFIED HELMHOLTZ EQUATION

The l.h.s. of equation (8) contains a linear operator and is often referred to as the diffusion-reaction operator. Problems of this type are called the modified Helmholtz problem due to the presence of the nonhomogeneous term on the r.h.s. of (8). The presence of the linear operator on the l.h.s. makes the equation amenable to solution by the dual reciprocity boundary elements or by the particular solution combined with the MFS method. Both methods are based on boundary collocation and have been successfully applied to similar problems. Here we use the MFS since this method avoids the evaluation of boundary integrals. The method is described in detail below.

Equation (8) is expressed in the following manner:

$$\nabla^2 u^* - \lambda^2 u^* = F,$$  

where

$$\lambda^2 = \frac{2}{\Delta t}.$$  

$$\nabla^2 u^* - \lambda^2 u^* = F,$$  

where

$$\lambda^2 = \frac{2}{\Delta t}.$$  

and

\[ F = -\left(1.5f^n - 0.5f^{n-1}\right) - \frac{2u^n}{\Delta t}. \]

The solution is expressed as

\[ u^* = v + w, \]

where \( v \) is the solution to the homogeneous part of equation (9), and \( w \) is a particular solution to the complete differential equation. The governing differential equation for \( v \) is given by

\[ \nabla^2 v - \lambda^2 v = 0, \]

together with the boundary conditions

\[ v = u^* - w, \quad \text{on } \Gamma_1, \]
\[ \frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}, \quad \text{on } \Gamma_2. \]

The governing equation for the particular solution is given by

\[ \nabla^2 w - \lambda^2 w = F. \]

The solutions \( v \) and \( w \) are now presented. The solution to equation (12) can be expressed as a linear combination of \( F \)-Trefftz functions (fundamental solutions) as

\[ v = \sum_{i=1}^{nb} a_i G_i, \]

where \( nb \) is the number of boundary collocation points and \( G_i \) is the fundamental solution to the linear diffusion reaction equation and is given by

\[ G_i = K_0(\lambda r_i), \]

where \( K_0 \) is the modified Bessel function of the second kind, and \( r_i \) is the distance between any field point in the domain and the \( i^{th} \) source point located outside the domain of consideration.

In general, the particular solution cannot be obtained exactly. In order to find the approximate particular solution, one has to first approximate the forcing function \( F \) in the domain of consideration. One can do this using radial basis functions in the form

\[ F = \sum_{i=1}^{nt} \phi_i \alpha_i, \]

where \( \{\phi_i\} \) represents a set of suitable basis functions which spans the domain, and \( \{\alpha_i\} \) are the corresponding interpolating coefficients. The particular solution to the problem is now represented as

\[ w = \sum_{k=1}^{nt} \psi_k \alpha_k, \]

where \( \{\psi_k\} \) is the set of particular solutions to the basis functions satisfying the following differential equations:

\[ \nabla^2 \psi_k - \lambda^2 \psi_k = \phi_k. \]
The commonly used radial basis functions are the thin plate splines and multiquadrics. The problem with these choices is the difficulty in finding the particular solution functions. This difficulty arises because we now have the diffusion-reaction operator on the l.h.s. of equation (18), rather than the usual Laplace operator for steady-state Poisson problems. One method of circumventing this problem is to choose a form for $\psi_k$ directly and evaluate $\phi_k$ from equation (18). We use this method for evaluating the particular solutions in this work. Thus, the particular solutions are directly chosen as follows:

$$\psi_k = \frac{r_k^2}{4} + \frac{r_k^2}{9}, \quad (19)$$

where $r_k$ is the distance function from the knot or interpolation point $k$. Substituting the above expression in equation (18) and differentiating twice, we obtain the interpolating functions as

$$\phi_k = 1 + r_k - \lambda^2 \left( \frac{r_k^2}{4} + \frac{r_k^3}{9} \right). \quad (20)$$

The interpolating functions now depend only on $\Delta t$. This procedure is similar to that used by Partridge et al. [1] in the context of the dual reciprocity method for convection-diffusion problem. Recently, the mathematical procedure to find the particular solutions for thin plate splines and the diffusion-reaction operator has been developed by Golberg and Chen [34]. This method is called the annihilator method where $\psi_k$ for the augmented thin plate spline are evaluated analytically for the case where $\phi_k$ is the augmented ATPS. This methodology can be extended similarly to three dimensions and for higher-order splines, and can be used to increase the accuracy of algorithms for these types of problems as well as for transient problems as pointed out in [34]. The advantage of the annihilator method used by Golberg and Chen is that the standard radial basis functions (polyharmonic splines) can be used for interpolation, and the error analysis for such cases is established (e.g., [35]). We have not used this (annihilator) method here, since the method is rather recent. Here we choose directly a form for the particular solution and use the corresponding interpolating functions as given by equation (20). A preliminary comparison of the two approaches (not reported in this paper), for one of the test cases, showed that both the methods give almost similar results for the evaluation of the particular solution. Hence, the procedure used here appears to be satisfactory. Further work should address the comparison of the two approaches and examine the effect of interpolating functions on the solution accuracy. Also the details of the stability, error analysis, and the condition number of the interpolating matrix generated by equation (20) need to be further investigated, since our results show that this interpolation procedure leads to good results. Such results are available only for the standard type of RBFs (for example, in [37]) and not for the functions of the type given by equation (20).

The coefficients $\alpha_i$ needed in equation (17) are found as follows. One selects $n_t$ total interpolation points in the domain of consideration. The interpolation equations at each of these points are given by

$$\sum_{i=1}^{nt} \alpha_i \phi_{k_i} = \beta_m, \quad m = 1, 2, \ldots, nt. \quad (21)$$

The vector $\alpha$ of coefficients $\{\alpha_i\}$ are then found by inverting the system of linear equations obtained, i.e.,

$$\alpha = \Phi^{-1} \vec{F}, \quad (22)$$

where $\Phi$ is called the interpolation matrix and $\vec{F}$ is the vector that consists of the values of the forcing function at the computational nodes. The particular solutions can now be expressed in the Lagrangian form by using the above equation in equation (17). Thus,

$$w = \sum_{m=1}^{nt} \beta_m F_m, \quad (23)$$
where $\beta_m$ is given by

$$\beta_m = \sum_{k=1}^{nt} \psi_k \Phi_{km}^{-1}. \quad (24)$$

This form allows for ease in computations. Once the particular solution is expressed in the Lagrangian form, one can write the composite solution in the form

$$u^* = \sum_{i=1}^{nb} a_i G_i + \sum_{m=1}^{nt} \beta_m F_m. \quad (25)$$

The last step involves fitting the boundary condition. This is done by collocating at the boundary points. For a Dirichlet boundary, e.g., if $u^* = u_0$ at the collocation point $j$, then the collocation representation of equation (25) is used at this point.

$$\sum_{i=1}^{nb} a_i G_{ji} = u_0 - \sum_{m=1}^{nt} \beta_m F_m, \quad (26)$$

where the unknown coefficients on the left-hand side, $a_i$, are obtained by the solution of the linear system. The expression can be put in a vector-matrix form as

$$\mathbf{G} \mathbf{a} = \mathbf{u}_0 - \mathbf{\beta} \mathbf{F}. \quad (27)$$

The matrix, $\mathbf{G}$, is known as the particular solution matrix [21] which is useful for a direct estimate of the particular solution in terms of the function values at the nodes. The matrix is independent of the function, $F$, and depends only on the locations of the interpolation points. Hence, it can be computed once and stored and used for the same node placements for a different problem. This is one of the advantages of the Lagrangian formulation.

For the Neumann boundary, the collocation condition is applied after taking the derivative of equation (25) in the normal direction, i.e.,

$$\sum_{i=1}^{nb} a_i \frac{\partial G_i}{\partial n} = \frac{\partial u_0}{\partial n} - \sum_{m=1}^{nt} \frac{\partial}{\partial n} (\beta_m) F_m. \quad (28)$$

The equation is collocated at the boundary point, $j$, using the given value of the normal gradient in a similar manner to equation (26). The unknown vector, $\mathbf{a}$, of coefficients $a_i$ are evaluated by solving the linear algebraic problem

$$\mathbf{A} \mathbf{a} = \mathbf{r}, \quad (29)$$

where $\mathbf{r}$ is a known vector containing the right-hand side values of equation (26) or (28) for the Dirichlet or Neumann cases, respectively. This vector varies with each time step. Matrix $\mathbf{A}$ is the corresponding $(nb \times nb)$ coefficient matrix, generated by the l.h.s. of equation (26) or (28) and is independent of the time step. Hence, this matrix $\mathbf{A}$ can be LU decomposed just once and stored. Consequently, the time marching requires only a matrix-vector multiplication at each time step. Moreover, only boundary collocation is involved since $F$ is explicitly known. This provides for a scheme with high computational efficiency. An illustration of the collocation, source, and interpolation points is given in Figure 1 for a simple case of a circular geometry. Note that the method is mesh free and can be easily extended to three-dimensional problems. The ease of handling any complex geometry should also be noted.

4. TRANSIENT CONVECTION DIFFUSION REACTION PROBLEMS

In this section, we extend the operator splitting methodology to a transient convection diffusion reaction problem, the governing equation being of the form

$$\frac{\partial u}{\partial t} + v_x \frac{\partial u}{\partial x} + v_y \frac{\partial u}{\partial y} = D_x \nabla^2 u - R(u), \quad (30)$$
where \( v_x = v_x(x, y) \) and \( v_y = v_y(x, y) \) are the velocities in the \( x \) and \( y \) directions, respectively, and \( R(u) \) is the reaction rate term. The solution to this problem is formulated in the same manner as before by explicitly advancing all the nonlinear and quasi-linear terms, i.e., the convection and reaction terms. The solution procedure is similar, but an additional feature required is interpolation of \( u \) and the calculation of its spatial derivatives. This is done as follows. For the sake of simplicity, we set \( D_e = 1 \) in this section and assume that all the variables are scaled accordingly.

Equation (27) can be represented in a similar form to equation (1) with \( f \) defined as

\[
f = -v_x \frac{\partial u}{\partial x} - v_y \frac{\partial u}{\partial y} - R(u).\]

(31)

In order to advance the convection term, one needs the derivative of the dependent variable. In order to preserve the generality of the OS-RBF method, we use radial basis functions to interpolate the known values of the concentration variable over the domain. Thus, we have

\[
\sum_{k=1}^{n_t} \gamma_k c_k, \tag{32}
\]

where, as before, \( \gamma_k \) is now the radial basis function used to interpolate the dependent variables, and \( c_k \) are interpolation coefficients. Since interpolation is needed only for the purpose of calculation of the derivatives, the basis functions for interpolation of \( u \) can be chosen differently from that needed to calculate the particular solutions in the earlier section. In fact, the basis functions should be differentiable at all points. The function defined earlier by equation (20) does not have this property. Hence, we choose the standard multiquadrics [36,37] for the interpolation of \( u \).

Thus,

\[
\gamma_k = \sqrt{s^2 + r_k^2}, \tag{33}
\]

where \( s \) is a shift parameter. Again, it is convenient to express the vector of interpolation coefficients, \( \tilde{c} \), in the Lagrange form as

\[
\tilde{c} = \tilde{\gamma}^{-1} \tilde{u}. \tag{34}
\]

Hence, the interpolation for \( u \) at any node \( i \), \( u_i \), can be expressed as

\[
u_i = \{ \tilde{\gamma}^T \tilde{\gamma}^{-1} \tilde{u} \}_i. \tag{35}\]
Since the multiquadrics are continuous and differentiable for all $r \geq 0$, the derivative terms can be calculated as

$$\frac{\partial u}{\partial x} = \left\{ \frac{\partial \gamma}{\partial x} \right\}^T \gamma^{-1} \vec{u}, \quad (36a)$$

$$\frac{\partial u}{\partial y} = \left\{ \frac{\partial \gamma}{\partial y} \right\}^T \gamma^{-1} \vec{u}, \quad (36b)$$

where the derivatives of the radial basis functions are given by

$$\frac{\partial \gamma_k}{\partial x} = \frac{x - x_k}{\sqrt{s^2 + r_k^2}} \quad (37a)$$

and

$$\frac{\partial \gamma_k}{\partial y} = \frac{y - y_k}{\sqrt{s^2 + r_k^2}} \quad (37b)$$

The forcing function $f$ for the convection-reaction (equation (31)) case can now be calculated using the above expressions for the $x$ and $y$ derivatives. The solution procedure parallels that in Section 3.

5. CASE STUDIES AND ILLUSTRATIVE RESULTS

In this section, we demonstrate the method on some standard test problems to establish its validity and accuracy. Sections 5.1-5.3 consider cases with no convection, while Section 5.4 provides the test of the method for a simple convection with reaction problem. The geometry chosen were a rectangle, circle, and a more complex shape of a trilobe in order to show the ease of applicability of the method for various shapes. The results are compared with known analytical solutions wherever possible.

5.1. Pure Transient Diffusion

We first consider a simple transient diffusion problem (equation (1) with $f = 0$) in a unit square, with initial conditions, $u = 1$ at $t = 0$, and half of right vertical side, BC, kept at $u = 0$, the rest of the perimeter being insulated (Figure 2). The node placement is shown in Figure 3. The sources were placed on a larger square with an offset distance of 0.1. The time step used was 0.005. The time-dependent profiles along the left vertical side AB were compared with those obtained by Karur [6] using the dual reciprocity boundary element method and are shown in Figure 4. As one sees, the comparisons are excellent. This test case illustrates the method for a mixed type (Dirichlet-Neumann) boundary condition along the side BC.

![Figure 2. Geometry and boundary conditions on the unit square for Case 1.](image)
5.2. Transient Diffusion with Reaction

In this section, we show the application of the solution procedure to the problem of diffusion with reaction in a catalyst pellet. The problem is of importance in the design of chemical reactors, and a similar type of differential equation is encountered in many areas of physics (e.g., Liouville equation, thermal runaway problem). Of particular interest here is the use of the method for mixed type of boundary conditions. The first geometry is taken as a circle so that the results can be compared to the solutions obtained in earlier studies. The solution procedure, however, is not restrictive to these geometries, and one of the main advantages of the method is the ease of handling domains of complex shape. Hence, the second geometry is a more complex domain of a trilobe-shaped catalyst.

We first consider a unit circle with the whole perimeter being maintained at $u = 1$. The initial condition is $u = 0$ at $t = 0$. Two cases were considered for comparison are those corresponding to a first-order reaction, $f = -25u$, and a second-order reaction, $f = -25u^2$. Comparison of the solutions is made with a one-dimensional boundary element solution from Ramachandran [2].
Figure 5. Collocation point (*) and source point (o) locations for the unit circle.

(a) Comparison of 1-D and operator splitting solution for \( f = -25u \). (o, *, \( \nabla \) denote the MFS solutions; solid lines—1-D BEM.)

(b) Comparison of 1-D and operator splitting solution for \( f = -25u^2 \). (o, *, \( \nabla \) denote the MFS solutions; solid lines—1-D BEM.)
Figure 7. Collocation point (*) and source point (o) for the cusp of a trilobe (shaded regions denote no flux boundaries).

Figure 8. Concentration profiles for a second-order reaction $f = -25u^2$ in a cusp of a trilobe at $t = 0.05$ and $t = 0.15$.

The discretization for the circle is shown in Figure 5. Results are shown in Figures 6a and 6b for first-order and second-order reactions, respectively. The results of one-dimensional simulation are also indicated in these figures as dotted lines. As one can see (Figure 6), the comparison of the time-dependent solutions are excellent.

In order to illustrate the suitability of the OS-RBF method for nonregular geometries, we consider a cusp of a trilobe where Dirichlet and Neumann boundary conditions are imposed over different parts of the perimeter (Figure 7). The time-dependent profiles for a second-order reaction $f = 25u^2$ are shown in Figure 8 for $t = 0.05$ and $0.15$. The results show the trends that are consistent with physical considerations. No benchmark results were available for comparison for this case.

5.3. Microwave Heating of a Square Slab

As our third example, we consider the problem of heating a square slab using microwave radiation. This problem is illustrative of a system with a thermal boundary layer. The process
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\[ \nabla^2 u = \frac{\partial u}{\partial t} - n(u)|E|^2, \]  

(38)

where \( n(u) \) is the (temperature dependent) thermal absorption coefficient and \(|E|\) is the amplitude of the electric field. A power law model is often used for absorption coefficient. Thus, we choose \( n(u) = \beta u^n \). Typical values of \( n = 2 \) or 3 are used in practical modeling work. A constant thermal diffusivity is used in the model, and hence, \( t \) represents a scaled time. The amplitude of the electric field is assumed to be an exponentially decaying function in one of the spatial directions. Thus, we model the spatial variation of the electric field as \(|E| = \exp(-\gamma x)\) where \( x = 0 \) is the face exposed to microwave radiation and \( \gamma \) is the decay constant. Combining this information, the model equation is represented as

\[ \nabla^2 u = \frac{\partial u}{\partial t} - \beta e^{-\gamma x} u^n, \]  

(39)

with a boundary condition of \( u = 1 \) on the sides of the square and \( u = 1 \) as the initial condition.

This test problem was chosen to illustrate the accuracy of the solution procedure by comparison with a nonlinear DRM solution in [10] who used a completely implicit solution procedure for the nonlinear terms, and thus, a large system of nonlinear equations has to be solved at each time step. In contrast, the OS-RBF method provides for a method where only a linear system of equations involving only the boundary nodes has to be solved. Thus, a finer discretization of the interior can be afforded and a smaller time step since it involves only an increase in multiplication operations (after the initial calculations of the matrix coefficients). In Table 1, the comparison of the OS-RBF solution with the DRM solution at selected points for given values of \( \beta \) and \( \gamma \) and \( n = 2,3 \) for \( t = 1 \) is given. The node locations for the square were identical to Case 1. It can be seen from Table 1 that the OS-RBF and the more expensive DRM methods yield solutions that agree with each other correct to two decimal digits.

### Table 1. Comparison of operator splitting and DRM solutions for microwave heating.

<table>
<thead>
<tr>
<th>Coordinates ((x,y))</th>
<th>Case a (\beta - 11, \gamma - 2, n = 2) (DRM)</th>
<th>Case a (\beta - 12.3, \gamma - 4, n = 3) (OS-RBF)</th>
<th>Case b (\beta - 12.3, \gamma - 4, n = 3) (DRM)</th>
<th>Case b (\beta - 12.3, \gamma - 4, n = 3) (OS-RBF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1429, 0.6571)</td>
<td>1.332</td>
<td>1.332</td>
<td>1.240</td>
<td>1.247</td>
</tr>
<tr>
<td>(0.2857, 0.7142)</td>
<td>1.817</td>
<td>1.829</td>
<td>1.510</td>
<td>1.536</td>
</tr>
<tr>
<td>(0.4285, 0.5714)</td>
<td>2.027</td>
<td>2.042</td>
<td>1.550</td>
<td>1.577</td>
</tr>
<tr>
<td>(0.7142, 0.4285)</td>
<td>1.599</td>
<td>1.603</td>
<td>1.252</td>
<td>1.261</td>
</tr>
<tr>
<td>(0.8571, 0.2857)</td>
<td>1.253</td>
<td>1.252</td>
<td>1.098</td>
<td>1.101</td>
</tr>
<tr>
<td>(0.4285, 0.1428)</td>
<td>1.482</td>
<td>1.480</td>
<td>1.256</td>
<td>1.265</td>
</tr>
</tbody>
</table>

### 5.4. Convection Diffusion Reaction Problem

Finally, we consider a convection diffusion reaction problem in a rectangle of length 6 units \( \times 0.7 \) units, with the flow being in the \( x \)-direction at a constant velocity. We consider this case to compare the operator splitting method with the solutions obtained using DRM by Partridge et al. [1]. The rate form is linear here, i.e., \( R(u) = ku \). The node placement for the solution is shown in Figure 9. The boundary conditions imposed are \( u = 1 \) at \( x = 0 \) and \( \frac{\partial u}{\partial n} = 0 \) over the rest of the boundary, and initial condition is given by \( u = 0 \). Though the problem is essentially one dimensional, we illustrate the accuracy of the method at high Peclet numbers. Using the OS-RBF formulation, we avoid the iterative solution procedure involved in DRM, reducing the computational effort substantially. Figures 10 and 11 provide a comparison of the two methods...
at \( v_x = 1 \) m/s and \( v_x = 6 \) m/s for \( k = 0.278 \text{sec}^{-1} \). Since multiquadric interpolation has been used for the convection terms, a shift parameter \( s = 0.5 \) is used. Once again, it can be seen that solutions obtained from the DRM and the OS-RBF methods are in good agreement with each other.

6. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we present a novel methodology (OS-RBF) for solving time-dependent, nonlinear Poisson problems. We have combined the operator splitting approach (used extensively in other numerical methods) with a boundary only solution procedure called the method of fundamental solutions. The resulting Helmholtz equation is solved at every time step using radial basis functions for the interpolation of nonlinear terms. Consequently, this method is mesh free and dimension independent. Another feature of the method is that it requires only two LU decompositions (three if convective terms are present) to obtain the entire solution in space and time. Thus, once these coefficient matrices have been LU-factored, time stepping can be accomplished via a series of matrix-vector multiplication that substantially reduce the computational effort, as compared to traditional DRM. A number of prototypical linear and nonlinear Poisson problems are solved in order to benchmark this new OS-RBF method. The results are in close agreement with solutions presented in the literature. Although the OS-RBF method is only illustrated here for two-dimensional problems, its extension to three-dimensional problems is straightforward and requires only minor reprogramming.

A number of issues can be addressed as topics for future investigations in this area, and we mention a few of these here. The splitting method presented is second-order accurate in time, and the error accumulation with every time step might not be satisfactory for some complex flow problems. For such cases, this method can be extended using higher-order splitting methods that provide greater accuracy. Furthermore, radial basis functions with compact support [38] can be used for interpolation so that the interpolation matrices obtained will be sparse and iterative solvers can be used. The comparison of results obtained using different types of radial basis functions is also of interest. Extension to systems of Poisson equations is of importance to engineering problems that involve simultaneous solution of mass, momentum, and energy balances along with conservation equations for multiple chemical species. The OS-RBF method can be easily extended to such problems without a substantial increase in programming effort, since the generic linear framework of the equations obtained after operator splitting remains the same. Free and moving boundary problems present another useful application area and being a
grid-free method, one can easily apply the OS-RBF method to such problems, where a moving interface can be tracked by a collection of points without requiring extensive remeshing.

REFERENCES